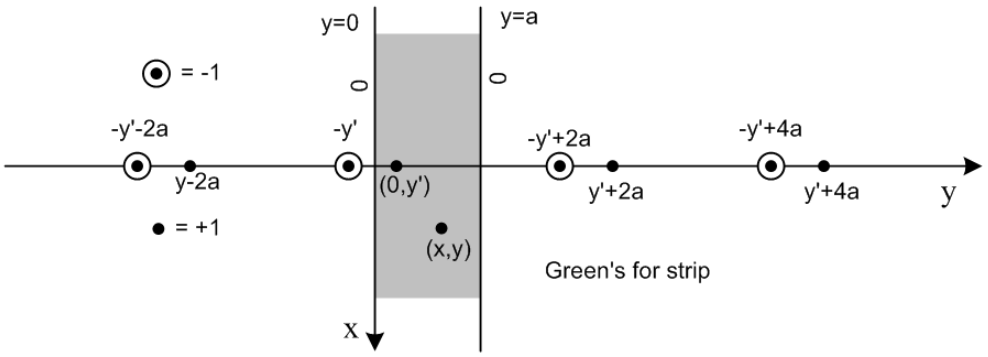
Exercise 6.35 v2 PhL 9.24.09

This has turned out to be a true "exercise" for the student "analyst" in that several non-trivial corners of analysis are touched. See history at the end. [This is Stakgold Vol II page 166.]

(1) Setting up the problem. **1**
(2) Convergence of the series. **1**
(3) Symmetries. **2**
(4) An Euler formula solves the problem. **2**
(5) Project the result onto Fourier components sin(nπy/a) **3**

(1) Setting up the problem. Method of images for a strip (2D) (166)



Here I have rotated the picture described in 6.120 so uses horizontal paper space. We put the Green's official point charge on the y axis so coordinates are (0,y'). The observation point is (x,y). I show where the claimed image charges should be located, an infinite number of each polarity. My y' = his η. The thing we seek is then g(**r**|**r'**) = g(x,y |0,y'). The sum of the fundies shown above is,

$$g(x,y\,|0,y') = -(1/4\pi)\Sigma_n \ln [\, x^2 + (y - \{y'+2na\})^2 \,] + (1/4\pi)\Sigma_n \ln [\, x^2 + (y - \{-y'+2na\})^2 \,] \qquad 1.1$$

where the Σ_n is over all integers 0,±1,±2.... Our assignment is to show that this formula is equivalent to an expression derived in (6.127) which is this

$$g(x,y\,|0,y') = \Sigma_{n=1}^{\infty} (1/n\pi)\, e^{-(n\pi/a)|x|} \sin(n\pi y/a) \sin(n\pi y'/a) \qquad 1.2$$

and we note in passing that this result is symmetric under y↔y' and is odd under y→-y.

(2) Convergence of the series. For large n, each series ends as $\Sigma_n \ln[4a^2n^2]$ and so is log divergent, so the first reasonable action is to combine the series to get something that converges.

$$g(x,y\,|0,y') = -(1/4\pi)\Sigma_n \ln \{ \; [\, x^2 + (y - y' - 2na)^2 \,] / [\, x^2 + (y + y' - 2na)^2 \,] \; \} \qquad 2.1$$

For large n, a term in the series approaches ln{1}, so with x or y or y' as the "variable", this series has a convergence radius of $1/R = \ln\{1\}^{1/n} = 0^{1/n} = 0$ so $R = \infty$ by Hadamard Ahlfors p 39. So it is very convergent indeed. You can put (x,y) wherever you like in the picture shown above.

(3) Symmetries. The log series form is the same if we take n→-n in either or both of the two square bracket expressions (or both at once). This follows at once from 1.1. This fact lets us show that 2.1 has the two symmetries mentioned above which 1.2 has. First, if we take y↔y' the quantity $(y - y'–2na)^2$ becomes $(y' - y–2na)^2$ and n→-n makes it $(y' - y+2na)^2 = (y - y'-2na)^2$ which is the starting point, so 2.1 is symmetric under y↔y'. Second, if we take y→-y, then $(y - y'–2na)^2 \rightarrow (y + y'+2na)^2$ and n→-n makes this become the denominator quantity $(y + y'- 2na)^2$. Similarly, the original denominator factor becomes the numerator factor, so the ratio flips under y→-y and the ln → -ln and so 2.1 is odd in y. The same would be true under y'→-y'. So our conclusion is that 2.1 has the same basic symmetries as 1.2, which is at least encouraging.

(4) An Euler formula solves the problem. We can at once rewrite 2.1 in terms of an infinite product

$$g(x,y\,|0,y') = -(1/4\pi)\ln\left(\Pi_n \{ [(y - y'–2na)^2 + x^2] / [(y +y'–2na)^2 + x^2] \} \right) \qquad 4.1$$

For large n, as noted above, the ratio → 1, so it is at least possible that the product is a finite quantity. Of course we know this is true since the series form above converges.

At this point, we quote the "four sine formula" which I derived in a separate document about infinite products. That formula is this:

$$\Pi_n [(n-x_1)(n-x_2)] / [(n-y_1)(n-y_2))] = [\sin(\pi x_1)\sin(\pi x_2)] / [\sin(\pi y_1)\sin(\pi y_2)] \qquad 4.2$$

which has this special case which applies to us

$$\Pi_n [(n-A)^2 + B^2] / [(n-C)^2 + D^2] = [\sin(\pi(A+iB))\sin(\pi(A-iB))] / [\sin(\pi(C+iD))\sin(\pi(C-iD))] \qquad 4.3$$

$$= [\mathrm{ch}(2\pi B) - \cos(2\pi A)] / [\mathrm{ch}(2\pi D) - \cos(2\pi B)]$$

To use this formula, we divide everything in 2.1 by $4a^2$ to get

$$g(x,y\,|0,y') = -(1/4\pi)\ln\left(\Pi_n \{ [([y - y']/2a–n)^2 + (x/2a)^2] / [([y + y']/2a–n)^2] + (x/2a)^2] \} \right) \qquad 4.4$$

We can then identify:

$$A = [y - y']/2a \qquad B = x/2a \qquad 4.5$$
$$C = [y + y']/2a \qquad D = x/2a$$

So we end up then with

$$g(x,y\,|0,y') = -(1/4\pi)\ln \{ [\mathrm{ch}(\pi x/a) - \cos(\pi(y-y')/a))] / [\mathrm{ch}(\pi x/a) - \cos(\pi(y+y')/a)] \} \qquad 4.6$$

which is a pretty nice zero-summation result, which Stakgold does not give back near p 163 where he is fiddling with this strip problem. Notice that this result maintains the required symmetries. If y→-y, get a -1. If y↔y', it stays the same.

Footnote: I derived the "multiple sine formulas" in my document on infinite products. I was led to the above A,B,C,D formula by a web site quoted op cit, namely

$$\prod_{k=-\infty}^{\infty} \frac{(k-z1)^2+z3^2}{(k-z2)^2+z4^2} = \frac{\text{Sin}[\pi\,(z1+i\,z3)]\,\text{Sin}[\pi\,(z1-i\,z3)]}{\text{Sin}[\pi\,(z2+i\,z4)]\,\text{Sin}[\pi\,(z2-i\,z4)]} = \frac{\text{Cosh}[2\,\pi\,z3]-\text{Cos}[2\,\pi\,z1]}{\text{Cosh}[2\,\pi\,z4]-\text{Cos}[2\,\pi\,z2]};$$

I wonder if Stakgold was expecting students to do this problem some other way. Maybe you can just directly project 2.1 onto the sin(ny). I like my way of doing it because it gives the compact result 4.6.

(5) Project the result onto Fourier components sin(nπy/a)

Our remaining task is to show that the summation result

$$g(x,y\,|0,y') = \Sigma_{n=1}^{\infty}\,(1/n\pi)\,e^{-(n\pi/a)|x|}\,\sin(n\pi y/a)\,\sin(n\pi y'/a) \qquad 1.2$$

reduces to our answer. Let's switch over to scaled variables. I could give them new names, but here I just give them their same names, understanding that later I have to reinsert factors. So here are our two formulas we need to show are the same:

$$f_1(x,y,y') = -(1/4\pi)\,\ln\{\ [\text{ch}(x)-\cos(y-y')]\,/\,[\text{ch}(x)-\cos(y+y')]\ \}$$

$$f_2(x,y,y') = \Sigma_{n=1}^{\infty}\,(1/n\pi)\,e^{-n|x|}\,\sin(ny)\,\sin(ny')$$

Since things are odd in y and y', we expect that f_2 is just a double Fourier Series expansion of f_1.

So let's expand f_1 onto the sin(ny) functions first, using Schaum p 131 using L = π. Then

$$b_n(x,y') = (1/\pi)\ \int_{-\pi}^{\pi} dy\,\sin(ny)\{\ -(1/4\pi)\,\ln\{\ [\text{ch}(x)-\cos(y-y')]\,/\,[\text{ch}(x)-\cos(y+y')]\ \}\ \}$$

$$= -(1/4\pi^2)\ \int_{-\pi}^{\pi} dy\,\sin(ny)\ \ln\{\ [\alpha-\cos(y-y')]\,/\,[\alpha-\cos(y+y')]\ \} \qquad \alpha = \text{ch}(x)$$

If we do parts at this point, "the parts" vanish because sin(ny) = 0 at both endpoints. Then

$\sin(ny) = (-1/n)\partial_y \cos(ny)$ so we get

$$b_n(x,y') = -(1/4\pi^2)\,(-1/n)(-)\ \int_{-\pi}^{\pi} dy\,\cos(ny)\,\partial_y\,\ln\{\ [\alpha-\cos(y-y')]\,/\,[\alpha-\cos(y+y')]\ \}$$

But we find that (Maple)

$$\partial_y \ln \{ [\alpha - \cos(y-y')] / [\alpha - \cos(y+y')] \} = \sin(y-y')/ [\alpha - \cos(y-y')] - \sin(y+y')/ [\alpha - \cos(y+y')]$$

$$= \sin(y-y')/ [\alpha - \cos(y-y')] - (y' \rightarrow -y')$$

Let's **ignore this second term for now (** subtract $y' \rightarrow -y'$) , so we then have

$$b_{n,1}(x,y') = - (1/4\pi^2 n) \int_{-\pi}^{\pi} dy \cos(ny) \sin(y-y')/ [\alpha - \cos(y-y')]$$

Now write

$$2\sin(y-y')\cos(ny) = \sin(y-y'-ny) + \sin(y-y'+ny)$$

Let's for the moment **ignore *this* second term** (add $n \rightarrow -n$ BUT there is a factor 1/n outside, so you really will want to *subtract* $n \rightarrow -n$!] and look at the first term

$$b_{n,1,1}(x,y') = - (1/8\pi^2 n) \int_{-\pi}^{\pi} dy \sin(y-y'-ny) / [\alpha - \cos(y-y')]$$

The integrand is periodic in y with period 2π, so the <u>end points are arbitrary</u> as long as they cover the range. Then we try $y'' = y-y'$ do that $dy'' = dy$ and we have

$$y-y'-ny = y'' - n(y''+y') = y''(1-n) - ny'$$

$$b_{n,1,1}(x,y') = - (1/8\pi^2 n) \int_{-\pi}^{\pi} dy'' \sin(y''(1-n) - ny') / [\alpha - \cos(y'')]$$

$$= - (1/8\pi^2 n) \int_{-\pi}^{\pi} dx \sin((1-n)x - ny') / [\alpha - \cos(x)]$$

GR on page 366 circa want the denominator to have the form

$$\text{den} \equiv (1+a^2) - 2a\cos(x) = 2a [(1+a^2)/2a - \cos(x)] = 2a [\alpha - \cos(x)]$$

where

$$\alpha = (1+a^2)/2a = (a^{-1}+ a)/2 = \text{ch}(x) \qquad \Rightarrow a = e^{-|x|} < 1 \qquad !$$

which certainly looks promising. We then have

$$b_{n,1,1}(x,y') = - (2a/8\pi^2 n) \int_{-\pi}^{\pi} dx \sin((1-n)x - ny') / \text{den}$$

But GR want to see 0 to π integrals, so write this as

$$\int_{-\pi}^{\pi} dx\ \sin((1-n)x - ny') / den = \int_{0}^{\pi} dx\ \sin((1-n)x - ny') / den(x) + \int_{-\pi}^{0} dx\ \sin((1-n)x - ny') / den(x)$$

$$= \int_{0}^{\pi} dx\ \sin((1-n)x - ny') / den(x) + \int_{0}^{\pi} dx\ \sin(-(1-n)x - ny') / den(-x)$$

$$= \int_{0}^{\pi} dx\ \{ \sin((1-n)x - ny') + \sin(-(1-n)x - ny')\} / den(x)$$

$$= 2 \int_{0}^{\pi} dx\ \{\sin(-ny')\cos((1-n)x) / den(x) = -2\sin(ny') \int_{0}^{\pi} dx\ \cos((1-n)x) / den(x)$$

This integral finally we can look up

$$\int_{0}^{\pi} dx\ \cos((n-1)x) / den(x) = \pi\, a^{n-1} / [(1-a^2)] \qquad a<1$$

GR p 366 second from bottom

so our integral is then

$$\int_{-\pi}^{\pi} dx \cos((1-n)x - ny') / den = -2\pi\sin(ny')\, a^{n-1} / [(1-a^2)]$$

and so we have found that

$$b_{n,1,1}(x,y') = -(2a/8\pi^2 n)(-2\pi)\sin(ny')\, a^{n-1} / [(1-a^2)] = (a/2\pi n)\sin(ny')\, a^{n-1} / (1-a^2)$$

Now our "second term" is obtained by subtracting $n \rightarrow -n$, BUT be careful. If we trace down through the GR integral, we will be getting

$$\int_{0}^{\pi} dx\ \cos((-n-1)x) / den(x) = \int_{0}^{\pi} dx\ \cos((n+1)x) / den(x) = \pi\, a^{n+1} / [(1-a^2)] \qquad a<1$$

so we really need to replace a^{n-1} by a^{n+1} when we add in this second term. The other two factors both change sign resulting in the same factor.

Note: The GR integral is really only valid for $n \geq 0$, something clarified on p 97 of Schaum. Ie, the LHS or GR is the same for -n, so the RHS must also be the same. So

$$b_{n,1}(x,y') = (a/2\pi n)\sin(ny')\,(1-a^2)^{-1}\,[\,a^{n-1} - a^{n+1}\,]$$

$$= (1/2\pi n)\sin(ny')\, a^n\,[\,1 - a^2]\,/\,(1-a^2)$$

$$= (1/2\pi n)\sin(ny')\, a^n = (1/2\pi n)\sin(ny')\, e^{-|x|}$$

Now we include our "other second term" – (y'→ –y'), but this just doubles the result, so here is our final answer:

$$b_n(x,y') = (1/\pi n)\sin(ny')\, a^n = (1/\pi n)\sin(ny')\, e^{-n|x|}$$

If we now go back to our target function

$$f_2(x,y,y') = \Sigma_{n=1}^{\infty} (1/n\pi)\, e^{-n|x|} \sin(ny) \sin(ny')$$

We see that we our answer agrees! Of course then we could go back and insert all the π/a factors, but we know we are done!

Conclusion: I have found several ways to express the answer to this problem:

$$g(x,y\,|0,y') = -(1/4\pi)\Sigma_n \ln \{ [x^2 + (y - y'-2na)^2] / [x^2 + (y +y'-2na)^2] \} \quad 2.1$$

$$g(x,y\,|0,y') = -(1/4\pi) \ln(\Pi_n \{ [(y - y'-2na)^2 + x^2] / [(y +y'-2na)^2 + x^2] \}) \quad 4.1$$

$$g(x,y\,|0,y') = -(1/4\pi) \ln \{ [ch(\pi x/a) - \cos(\pi(y-y')/a))] / [ch(\pi x/a) - \cos(\pi(y+y')/a)] \} \quad 4.6$$

$$g(x,y\,|0,y') = \Sigma_{n=1}^{\infty} (1/n\pi)\, e^{-(n\pi/a)|x|} \sin(n\pi y/a) \sin(n\pi y'/a) \quad 1.2$$

I have shown that all four forms are the same. Stak provides another integral form in 6.123.

9.24.09 Spent about 8 hours flailing on this problem, in line in the raw notes
Plans A through G, just not clear what to do with the sum. Then started "v1" doc dedicated to this problem.

9.25.09 Started separately into my "v1" doc just to solve this problem. I tried projecting the image sum formula onto sin(mπy/a). Ended up with Ci and Si functions. I then stumbled onto the 4-sine identity on a web site. Got the cosh result and tried then projecting onto sin(mπy/a) but went astray down various wrong paths.
I am now deleting this document since it is not worth saving.

6.26.09 Decided to learn about infinite products in attempt derive the 4-sine thing. Started document "theory of infinite products.doc".

At some point I worried about the convergence of these two series, see doc "convergence question.doc:

A. $\Sigma_{n=1}^{\infty} \ln [(n-x)/(n-y)] = \Sigma_{n=1}^{\infty} a_n$

B. $\Sigma_n \ln [(n-x)/(n-y)] = \ln(x/y) + \Sigma_{n=1}^{\infty} (a_n + a_{-n})$ $\qquad a_n = \ln [(n-x)/(n-y)]$

The integral test said A diverges and B converges, and B is what is in the 2-sine formula.

6.27.09 dinner prep day, no work done

6.28.09 some more work on the infinite product document

6.29.09 final work on infinite products, learned ways to derive Euler's product for the sine from which all my other formulas of interest descend. Then resumed on this problem in this "v2" document and finally nailed it. So 6 days of work, most full time.
"*Never never never give up*" -WC