Green's functions and change of coordinates PhL 3.26.05

For some reason, I have allowed myself to be confused in this matter.

A. Surfaces in 3D. Consider this function f: $R^2 \rightarrow R$.

$$f(x,y) = 3x^2 + 2y + 5$$

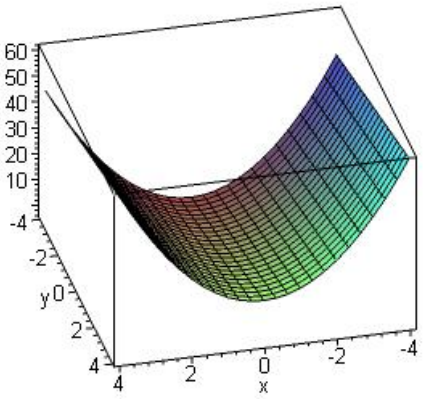
Let x,y be coordinates in system S, and let x',y' be coordinates in system S'. Assume there is some invertible transformation s: $R^2 \rightarrow R^2$ so that $(x',y') = s(x,y)$ meaning $x' = s_1(x,y)$ and $y' = s_2(x,y)$. Let the inverse map be called h so that $(x,y) = h(x',y')$ meaning $x = h_1(x',y')$ and $y = h_2(x',y')$. We can certainly write the above equation as follows:

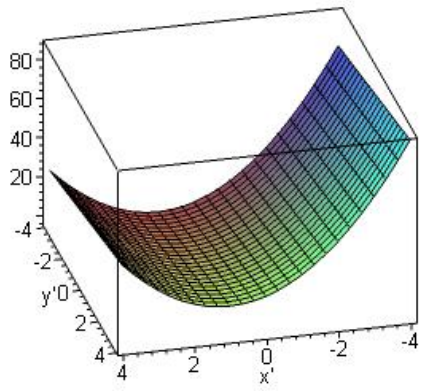
$$f(h_1(x',y'), h_2(x',y')) = 3\, h_1(x',y')^2 + 2\, h_2(x',y') + 5 \quad \equiv \quad F(x',y')$$

where on the right we have defined a function $F:R^2 \rightarrow R$ where $F = f\, s$.

Question: what exactly *is* this "new equation"?

I like to think of $f(x,y) = 3x^2 + 2y + 5$ as a "surface" plotted above the x,y plane. It is "the surface you see" in the S coordinate system, shown on the left, the valley minimum is along x = 0





Suppose g is a simple translation in the x direction: $x' = x + 1$ and $y' = y$. Then

$$F(x',y') = 3(x'-1)^2 + 2y' + 5$$

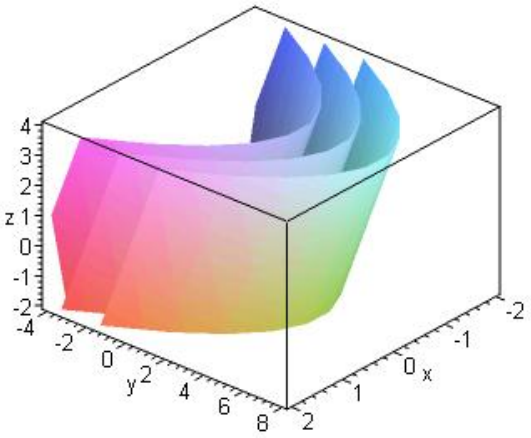
This is the surface shown on the right, where now the valley is along x' = -1. I think it is fair to say that there is a rigid surface here and on the left we view it in system S, and on the right we view it in S'. It is the same surface in both cases. As long as the coordinate transformation g is a rigid body motion meaning rotation and/or transformation, the surface will not be distorted, it is just viewed from a different "frame of reference". If we do some other transformation, such as x' = 2x and y' = 3y, then the surface viewed in the new frame has a different shape. Still, it is the same surfaced "viewed" in a new system.

B. Surfaces in 4D. Consider this function f: $R^3 \rightarrow R$.

$f(x,y,z) = 3x^2 + 2y + 5 - z$

We cannot plot this surface in the same way as in A. The traditional plotting method is to pick a number for function f, and then plot all points in R^3 which give that number. Here is an example of picking three numbers for f, creating 3 surfaces in Maple (the numbers were f = 8,12 and 16)

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with(plots):implicitplot3d( {seq(3*x^2 + 2*y + 5 - z = 4*N,N=2..4)}, x=-2..2, y=-4..8, z=-2..4, style=patchnogrid);
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As before, we can define transformation s and inverse h and thereby obtain

$f(x,y,z) = 3x^2 + 2y + 5 - z$

$f(h_1(x',y',z'), h_2(x',y',z'), h_3(x',y',z')) = 3\, h_1(x',y',z')^2 + 2\, h_2(x',y',z') + 5 - h_2(x',y',z')$

$\equiv F(x',y',z')$

We would say that F(x',y',z') was the same we started with, but viewed in a new frame S'. In frame S we have certain surfaces where f = 8,12,16 as plotted above, and in S' we could plot F = 8,12,16 to see where these same surfaces are now located.

C. Application: 2D Green's Functions

Suppose we have some Green's Function that is 0 on some boundary σ, viewed in reference frame S

$g(x,y|x_o,y_o) = f(x,y,x_0,y_0)$

We want to view this in reference frame S' as described above in A. We can write:

$g(h_1(x',y'), h_2(x',y')|\, h_1(x_0',y_0'), h_1(x_0',y_0'))$

$= f(h_1(x',y'), h_2(x',y'), h_1(x_0',y_0'),y_0)\ h_1(x_0',y_0')$

$\equiv\ F(x',y',x_0',\ y_0')\ \equiv G(x',y'|x_o',y_o')$

The new Green's Function G will be 0 on boundary σ' which is the same boundary viewed in the new frame S'.

But how do we know that G will really be a "Green's Function"? It will certainly vanish on σ', but how do we know it satisfies the new Green's Equation? Here is the original Green's Equation:

$L(x,y)\ g(x,y|x_o,y_o)\ = \delta(x-x_0)\ \delta(y-y_0)$ $\qquad\qquad$ $g = 0$ on σ

$L(\mathbf{r})\ g(\mathbf{r}|\mathbf{r}_o)\ = \delta(\mathbf{r}-\mathbf{r}_0)$

Our transformation is $\mathbf{r}' = R\,\mathbf{r} + \mathbf{a}$ where we *limit ourselves* to rotation plus translation. This will have some inverse which write as $\mathbf{r} = Q\,\mathbf{r}' + \mathbf{b}$. Let's write these symbolically as $\mathbf{r}' = S\mathbf{r}$ and $\mathbf{r} = H\mathbf{r}'$ where S and H are *not* square matrices, they are the transformations just shown. Then:

$L(\mathbf{r})\ g(\mathbf{r}|\mathbf{r}_o)\ = \delta(\mathbf{r}-\mathbf{r}_0)$
$L(H\mathbf{r}')\ g(H\mathbf{r}'|\ H\mathbf{r}'_0)\ = \delta(H\mathbf{r}'\ H\mathbf{r}'_0)$

How if L is the Laplacian, and if H is just a translation, we know that the above two equations are exactly the same. In this case, we feel confident that $G(x',y'|x_o',y_o')$ is in fact the Green's Function in frame S'. But we also are pretty confident this also works for H being an arbitrary rotation, and then G is just the Green's Function in the new rotated frame S', and we know that L is rotationally invariant in form. So without being too rigorous, I think we can conclude that $G(x',y'|x_o',y_o')$ solves the Laplace Green's equation in the new frame S' arrived at by some combination of translation and rotation. We would also conclude this was the case in 3D, though today we are looking at 2D.

D. 2D Green's Function Example

Suppose you have this Green's function for a full strip running from y = 0 to y = a:

$g(x,y\ |x_0,y_0) = \Sigma_{n=1}^{\infty}\ (1/n\pi)\ e^{-(n\pi/a)|x-x0|}\ \sin(n\pi y/a)\ \sin(n\pi y_0/a)$ $\quad=\ f(x,y,x_0,y_0)$ in this case.

We know that g = 0 on σ which is the boundary consisting of lines at y = 0 and y = a.

Now go to a new frame S' defined by y' = y-a/2. Now the y = 0 boundary is at y' = -a/2, and the y=a boundary is at +a/2. We then write: $\qquad y = y' + a/2$

$G(x',y'|x_o',y_o')\ =\ f(x',y'+a/2,x_0',y_0' + a/2)$

$= \Sigma_{n=1}^{\infty}\ (1/n\pi)\ e^{-(n\pi/a)|x'-x0'|}\ \sin(n\pi[y'+a/2]/a)\ \sin(n\pi[y_0'+a/2]/a)$

This is not rocket science. We have for this problem two other "forms" of g, as follows. Here is one of those other forms:

$$g(x,y\,|x_0,y_0) = -(1/4\pi)\Sigma_n \ln [\,(x-x_0)^2 + (y - \{y_0+2na\})^2\,] + (1/4\pi)\Sigma_n \ln [(x-x_0)^2 + (y - \{-y_0+2na\})^2\,]$$

$$= -(1/4\pi)\Sigma_n \ln \{\,[\,(x-x_0)^2 + (y - \{y_0+2na\})^2\,] / [(x-x_0)^2 + (y - \{-y_0+2na\})^2\,]$$

In our new frame S', we have at once:

$$G(x',y'|x_o',y_o') = (1/4\pi)\Sigma_n \ln \{\,[\,(x'-x'_0)^2 + (y' - \{y'_0+2na\})^2\,] / [(x'-x'_0)^2 + (y' + a - \{-y'_0+2na\})^2\,]$$

Notice that the main change of interest is in the denominator factor where we pick up +a .

The third form is this:

$$g(x,y\,|x_0,y_0) = -(1/4\pi) \ln \{\,[ch(\pi(x-x_0)/a) - \cos(\pi(y-y_0)/a))] / [ch(\pi(x-x_0)/a) - \cos(\pi(y+y_0)/a)]\,\}$$

and viewed in the S' frame this becomes

$$G(x',y'|x_o',y_o') = -(1/4\pi) \ln \{\,[ch(\pi(x'-x'_0)/a) - \cos(\pi(y'-y'_0)/a))] / [ch(\pi(x'-x'_0)/a) - \cos(\pi(y'+y'_0 +a)/a)]\,\}$$

and once again, the main change of interest is the extra +a in the denominator cosine. We know that $\cos(A + \pi) = -\cos A$, so we could rewrite this last result as

$$G(x',y'|x_o',y_o') = -(1/4\pi) \ln \{\,[ch(\pi(x'-x'_0)/a) - \cos(\pi(y'-y'_0)/a))] / [ch(\pi(x'-x'_0)/a) + \cos(\pi(y'+y'_0)/a)]\,\}$$

We now summarize what we know. In frame S we have:

$$g(x,y\,|x_0,y_0) = \Sigma_{n=1}^{\infty} (1/n\pi)\, e^{-(n\pi/a)|x-x0|} \sin(n\pi y/a) \sin(n\pi y_0/a)$$

$$g(x,y\,|x_0,y_0) = -(1/4\pi)\Sigma_n \ln \{\,[\,(x-x_0)^2 + (y - \{y_0+2na\})^2\,] / [(x-x_0)^2 + (y - \{-y_0+2na\})^2\,]$$

$$g(x,y\,|x_0,y_0) = -(1/4\pi) \ln \{\,[ch(\pi(x-x_0)/a) - \cos(\pi(y-y_0)/a))] / [ch(\pi(x-x_0)/a) - \cos(\pi(y+y_0)/a)]\,\}$$

and in the new frame S' obtained by y' = y-a/2 and x' = x we have

$$G(x',y'|x_o',y_o') = \Sigma_{n=1}^{\infty} (1/n\pi)\, e^{-(n\pi/a)|x'-x0'|} \sin(n\pi[y'+a/2]/a) \sin(n\pi[y_0'+a/2]/a)$$

$$G(x',y'|x_o',y_o') = (1/4\pi)\Sigma_n \ln \{\,[\,(x'-x'_0)^2 + (y' - \{y'_0+2na\})^2\,] / [(x'-x'_0)^2 + (y' + a - \{-y'_0+2na\})^2\,]$$

$$G(x',y'|x_o',y_o') = -(1/4\pi) \ln \{\,[ch(\pi(x'-x'_0)/a) - \cos(\pi(y'-y'_0)/a))] / [ch(\pi(x'-x'_0)/a) + \cos(\pi(y'+y'_0)/a)]\,\}$$

Just for completeness, let's restate these last three for a ***vertical*** **strip x = -a/2 to a/2**. This just means swapping all x's and y's on the RHS

$$G(x',y'|x_o',y_o') = \Sigma_{n=1}^{\infty} (1/n\pi)\, e^{-(n\pi/a)|y'-y0'|} \sin(n\pi[x'+a/2]/a) \sin(n\pi[x_0'+a/2]/a)$$

$$G(x',y'|x_o',y_o') = (1/4\pi)\Sigma_n \ln \{ [(y'-y'_0)^2 + (x' - \{x'_0+2na\})^2] / [(y'-y'_0)^2 + (x' + a - \{-x'_0+2na\})^2]$$

$$G(x',y'|x_o',y_o') = -(1/4\pi) \ln \{ [ch(\pi(y'-y'_0)/a) - \cos(\pi(x'-x'_0)/a))] / [ch(\pi(y'-y'_0)/a) + \cos(\pi(x'+x'_0)/a)] \}$$

And let's now **get rid of all primes** on these last three and regard them as basic results and change G to g

$$g(x,y|x_o,y_o) = \Sigma_{n=1}^{\infty} (1/n\pi)\, e^{-(n\pi/a)|y-y0|} \sin(n\pi[x+a/2]/a) \sin(n\pi[x_0+a/2]/a)$$

$$g(x,y|x_o,y_o) = (1/4\pi)\Sigma_n \ln \{ [(y-y_0)^2 + (x - \{x_0+2na\})^2] / [(y-y_0)^2 + (x + a - \{-x_0+2na\})^2]$$

$$g(x,y|x_o,y_o) = -(1/4\pi) \ln \{ [ch(\pi(y-y_0)/a) - \cos(\pi(x-x_0)/a))] / [ch(\pi(y-y_0)/a) + \cos(\pi(x+x_0)/a)] \}$$

Finally, to match earlier work, **let's replace $x_\circ$ with x'** (totally different meaning),

$$g(x,y|x',y') = \Sigma_{n=1}^{\infty} (1/n\pi)\, e^{-(n\pi/a)|y-y'|} \sin(n\pi[x+a/2]/a) \sin(n\pi[x'+a/2]/a)$$

$$g(x,y|\,x',y') = (1/4\pi)\Sigma_n \ln \{ [(y-y')^2 + (x - \{x'+2na\})^2] / [(y-y')^2 + (x + a - \{-x'+2na\})^2]$$

$$g(x,y|\,x',y') = -(1/4\pi) \ln \{ [ch(\pi(y-y')/a) - \cos(\pi(x-x')/a))] / [ch(\pi(y-y')/a) + \cos(\pi(x+x')/a)] \}$$

Often people then **set a = π,** so let's now do that as well

$$g(x,y|x',y') = \Sigma_{n=1}^{\infty} (1/n\pi)\, e^{-n|y-y'|} \sin(n[x+\pi/2]) \sin(n[x'+\pi/2])$$

$$g(x,y|\,x',y') = (1/4\pi)\Sigma_n \ln \{ [(y-y')^2 + (x - \{x'+2n\pi\})^2] / [(y-y')^2 + (x + \pi - \{-x'+2n\pi\})^2]$$

$$g(x,y|\,x',y') = -(1/4\pi) \ln \{ [ch(y-y') - \cos(x-x')] / [ch(y-y') + \cos(x+x')] \}$$