**Chapter 6 scraps** PhL 7.21.09

$h_n \equiv \int_0^\alpha d\varphi\, f(\varphi) \sin(\pi n\varphi/\alpha)$ $\qquad C_n\, b^{n\pi/\alpha} = (2/\alpha)\, h_n \qquad C_n = (2b^{-n\pi/\alpha}/\alpha)\, h_n$

$u(r,\varphi) = \Sigma_{n=1}^{\infty} C_n\, r^{n\pi/\alpha} \sin(\pi n\varphi/\alpha) = \Sigma_{n=1}^{\infty} (2b^{-n\pi/\alpha}/\alpha)\, h_n\, r^{n\pi/\alpha} \sin(\pi n\varphi/\alpha)$

so we have

$u(r,\varphi) = \Sigma_{n=1}^{\infty} b_n (r/b)^{n\pi/\alpha} \sin(\pi n\varphi/\alpha)$

$b_n \equiv (2/\alpha) \int_0^\alpha d\varphi\, f(\varphi) \sin(\pi n\varphi/\alpha)$

where here $b_n$ is the "official" Fourier sine coefficient, and this "b" is unrelated to radius b.

First our "periodic range" is $2L = \alpha$ so that $L = \alpha/2$. This is because $f(\varphi)$ is only defined on $(0,\varphi)$. The series we see on page 131 is then this:

$f(\varphi) = a_o/2 + \Sigma_{n=1}^{\infty} ( a_n \cos(2n\pi\varphi/\alpha) + b_n \sin(2n\pi\varphi/\alpha) )$

but this does not agree with what we have in our current problem. So define $2n = m$ and write the above Fourier form as

$f(\varphi) = a_o/2 + \Sigma_{m=2,4,..}^{\infty} ( a_m \cos(m\pi\varphi/\alpha) + b_m \sin(m\pi\varphi/\alpha) ) = \Sigma_{m=2,4,..}^{\infty} b_m \sin(m\pi\varphi/\alpha)$

and this does match our form, but is then missing half the terms!

Notice that this is a partial Fourier Series. The cos terms are missing because $u = 0$ on $\varphi = 0$, and this statement includes the constant term. If we look at Schaum p 131, we see the right form for a FS and we can read of the projection formula

$C_n\ b^{\pi n/\alpha} = (1/\alpha) \int_0^{2\alpha} d\varphi\ f(\varphi) \sin(\pi n\varphi/\alpha)$ n = 1,2,3...

HOWEVER, something is not quite right here because f(φ) is only defined in the range (0,α).

The condition that R(0) = 0 requires that $C_0 = 0$, so we end up with these as the basis solutions to our problem:

$\Phi_\sigma(\varphi) = \sin(\pi n\varphi/\alpha)$
$R_\sigma(r) = r^{\pi n/\alpha}$

Our general solution is then $u = \Sigma_\sigma\ R_\sigma(r)\ \Phi_\sigma(\varphi) = \Sigma_{n=1}^{\infty}\ A_n\ r^{\pi n/\alpha} \sin(\pi n\varphi/\alpha)$ -- there is no n=0 contribution for the reason just noted. Actually, n = 0 kills

Solve Laplace's equation for a disk **wedge** where we are given certain values all around the wedge boundary. I screwed this up and only got cleared up with Plan G, so you should skip all the other stuff and jump down to that point. I keep the raw screw up notes anyway.

**Plan A.** So here is one approach:

$u(r,\varphi) = \Sigma\ f_n\ r^{|n|} e^{in\varphi}$ // wrong form for these BC's !!!!

1. $u(r,0) = \Sigma\ f_n\ r^{|n|} = 0$ for any r
2. $u(r,\alpha) = \Sigma\ f_n\ r^{|n|} e^{in\alpha} = 0$ for any r
3. $u(1,\varphi) = \Sigma\ f_n\ e^{in\varphi} = f(\varphi)$ for $\varphi = 0$ to $\alpha$

Question: the first of these conditions would seem to be a different equation for each value or r, so it is really an infinite number of equations. But $a_n$ is an infinite number of coefficients. Not sure how to proceed here.

**Plan B** Wedge **Superposition**? Call the wedge solution u(r,φ). Suppose there were N wedges around the circle. Then the BC's superpose OK so we could say this:

$u(r,\varphi) + u(r,\varphi+(2\pi/N)) + u(r,\varphi+2(2\pi/N)) + ... + u(r,\varphi+(N-1)(2\pi/N)) = 6.11$

where we set $f(\psi) = f(\psi+i(2\pi/N))$ and so is periodic. i = 0,1,2...N-1 But this is not right, because we have extra slit BC's where u = 0 is being forced, so this superposition does not work right.

**Plan C. Real and Imaginary** with original idea of Plan A (which is wrong!): Let $f_n = a_n + i\ b_n$

1. $\Sigma\ a_n\ r^{|n|} + \Sigma\ i\ b_n\ r^{|n|} = 0 \quad \Rightarrow \quad \Sigma\ a_n\ r^{|n|} = 0$ AND $\Sigma\ b_n\ r^{|n|} = 0$

2. $\Sigma (a_n + i\, b_n)\, r^{|n|}( \cos(n\alpha) + i\sin(n\alpha)) = 0$

$\Rightarrow \quad \Sigma\, r^{|n|}\, [a_n \cos(n\alpha) - b_n \sin(n\alpha)] = 0$

$\Rightarrow \quad \Sigma\, r^{|n|}\, [b_n \cos(n\alpha) + a_n \sin(n\alpha)] = 0$

3. $\Sigma (a_n + i\, b_n)\, ( \cos(n\varphi) + i\sin(n\varphi)) = f(\varphi)$ assume real

$\Rightarrow \Sigma\, [a_n \cos(n\varphi) - b_n \sin(n\varphi)] = f(\varphi)$ // both these on just 0 to $\alpha$
$\Rightarrow \Sigma\, [b_n \cos(n\varphi) + a_n \sin(n\varphi)] = 0$

So now I seem to have 6 equations to play with: I assume that $f(\varphi) = 0$ at its two endpoints.

$\Sigma\, a_n\, r^{|n|} = 0$ // r in 0,1
$\Sigma\, b_n\, r^{|n|} = 0$

$\Sigma\, r^{|n|}\, [a_n \cos(n\alpha) - b_n \sin(n\alpha)] = 0$ // r in 0,1
$\Sigma\, r^{|n|}\, [b_n \cos(n\alpha) + a_n \sin(n\alpha)] = 0$

$\Sigma\, [a_n \cos(n\varphi) - b_n \sin(n\varphi)] = f(\varphi)$ // both these on just 0 to $\alpha$
$\Sigma\, [b_n \cos(n\varphi) + a_n \sin(n\varphi)] = 0$

These should be valid with r = 1 I would guess, so have

1a $\Sigma\, a_n = 0$
1b $\Sigma\, b_n = 0$

2a $\Sigma\, [a_n \cos(n\alpha) - b_n \sin(n\alpha)] = 0$
2b $\Sigma\, [b_n \cos(n\alpha) + a_n \sin(n\alpha)] = 0$

3a $\Sigma\, [a_n \cos(n\varphi) - b_n \sin(n\varphi)] = f(\varphi)$ // both these on just 0 to $\alpha$
3b $\Sigma\, [b_n \cos(n\varphi) + a_n \sin(n\varphi)] = 0$

But if I combine 2a with 3a, I get $f(\varphi) = 0$ which is nonsense, so I guess r = 1 was no go. Let's try r = 1/2 instead:

$\Sigma\, a_n\, (1/2)^{|n|} = 0$
$\Sigma\, b_n\, (1/2)^{|n|} = 0$

$\Sigma\, (1/2)^{|n|}\, [a_n \cos(n\alpha) - b_n \sin(n\alpha)] = 0$
$\Sigma\, (1/2)^{|n|}\, [b_n \cos(n\alpha) + a_n \sin(n\alpha)] = 0$

$\Sigma\, [a_n \cos(n\varphi) - b_n \sin(n\varphi)] = f(\varphi)$ // both these on just 0 to $\alpha$
$\Sigma\, [b_n \cos(n\varphi) + a_n \sin(n\varphi)] = 0$

Well this is taking me nowhere and just seems all wrong somehow.

**Plan D.** Can I write a **Fourier transform on a wedge** instead of a circle. Here is the usual FT for a periodic function:

$$f(\varphi) = \Sigma_n f_n e^{in\varphi}$$

$$f_n = (1/2\pi) \int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi} d\varphi$$

Now change variables to $\psi$ so that integral goes from $-\alpha/2$ to $+\alpha/2$. Thus have $\psi = \varphi * (\alpha/2\pi)$. Then I guess we have: (page 94)

$$f(\varphi(\psi)) = \Sigma_n f_n e^{in(2\pi\psi/\alpha)} = \Sigma_n f_n e^{i(2\pi n/\alpha)\psi}$$

$$f_n = (1/2\pi) \int_{-\alpha/2}^{\alpha/2} f(\varphi(\psi)) e^{-i(2\pi n/\alpha)\psi} d\psi \ (2\pi/\alpha)$$

$$= (1/\alpha) \int_{-\alpha/2}^{\alpha/2} f(\varphi(\psi)) e^{-i(2\pi n/\alpha)\psi} d\psi$$

So let's call $f(\varphi(\psi)) = F(\psi)$ and it is defined only from $-\alpha/2$ to $+\alpha/2$. Our Fourier Series is this:

$$F(\psi) = \Sigma_n f_n e^{i(2\pi n/\alpha)\psi}$$

$$f_n = (1/\alpha) \int_{-\alpha/2}^{\alpha/2} d\psi \, F(\psi) e^{-i(2\pi n/\alpha)\psi}$$

Now on our wedge we are given some $F(\psi)$ specified on the rim of the wedge. We are at $r = 1$. Our general model for a harmonic solution is this:

$$u(r,\varphi) = \Sigma_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\varphi}$$ // wrong!

or

$$u(r,\varphi(\psi)) = \Sigma a_n r^{|n|} e^{i(2\pi n/\alpha)\psi}.$$

Then on our wedge edge we have

$$u(1,\varphi(\psi)) = \Sigma a_n e^{i(2\pi n/\alpha)\psi} = f(\psi)$$

Using our adjusted wedge FS, we then have

$$a_n = (1/\alpha) \int_{-\alpha/2}^{\alpha/2} d\psi\, f(\psi)\, e^{-i(2\pi n/\alpha)\psi}$$

which in turn implies

$$a_{-n} = (1/\alpha) \int_{-\alpha/2}^{\alpha/2} d\psi\, f(\psi)\, e^{+i(2\pi n/\alpha)\psi}$$

Now let's rewrite the series with r = 1 like this:

$$u(r,\varphi) = \Sigma\, a_n\, r^{|n|} e^{in\varphi} = a_0 + \Sigma_{n=1}^{\infty} r^n\, a_n\, e^{in\varphi} + \Sigma_{n=1}^{\infty} r^n\, a_{-n}\, e^{-in\varphi}$$

$$= a_0 + \Sigma_{n=1}^{\infty} r^n \{ (a_n + a_{-n}) \cos(n\varphi) + i\,(a_n - a_{-n}) \sin(n\varphi) \}$$

Now we have

$$a_n + a_{-n} = (1/\alpha) \int_{-\alpha/2}^{\alpha/2} d\psi\, f(\psi)\, 2 \cos(2\pi n\psi/\alpha)$$

$$a_n - a_{-n} = -\,(i/\alpha) \int_{-\alpha/2}^{\alpha/2} d\psi\, f(\psi)\, 2 \sin(2\pi n\psi/\alpha)$$

So this leaves us with

$$u(r,\varphi) = a_o + \Sigma_{n=1}^{\infty} r^n\, (1/\alpha) \int_{-\alpha/2}^{\alpha/2} d\psi\, f(\psi)\, 2 \{ \cos(2\pi n\psi/\alpha) \cos(n\varphi) + \sin(2\pi n\psi/\alpha) \sin(n\varphi) \}$$

The trig rule becomes just $\cos(2\pi n\psi/\alpha - n\varphi)$, so I then get

$$u(r,\varphi) = a_o + \Sigma_{n=1}^{\infty} r^n\, (2/\alpha) \int_{-\alpha/2}^{\alpha/2} d\psi\, f(\psi) \cos(2\pi n\psi/\alpha - n\varphi)$$

So I have something with the right kind of sum, but I am not getting the power $r^{n\pi/\alpha}$ , and other things are wrong as well.

**Plan E.** Go way back to top of page 92. Let's try changing variables at this point to $\psi = \varphi * (\alpha/2\pi)$ so that $d\psi = d\varphi * (\alpha/2\pi)$ and $\partial_\psi = \partial_\varphi\, (2\pi/\alpha)$. Then our first equation there becomes

$$-\partial_\psi^2 \Phi[\varphi(\psi)] = -(2\pi/\alpha)^2 \partial_\varphi^2 \Phi[\varphi] = (2\pi/\alpha)^2\, \lambda\, \Phi[\varphi(\psi)]$$

which we might write as

$$-\partial_\psi^2 \Psi(\psi) = (2\pi/\alpha)^2\, \lambda\, \Psi(\psi)$$

But now I don't have any requirement at boundaries on $\Psi(\psi)$ so nowhere to go. [ but that is the whole point, the wedge provides such BC's, see G below)

**Plan F.** Same point. What happens if you impose **"periodic BC's"** such that [ but what justification is there for doing this? The wedge does not have periodic BC's! ]

$\Phi(\varphi) = \Phi(\varphi + \alpha)$ and same for derivative

Then if we try $\Phi = e^{i\sigma\varphi}$ so that $\Phi' = i\sigma\,\Phi$ and $\Phi'' = -\sigma^2\Phi$, so what values can $\lambda = \sigma^2$ have?

$e^{i\sigma\varphi} = e^{i\sigma(\varphi+\alpha)}$ => $e^{i\sigma\alpha} = 1$ => $\sigma\alpha = n(2\pi)$ => $\sigma = n\,(2\pi/\alpha)$

Then $\lambda = n^2\,(2\pi/\alpha)^2$. I think the r solutions are then $r^{2\pi n/\alpha}$ which reduces to $r^n$ for $\alpha = 2\pi$. But Stak does not have that factor of 2 in his exponent, so this path is not going to the right place either.

**Plan G.** The correct solution (with a little help from the web)

http://quantumrelativity.calsci.com/Physics/EandM6.html

Look at Stak page 92 but, as above, write $\lambda = \sigma^2$ and we try $\Phi = e^{i\sigma\varphi}$ so that $\Phi' = i\sigma\,\Phi$ and $\Phi'' = -\sigma^2\Phi$, so what values can $\lambda = \sigma^2$ have? Let's DEFER this question for a moment and just think of $\sigma$ or $\lambda$ as a set of eigenvalues to be determined later. Then we know that our general solution is this

$$u(r,\varphi) = \Sigma_\sigma\, r^\sigma\,(\,A_\sigma \cos(\sigma\varphi) + B_\sigma \sin(\sigma\varphi)\,)$$

where we sum over reasonable eigenvalues. An eigenvalue is going to be a value of $\sigma^2 = \lambda$ which meets our boundary conditions, but now we are talking the wedge, not an open periodic function around the circle. So what do our BC conditions tell us:

1. $u(r,0) = 0$ for any r
2. $u(r,\alpha) = 0$ for any r
3. $u(1,\varphi) = f(\varphi)$ for $\varphi = 0$ to $\alpha$

The first condition says

$$u(r,0) = \Sigma_\sigma\, r^\sigma\,(\,A_\sigma\,1)\,) = 0 \quad \Rightarrow A_\sigma = 0$$

Second condition says

$$u(r,\alpha) = \Sigma_\sigma\, r^\sigma\,(\,B_\sigma \sin(\sigma\alpha)\,) = 0 \quad \Rightarrow \sigma\alpha = n\pi \quad \Rightarrow \sigma = n\pi/\alpha$$

So NOW we know that our eigenvalues are $\sigma = n\pi/\alpha$ where n = 0,1,2... We reject negative powers because they make us blow up at r = 0, So we then have

$$u(r,\varphi) = \Sigma_{n=0}^{\infty}\, r^{n\pi/\alpha}\,(\,b_n \sin(n\pi\varphi/\alpha)\,)$$

But the n=0 term gives nothing due to the sine, so we have

$u(r,\varphi) = \Sigma_{n=1}^{\infty} r^{n\pi/\alpha} ( b_n \sin(n\pi\varphi/\alpha) )$ // **agrees with p 107 A**

At r = 1 we then have

$u(1,\varphi) = \Sigma_{n=1}^{\infty} b_n \sin(n\pi\varphi/\alpha) = f(\varphi)$ which is condition 3.

But this stuff is only defined on (0,α) so I don't think we can use the normal FS inversion. Let's define a new variable x = (π/α)φ. Then an integral dφ from 0 to α would run x = 0 to π. So multiply both sides of the above by $\sin(m\pi\varphi/\alpha)$ and integrate dφ from 0 to α,

$$\int_0^{\alpha} d\varphi \sin(m\pi\varphi/\alpha) f(\varphi) = \int_0^{\alpha} d\varphi \sin(m\pi\varphi/\alpha) \Sigma_{n=1}^{\infty} b_n \sin(n\pi\varphi/\alpha)$$

$$= \Sigma_{n=1}^{\infty} b_n \int_0^{\alpha} d\varphi \sin(m\pi\varphi/\alpha) \sin(n\pi\varphi/\alpha)$$

$$= \Sigma_{n=1}^{\infty} b_n \int_0^{\pi} (\alpha/\pi)\, dx \sin(mx) \sin(nx) = \Sigma_{n=1}^{\infty} b_n (\pi/2)(\alpha/\pi)\delta_{n,m}$$ Schaum p 96

$$= (\alpha/2) b_m$$

So our projection formula must be

$b_m = (2/\alpha) \int_0^{\alpha} \sin(m\pi\varphi/\alpha) f(\varphi)$ // **agrees with p 107 B**

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annulus scraps

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Here we do have the normal "n" boundary conditions, so general solution is this:

$u(r,\varphi) = \Sigma_{n=1}^{\infty} (A_n(r/a)^n + B_n(r/a)^{-n})( \cos(\varphi n) + C_n \sin(\varphi n) ) + A_0 + B_0 \ln(r/a)$

where I want a < b. Our boundary conditions might be taken as u=f on the outside, u=0 on the inside:

$u(b,\varphi) = \Sigma_{n=1}^{\infty} (A_n(b/a)^n + B_n(b/a)^{-n})( \cos(\varphi n) + C_n \sin(\varphi n) ) + A_0 + B_0 \ln(b/a) = f(\varphi)$

$$u(a,\varphi) = \Sigma_{n=0}^{\infty} (A_n + B_n)( \cos(\varphi n) + C_n \sin(\varphi n) ) + A_0 + B_0 \ln(1) = 0$$

This last would seem to require that $B_n = - A_n$ for $n > 1$, $A_0 = 0$, an $B_0$ undetermined. From that condition only we learn that

$$u(r,\varphi) = \Sigma_{n=1}^{\infty} A_n ( (r/a)^n - (r/a)^{-n})( \cos(\varphi n) + C_n \sin(\varphi n) ) + B_0 \ln(r/a)$$

The first condition then says

$$u(b,\varphi) = \Sigma_{n=1}^{\infty} A_n ( (b/a)^n - (b/a)^{-n})( \cos(\varphi n) + C_n \sin(\varphi n) ) + B_0 \ln(b/a) = f(\varphi)$$

Let's define some new constants:

$$A_n ( (b/a)^n - (b/a)^{-n}) = a_n$$

$$A_n ( (b/a)^n - (b/a)^{-n}) C_n = b_n$$

Then we have

$$u(b,\varphi) = \Sigma_{n=1}^{\infty} \{ a_n \cos(\varphi n) + b_n \sin(\varphi n) \} + B_0 \ln(b/a) = f(\varphi)$$

where $\varphi$ ranges 0 to $2\pi$. We set $L = \pi$ on Schaum page 131 and we can then read off the coefficients, where we make the association that $B_0 \ln(b/a) = a_0/2$ :

$$a_n = (1/\pi) \int_0^{2\pi} d\varphi \, f(\varphi) \cos(n\varphi) \qquad n = 0,1,2... \qquad \Rightarrow a_0 = 2B_0 \ln(b/a) = (1/\pi) \int_0^{2\pi} d\varphi \, f(\varphi)$$

$$b_n = (1/\pi) \int_0^{2\pi} d\varphi \, f(\varphi) \sin(n\varphi) \qquad n = 1,2,3...$$

Notice that the first equation with n=0 determines the constant $B_0$

Our complete solution must then be

$$u(r,\varphi) = \Sigma_{n=1}^{\infty} A_n ( (r/a)^n - (r/a)^{-n})( \cos(\varphi n) + C_n \sin(\varphi n) ) + B_0 \ln(r/a)$$

where $A_n = a_n / [(b/a)^n - (b/a)^{-n}]$ and $A_n C_n = b_n / [(b/a)^n - (b/a)^{-n}]$. So install this to get

$$u(r,\varphi) = \Sigma_{n=1}^{\infty} [ (r/a)^n - (r/a)^{-n}] / [(b/a)^n - (b/a)^{-n}] ( a_n \cos(\varphi n) + b_n \sin(\varphi n) ) + B_0 \ln(r/a)$$

The last term is $B_0 \ln(r/a) = a_0/2 * 1/\ln(b/a)\} * \ln(r/a) = (a_0/2) \ln(r/a)/ \ln(b/a)$

We could now install the known coefficients and see what happens:

$$u(r,\varphi) = \Sigma_{n=1}^{\infty} [ (r/a)^n - (r/a)^{-n}] / [(b/a)^n - (b/a)^{-n}] + B_0 \ln(r/a)$$

$$\{ (1/\pi) \int_0^{2\pi} d\varphi' f(\varphi') \cos(n\varphi') \cos(\varphi n) + (1/\pi) \int_0^{2\pi} d\varphi' f(\varphi') \sin(n\varphi') \sin(\varphi n) ) \}$$

$$= (1/\pi) \sum_{n=1}^{\infty} [ (r/a)^n - (r/a)^{-n}] / [(b/a)^n - (b/a)^{-n}] \int_0^{2\pi} d\varphi' f(\varphi') \cos(n[\varphi-\varphi']) + B_0 \ln(r/a)$$

To verify, at r = a we get 0. At r = b we get

$$u(b,\varphi) = (1/\pi) \sum_{n=1}^{\infty} \int_0^{2\pi} d\varphi' f(\varphi') \cos(n[\varphi-\varphi']) + B_0 \ln(b/a)$$

$$= (1/\pi) \int_0^{2\pi} d\varphi' f(\varphi') \sum_{n=1}^{\infty} \cos(n[\varphi-\varphi']) + B_0 \ln(b/a)$$

where $B_0 \ln(b/a) = (a_0/2) = (1/2\pi) \int_0^{2\pi} d\varphi f(\varphi)$ so write as

$$= (1/\pi) \int_0^{2\pi} d\varphi' f(\varphi') \sum_{n=1}^{\infty} \cos(n[\varphi-\varphi']) + (1/2\pi) \int_0^{2\pi} d\varphi' f(\varphi')$$

$$= (1/2\pi) \int_0^{2\pi} d\varphi' f(\varphi') \{ 2 \sum_{n=1}^{\infty} \cos(n[\varphi-\varphi']) + 1 \}$$

so it must be that

$$1 + 2 \sum_{n=1}^{\infty} \cos(n\theta) = 2\pi \delta(\theta)$$

We did once have this on the real axis x from volume I page 45

$$\sum_{\text{int } k} \delta(x-k) = 1 + 2 \sum_{n=1}^{\infty} \cos(2n\pi x)$$

Take this last and change variables to $2\pi x = \theta$. We then get

$$1 + 2 \sum_{n=1}^{\infty} \cos(n\theta) = \sum_{\text{int } k} \delta(x-k) = \sum_{\text{int } k} 2\pi \delta(2\pi x-2\pi k) = \sum_{\text{int } k} 2\pi \delta(\theta-2\pi k)$$

$= 2\pi \delta(\theta)$ QED since only one δ hit in range $0 \le \theta < 2\pi$, k = 0

Now I don't see a simple way to "do" the sum as we did in the disk case. The $[(b/a)^n - (b/a)^{-n}]$ factor confounds things. Cannot write things as a power of some z as done page 94. So leave as is. Here is my final result:

$u(a,\varphi) = 0$ $u(b,\varphi) = f(\varphi)$ $b > a$

$$u(r,\varphi) = \sum_{n=1}^{\infty} [ (r/a)^n - (r/a)^{-n}] / [(b/a)^n - (b/a)^{-n}] ( f_n \cos(\varphi n) + g_n \sin(\varphi n) ) + (f_0/2) \ln(r/a)/ \ln(b/a)$$

$$f_n = (1/\pi) \int_0^{2\pi} d\varphi\, f(\varphi) \cos(n\varphi)$$

$$g_n = (1/\pi) \int_0^{2\pi} d\varphi\, f(\varphi) \sin(n\varphi)$$

**Now**, suppose we want to solve for an annulus which has g on the inner and 0 on the outer. Hopefully you just swap the role of a and b and the answer is this:

$u(b,\varphi) = 0 \quad u(a,\varphi) = g(\varphi) \qquad b > a$

$$u(r,\varphi) = \Sigma_{n=1}^{\infty} [ (r/b)^n - (r/b)^{-n}]/[(a/b)^n - (a/b)^{-n}] ( F_n \cos(\varphi n) + G_n \sin(\varphi n) ) + (F_0/2) \ln(r/b)/ \ln(a/b)$$

$$F_n = (1/\pi) \int_0^{2\pi} g(\varphi) \cos(n\varphi)\, d\varphi$$

$$G_n = (1/\pi) \int_0^{2\pi} g(\varphi) \sin(n\varphi)\, d\varphi$$

Now finally if we want f on the outer and g on the inner we can superpose our two solutions. This does the right things for BC's since we were careful to put 0 on "the other" circle. So here is the general solution:

$u(b,\varphi) = f(\varphi) \quad u(a,\varphi) = g(\varphi) \qquad b > a$

$$u(r,\varphi) = \Sigma_{n=1}^{\infty} [ (r/a)^n - (r/a)^{-n}] / [(b/a)^n - (b/a)^{-n}] ( f_n \cos(\varphi n) + g_n \sin(\varphi n) ) + (f_0/2) \ln(r/a)/ \ln(b/a)$$

$$+ \Sigma_{n=1}^{\infty} [ (r/b)^n - (r/b)^{-n}] / [(a/b)^n - (a/b)^{-n}] (F_n \cos(\varphi n) + G_n \sin(\varphi n) ) + (F_0/2) \ln(r/b)/ \ln(a/b)$$

$$f_n = (1/\pi) \int_0^{2\pi} f(\varphi) \cos(n\varphi)\, d\varphi$$

$$g_n = (1/\pi) \int_0^{2\pi} f(\varphi) \sin(n\varphi)\, d\varphi$$

$$F_n = (1/\pi) \int_0^{2\pi} g(\varphi) \cos(n\varphi)\, d\varphi$$

$$G_n = (1/\pi) \int_0^{2\pi} g(\varphi) \sin(n\varphi)\, d\varphi$$

You do wonder what happens as you take a limit $a \to 0$ say of the first solution f outside, 0 inside. The limit is some situation that has u = 0 at the center and so is not the correct MVT situation. So I don't see this limit as being relevant.

Recall from above (where I just now made up the second line)

$$u(b,\varphi) = \Sigma_{n=1}^{\infty} \{ f_n \cos(\varphi n) + g_n \sin(\varphi n) \} + (f_0/2) = f(\varphi)$$

$$u(a,\varphi) = \Sigma_{n=1}^{\infty} \{ F_n \cos(\varphi n) + G_n \sin(\varphi n) \} + (F_0/2) = g(\varphi)$$

This problem is treated in a book on the web here

http://books.google.com/books?id=QGw2EkI3WNIC&pg=PA7&lpg=PA7&dq=laplace+annulus&source=bl&ots=rvzNcD0nPJ&sig=tH6U4g9r57j6GcunK2_Ve9FZ50I&hl=en&ei=VhNlSqq2N4ygsgOQpIHqDg&sa=X&oi=book_result&ct=result&resnum=9

and I just copy some of the results:

$$u(\rho,\varphi) = a_0 + b_0 \ln \rho + \\ + \sum_{n=1}^{\infty} \left[ \left( a_n \rho^n + b_n \rho^{-n} \right) \cos(n\varphi) + \left( c_n \rho^n + d_n \rho^{-n} \right) \sin(n\varphi) \right].$$

This author did not use ratios the way I did, maybe that makes the results a lot simpler. The various coefficients in the above he then gives as

$$\begin{cases} a_0 + b_0 \ln R_1 = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f_1(s)\, ds, \\ a_0 + b_0 \ln R_2 = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f_2(s)\, ds, \end{cases} \tag{1.6$_1$}$$

(to be solved for $a_0$ and $b_0$);

$$\begin{cases} a_n R_1^n + b_n R_1^{-n} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f_1(s) \cos(ns)\, ds, \\ a_n R_2^n + b_n R_2^{-n} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f_2(s) \cos(ns)\, ds, \end{cases} \tag{1.6$_2$}$$

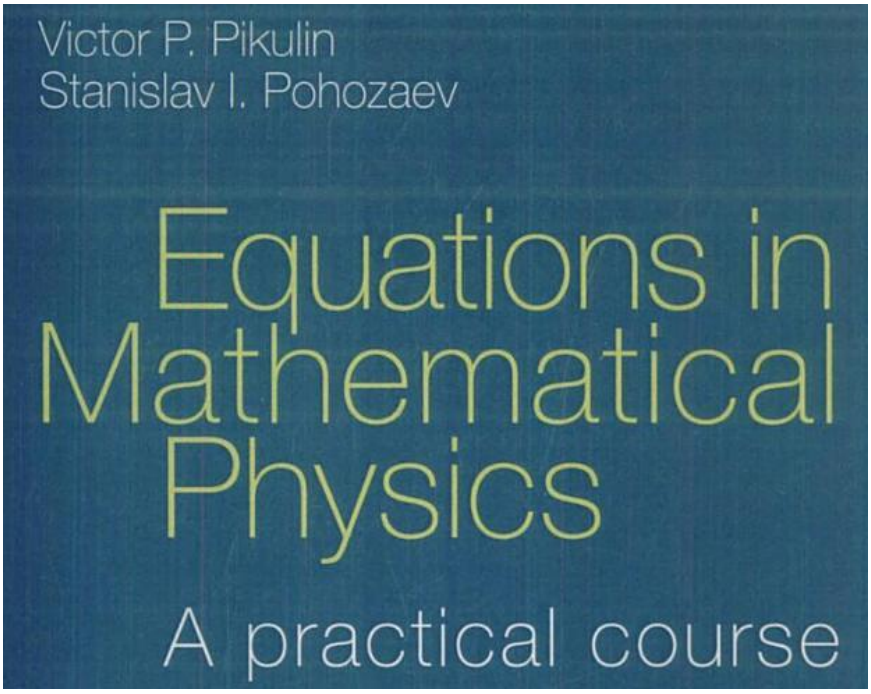
(to be solved for $a_n$ and $b_n$); and

$$\begin{cases} c_n R_1^n + d_n R_1^{-n} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f_1(s) \sin(ns)\, ds, \\ c_n R_2^n + bd_n R_2^{-n} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f_2(s) \sin(ns)\, ds, \end{cases} \tag{1.6$_3$}$$

(to be solved for $c_n$ and $d_n$).

where $f_1$ and $f_2$ are the boundary conditions on the two circles. He uses s in place of φ. So yes, his result is simpler without using ratios. But the general nature of this web author's solution confirms that I did

things pretty much on the right track. Were there some compact formula result, he would surely have stated it in his book, which, by the way, is this:




)_____

The first problem here is to the previous problem as the disk was to the wedge.

The start for either of the two parts here is this:

Write down the general form of the solution u(r,φ). That general form is this

$$u(r,\varphi) = \Sigma_\sigma \{ A_\sigma\ r^\sigma + B_\sigma\ r^{-\sigma} ) \cos(\sigma\varphi) + \Sigma_\sigma \{ C_\sigma\ r^\sigma + D_\sigma\ r^{-\sigma} ) \sin(\sigma\varphi) + A_0 + B_0\ \ln(r)$$

where the eigenvalues in the sum are $\sigma = n\pi/\alpha$ , so write as

$$u(r,\varphi) = \Sigma_{n=1}^{\infty} \{ A_n\ r^{n\pi/\alpha} + B_n\ r^{-n\pi/\alpha} ) \cos(n\pi\varphi/\alpha) + A_0 + B_0\ \ln(r)$$
$$+ \Sigma_{n=1}^{\infty} \{ C_n\ r^{n\pi/\alpha} + D_n\ r^{-n\pi/\alpha} ) \sin(n\pi\varphi/\alpha)$$

At this point, why not just treat r as a constant, and do a Fourier Series processing in φ. Use Schaum page 131 with these interpretations:

$L = \alpha$

$a_n(r) = A_n\ r^{n\pi/\alpha} + B_n\ r^{-n\pi/\alpha}$ n = 1,2,3...

$b_n(r) = C_n\ r^{n\pi/\alpha} + D_n\ r^{-n\pi/\alpha}$ n = 1,2,3...

$(a_0(r)/2) = A_0 + B_0\ \ln(r)$

The coefficients are given by

$$a_n(r) = (1/\alpha) \int_0^{2\alpha} d\varphi\, u(r,\varphi) \cos(n\pi\varphi/\alpha)$$

$$b_n(r) = (1/\alpha) \int_0^{2\alpha} d\varphi\, u(r,\varphi) \sin(n\pi\varphi/\alpha)$$

**Part (a)**

Put u = 0 on the inner radial part. We then get

$$0 = \Sigma_{n=1}^{\infty} \{ A_n\, a^{n\pi/\alpha} + B_n\, a^{-n\pi/\alpha} ) \cos(n\pi\varphi/\alpha) + A_0 + B_0\, \ln(a)$$
$$+ \Sigma_{n=1}^{\infty} \{ C_n\, a^{n\pi/\alpha} + D_n\, a^{-n\pi/\alpha} ) \sin(n\pi\varphi/\alpha)$$

This tells me that

$$A_n\, a^{n\pi/\alpha} + B_n\, a^{-n\pi/\alpha} = 0$$
$$C_n\, a^{n\pi/\alpha} + D_n\, a^{-n\pi/\alpha} = 0$$
$$A_0 + B_0\, \ln(a) = 0$$

because we have to have 0 coefficient on each φ dependent basis function. We can thus replace:

$$B_n = - A_n\, a^{2n\pi/\alpha}$$
$$D_n = - C_n\, a^{2n\pi/\alpha}$$
$$B_0 = - A_0/\ln(a)$$

So general solution with this BC is now given by

$$u(r,\varphi) = \Sigma_{n=1}^{\infty} \{ A_n\, r^{n\pi/\alpha} + B_n\, r^{-n\pi/\alpha} ) \cos(n\pi\varphi/\alpha) + A_0 + B_0\, \ln(r)$$
$$+ \Sigma_{n=1}^{\infty} \{ C_n\, r^{n\pi/\alpha} + D_n\, r^{-n\pi/\alpha} ) \sin(n\pi\varphi/\alpha)$$

$$= \Sigma_{n=1}^{\infty} A_n \{ r^{n\pi/\alpha} + B_n/A_n\, r^{-n\pi/\alpha} ) \cos(n\pi\varphi/\alpha) + A_0[1 + B_0/A_0\, \ln(r) ]$$
$$+ \Sigma_{n=1}^{\infty} C_n \{ r^{n\pi/\alpha} + D_n/C_n\, r^{-n\pi/\alpha} ) \sin(n\pi\varphi/\alpha)$$

$$= \Sigma_{n=1}^{\infty} A_n \{ r^{n\pi/\alpha} - a^{2n\pi/\alpha}\, r^{-n\pi/\alpha} ) \cos(n\pi\varphi/\alpha) + A_0[1 - \ln(r)/\ln(a) ]$$
$$+ \Sigma_{n=1}^{\infty} C_n \{ r^{n\pi/\alpha} - a^{2n\pi/\alpha}\, r^{-n\pi/\alpha} ) \sin(n\pi\varphi/\alpha)$$

Now put u = f on the outer radial part

$$u(r,\varphi) = \Sigma_{n=1}^{\infty} A_n \{ r^{n\pi/\alpha} - a^{2n\pi/\alpha}\, r^{-n\pi/\alpha} ) \cos(n\pi\varphi/\alpha) + A_0[1 - \ln(r)/\ln(a) ]$$
$$+ \Sigma_{n=1}^{\infty} C_n \{ r^{n\pi/\alpha} - a^{2n\pi/\alpha}\, r^{-n\pi/\alpha} ) \sin(n\pi\varphi/\alpha)$$

$$f(\varphi) = u(b,\varphi) = \Sigma_{n=1}^{\infty} A_n \{ b^{n\pi/\alpha} - a^{2n\pi/\alpha}\, b^{-n\pi/\alpha} ) \cos(n\pi\varphi/\alpha) + A_0[1 - \ln(b)/\ln(a) ]$$
$$+ \Sigma_{n=1}^{\infty} C_n \{ b^{n\pi/\alpha} - a^{2n\pi/\alpha}\, b^{-n\pi/\alpha} ) \sin(n\pi\varphi/\alpha)$$

But I think we have what we need already earlier,

$$a_n(b) = (1/\alpha) \int_0^{2\alpha} d\varphi\, u(b,\varphi) \cos(n\pi\varphi/\alpha) = (1/\alpha) \int_0^{2\alpha} d\varphi\, f(\varphi) \cos(n\pi\varphi/\alpha)$$

$$b_n(b) = (1/\alpha) \int_0^{2\alpha} d\varphi\, u(b,\varphi) \sin(n\pi\varphi/\alpha) = (1/\alpha) \int_0^{2\alpha} d\varphi\, f(\varphi) \sin(n\pi\varphi/\alpha)$$

$$a_n(b) = A_n\, b^{n\pi/\alpha} + B_n\, b^{-n\pi/\alpha} \qquad n = 1,2,3...$$
$$b_n(b) = C_n\, b^{n\pi/\alpha} + D_n\, b^{-n\pi/\alpha} \qquad n = 1,2,3...$$
$$(a_0(b)/2) = A_0 + B_0\, \ln(b)$$

Install our results from above that

$$B_n = -A_n\, a^{2n\pi/\alpha}$$
$$D_n = -C_n\, a^{2n\pi/\alpha}$$
$$B_0 = -A_0/\ln(a)$$

Then we have

$$a_n(b) = A_n\, b^{n\pi/\alpha} + B_n\, b^{-n\pi/\alpha} = A_n \left( b^{n\pi/\alpha} - a^{2n\pi/\alpha}\, b^{-n\pi/\alpha} \right) = A_n\, b^{n\pi/\alpha} [\, 1 - (a/b)^{2n\pi/\alpha}\, ]$$
$$b_n(b) = C_n\, b^{n\pi/\alpha} [\, 1 - (a/b)^{2n\pi/\alpha}\, ]$$
$$a_0(b) = 2\, A_0 [\, 1 - \ln(b)/\ln(a)\, ]$$

Thus, we do the dφ integrals shown above to get $a_n(b)$ and $b_n(b)$ and that tells us $A_n$ and $C_n$ and $A_0$. We then have the complete solution to our problem:

$$u(r,\varphi) = \sum_{n=1}^{\infty} A_n \{ r^{n\pi/\alpha} - a^{2n\pi/\alpha}\, r^{-n\pi/\alpha} ) \cos(n\pi\varphi/\alpha) + A_0 [1 - \ln(r)/\ln(a)\, ]$$
$$+ \sum_{n=1}^{\infty} C_n \{ r^{n\pi/\alpha} - a^{2n\pi/\alpha}\, r^{-n\pi/\alpha} ) \sin(n\pi\varphi/\alpha)$$

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For the case n = 0 and σ = 0, we do get that extra piece with the ln(r). So I guess this is our general solution that meets the radial BC's

$$u(r,\varphi) = \sum_{\sigma} \{ A_\sigma\, r^\sigma + B_\sigma\, r^{-\sigma} ) \cos(\sigma\varphi) + \sum_{\sigma} \{ C_\sigma\, r^\sigma + D_\sigma\, r^{-\sigma} ) \sin(\sigma\varphi) + A_0 + B_0\, \ln(r)$$

$$= \sum_{\sigma} A_\sigma \{ r^\sigma - a^{2\sigma}\, r^{-\sigma} ) \cos(\sigma\varphi) + \sum_{\sigma} C_\sigma \{ r^\sigma - a^{2\sigma}\, r^{-\sigma} ) \sin(\sigma\varphi) + A_0 + B_0\, \ln(r)$$

The log term looks fishy. We need $A_0 + B_0\, \ln(r)$ to be 0 at both r = a and r = b.

$A_0 + B_0 \ln(a) = 0$
$A_0 + B_0 \ln(b) = 0$

Subtract to get $B_0 \ln(a/b) = 0$ so $B_0 = 0$ and then we need $A_0 = 0$. So here is our solution now:

$$u(r,\varphi) = \Sigma_\sigma A_\sigma \{ r^\sigma - a^{2\sigma} r^{-\sigma} ) \cos(\sigma\varphi) + \Sigma_\sigma C_\sigma \{ r^\sigma - a^{2\sigma} r^{-\sigma} ) \sin(\sigma\varphi)$$

where $\sigma = n \pi / \ln(b/a)$ with $n = \pm 1, \pm 2, \ldots$ . Lets define $1/\ln(b/a) = c$. Then we have $\sigma = n\pi c$

$$u(r,\varphi) = \Sigma'_{n=-\infty}^{\infty} \{A_n ( r^{n\pi c} - a^{2\pi nc} r^{-n\pi c} ) \cos(\sigma\varphi) + C_n ( r^{n\pi c} - a^{2\pi nc} r^{-n\pi c} ) \sin(\sigma\varphi) \}$$

$$= \Sigma'_{n=-\infty}^{\infty} ( r^{n\pi c} - a^{2\pi nc} r^{-n\pi c} ) \{ A_n \cos(\sigma\varphi) + C_n \sin(\sigma\varphi) \}$$

Now at last we can impose our condition at $\varphi = 0$ which says $u = 0$

$$u(r,0) = \Sigma'_{n=-\infty}^{\infty} ( r^{n\pi c} - a^{2\pi nc} r^{-n\pi c} ) \{ A_n \} = 0$$

I think this is saying $A_n = 0$, so then our general solution has this form:

$$u(r,\varphi) = \Sigma'_{n=-\infty}^{\infty} ( r^{n\pi c} - a^{2\pi nc} r^{-n\pi c} ) C_n \sin(n\pi c\varphi)$$

and then our condition at $\varphi = \alpha$ is this:

$$u(r,\alpha) = \Sigma'_{n=-\infty}^{\infty} ( r^{n\pi c} - a^{2\pi nc} r^{-n\pi c} ) C_n \sin(n\pi c\alpha) = f(r)$$

Now I am dead in the water because I don't know how to solve this for $C_n$. We really need some kind of orthonormal set on the interval (a,b) in terms of these powers, like Legendres. So I think I have done enough on this particular problem. Let's move on!

Suppose we start off by imposing the condition at r = a where u = 0, as we did above, leaving us with this general form:

$$u(r,\varphi) = \Sigma_{n=1}^{\infty} A_n \{ r^{n\pi/\alpha} - a^{2n\pi/\alpha} r^{-n\pi/\alpha} ) \cos(n\pi\varphi/\alpha) + A_0[1 - \ln(r)/\ln(a) ]$$
$$+ \Sigma_{n=1}^{\infty} C_n \{ r^{n\pi/\alpha} - a^{2n\pi/\alpha} r^{-n\pi/\alpha} ) \sin(n\pi\varphi/\alpha)$$

Now let's impose that this be 0 at r = b:

$$u(b,\varphi) = \Sigma_{n=1}^{\infty} A_n \{ b^{n\pi/\alpha} - a^{2n\pi/\alpha}\ b^{-n\pi/\alpha} ) \cos(n\pi\varphi/\alpha) + A_0[1 - \ln(b)/\ln(a) ]$$
$$+ \Sigma_{n=1}^{\infty} C_n \{ b^{n\pi/\alpha} - a^{2n\pi/\alpha}\ b^{-n\pi/\alpha} ) \sin(n\pi\varphi/\alpha) \ = 0$$

But this seems to force $A_n = 0$ and $C_n = 0$ and $A_0 = 0$, which is obviously wrong.

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Meanwhile, our equation 6.55 has this form

$$u(\mathbf{r}) = -(1/8\pi) \int d\Omega'\ b(\mathbf{r}') / |\mathbf{r} - \mathbf{r}'| \qquad \text{// WRONG!}$$

where **r** is inside the sphere, and **r'** is on the sphere. We can project this onto Y's analogously to how we did it above, but now |r| = r and not 1, so there is a slight difference. Let's review our projection above:

$$f(\mathbf{r}) = -b(\mathbf{r})/2 - 1/8\pi \int b(\mathbf{r}')d\Omega' / \ |\mathbf{r}-\mathbf{r}'| \qquad R = |\mathbf{r}-\mathbf{r}'|$$
$$f_{mn} = -b_{mn}/2 + (1/8\pi)\ N_{0n}\ b_{mn}$$

$$u(\mathbf{r}) = -(1/8\pi) \int d\Omega'\ b(\mathbf{r}') / |\mathbf{r} - \mathbf{r}'|$$
$$u_{mn} = -(1/8\pi)\ N_{0n}\ b_{mn}\ r^n \quad = -(1/2)\ b_{mn}\ r^n/(2n+1) \qquad \text{// my guessed result, confirmed below}$$

$$= -(1/2)\ \{-(2n+1)/(n+1) * f_{mn}\}\ r^n/(2n+1)$$

$$= +(1/2(n+1)) * f_{mn}\ r^n$$

Then our u(r) is given by

$$u(\mathbf{r}) = \Sigma_{nm} Y_{nm}(\Omega)\ u_{mn} = \Sigma_{nm} Y_{nm}(\Omega)\ (+(1/2(n+1)) * f_{mn}\ r^n\}$$

$$= +\Sigma_{nm}\ r^n\ (1/2(n+1))\ f_{mn}\ Y_{nm}(\Omega)$$

Now if we want to have this be an integral over angles, we could do this

$$f_{mn} = (1/N_{nm}) \int d\Omega'\ Y_{nm}^*(\Omega')\ f(\Omega')$$

Then our final result would be

$$u(\mathbf{r}) = \Sigma_{nm}\ r^n\ (1/2(n+1))\ Y_{nm}(\Omega)\{\ (1/N_{nm}) \int d\Omega'\ Y_{nm}^*(\Omega')\ f(\Omega')\}$$

$$= \int d\Omega' \, f(\Omega') \; \{ \Sigma_{nm} \, r^n \, (1/2(n+1)) \, Y_{nm}(\Omega) \, Y_{nm}^*(\Omega')/N_{nm} \}$$

If we compare this to 6.26, we would have to conclude that

$$\{ \Sigma_{nm} \, r^n \, (1/2(n+1)) \, Y_{nm}(\Omega) \, Y_{nm}^*(\Omega')/N_{nm} \} = - (r^2-1)/4\pi \; * \; 1/(1+r^2-2r\cos\gamma)^{3/2}$$

But from A.14 we do know this: (true for any angle, so replace θ with γ)

$$(r^2-1) \, /(1+r^2-2r\cos\gamma)^{3/2} = - \Sigma_n (2n+1) \, r^n P_n(\cos\gamma)$$

so our task would then be to show that

$$\{ \Sigma_{nm} \, r^n \, (1/2(n+1)) \, Y_{nm}(\Omega) \, Y_{nm}^*(\Omega')/N_{nm} \} = (1/4\pi) \, \Sigma_n (2n+1) \, r^n P_n(\cos\gamma)$$

or for each $r^n$ term that

$$\Sigma_m \; (1/2(n+1)) \, Y_{nm}(\Omega) \, Y_{nm}^*(\Omega')/N_{nm} = (1/4\pi) \, (2n+1) \, P_n(\cos\gamma) \qquad (*)$$

But from A.11 or p 126 A we do know that

$$P_n(\cos\gamma) = N_{0n} \Sigma_m \, Y_{nm}(\Omega) \, Y_{nm}^*(\Omega')/N_{nm}$$

so the RHS of (*) becomes

$$(1/4\pi) \, (2n+1) \, N_{0n} \Sigma_m \, Y_{nm}(\Omega) \, Y_{nm}^*(\Omega')/N_{nm}$$

$$= \Sigma_m \; \{(1/4\pi) \, N_{0n} \, (2n+1)\} \, Y_{nm}(\Omega) \, Y_{nm}^*(\Omega')/N_{nm}$$

We would then have our desired equality if it were true that

$$(1/2(n+1)) = (1/4\pi) \, N_{0n} \, (2n+1)$$

But $N_{0n} = 4\pi/(2n+1)$ so this would say

$$(1/2(n+1)) = 1$$

so we are close but no cigar. Somewhere I have made an error. I have already found several errors but some still remain somewhere!

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How are we going to handle this $1/R^3$ factor ?? I can set $\cos\gamma = P_1(\cos\gamma)$ and expand that on Y's, and the same for b(r'), but we still have to deal with $1/R^3$. Here is an idea. Let's consider A.8 which says:

$$(1/4\pi)R^{-1} = \Sigma_{nm}\, r^n\, Y_{nm}(\Omega)Y^*_{nm}(\Omega')/\,[\,(2n+1)N_{mn}]$$

Differentiate this with respect to r to get

$$-\,(1/4\pi)\,R^{-2}\,\partial_r R \;\; = \Sigma_{nm}\, n\, r^{n-1}\, Y_{nm}(\Omega)Y^*_{nm}(\Omega')/\,[\,(2n+1)N_{mn}]$$

But $R = [1 + r^2\text{-}2r\cos\gamma]^{1/2}$ so $\partial_r R = 1/2\;\; [1 + r^2\text{-}2r\cos\gamma]^{-1/2}\,(2r - 2\cos\gamma) = (r\text{-}\cos\gamma)/R$

which tells us then that

$$-\,(1/4\pi)\,(r\text{-}\cos\gamma)/R^3 \;\; = \Sigma_{nm}\, n\, r^{n-1}\, Y_{nm}(\Omega)Y^*_{nm}(\Omega')/\,[\,(2n+1)N_{mn}]$$

which says:

$$1/4\pi R^3 = -\,(r\text{-}\cos\gamma)^{-1}\,\Sigma_{nm}\, n\, r^{n-1}\, Y_{nm}(\Omega)Y^*_{nm}(\Omega')/\,[\,(2n+1)N_{mn}]$$

and then we have

$$u(\mathbf{r}) = -\int d\Omega'\,(r\cos\gamma - 1)/(r - \cos\gamma)\; b(\mathbf{r}')\;\; \Sigma_{nm}\, n\, r^{n-1}\, Y_{nm}(\Omega)Y^*_{nm}(\Omega')/\,[\,(2n+1)N_{mn}]$$

But this is a huge mess that I don't like! So I am now blocked.

Anther approach is geometry which shows this:

$$\cos\psi = (r^2\text{-}\, 1 - R^2)/2R$$

so I would then have

$$u(\mathbf{r}) = \int \cos(\mathbf{r}\text{-}\mathbf{r}',\mathbf{n} \text{ at } r')\, b(\mathbf{r}')\, dS_{r'} \,/\, 4\pi\, R^2 \qquad\qquad R = |\mathbf{r} - \mathbf{r}'|$$

$$= \int d\Omega'\; b(\mathbf{r}')\;(r^2\text{-}\, 1 - R^2)\;\; * \; 1/8\pi\, R^3$$

but then what??

So time to activate **"the Master Debugger"**. We need to produce a factor of (n+1) somewhere, or show that this (n+1) should really be 1.2.

(1) Let's check on "my guessed result" and see if somehow that is where the error lies:

$$u(\mathbf{r}) = -(1/8\pi) \int d\Omega' \, b(\mathbf{r}') / |\mathbf{r} - \mathbf{r}'|$$

$(1/N_{mn}) \int d\Omega \, Y^*_{nm}(\Omega)$ to both sides, giving

$$u_{mn} = -(1/8\pi) (1/N_{mn}) \int d\Omega \, Y^*_{nm}(\Omega) \int d\Omega' \, b(\mathbf{r}') / |\mathbf{r} - \mathbf{r}'|$$

$$= -(1/2) \int b(\mathbf{r}') \, d\Omega' (1/N_{mn}) \int d\Omega \, Y^*_{nm}(\Omega) / 4\pi|\mathbf{r} - \mathbf{r}'|$$

Now use A.8 with $r_< = r$ and $r_> = 1$ to get

$$1/4\pi|\mathbf{r} - \mathbf{r}'| = \Sigma_{n',m'} \, r^{n'} \, Y_{n',m'}(\Omega) \, Y^*_{n',m'}(\Omega') / \{ (2n'+1)N_{m',n'} \}$$

Then we have

$$u_{mn} == -(1/2) \int b(\mathbf{r}') \, d\Omega' (1/N_{mn}) \int d\Omega \, Y^*_{nm}(\Omega) \{ \Sigma_{n',m'} \, r^{n'} \, Y_{n',m'}(\Omega) \, Y^*_{n',m'}(\Omega') / [ (2n'+1)N_{m',n'} ] \}$$

$$= -(1/2) \int b(\mathbf{r}') \, d\Omega' (1/N_{mn}) \{ \Sigma_{n',m'} \, r^{n'} \, Y^*_{n',m'}(\Omega') / [ (2n'+1)N_{m',n'} ] \int d\Omega \, Y^*_{nm}(\Omega) Y_{n',m'}(\Omega) \}$$

$$= -(1/2) \int b(\mathbf{r}') \, d\Omega' (1/N_{mn}) \{ \Sigma_{n',m'} \, r^{n'} \, Y^*_{n',m'}(\Omega') / [ (2n'+1)N_{m',n'} ] \, \delta_{nn'}\delta_{mm'}N_{nm} \}$$

$$= -(1/2) \int b(\mathbf{r}') \, d\Omega' (1/N_{mn}) \{ \, r^n \, Y^*_{nm}(\Omega') / [ (2n+1) ] \}$$

$$= -(1/2) * r^n / [ (2n+1) ] (1/N_{mn}) \int b(\mathbf{r}') \, d\Omega' \, Y^*_{nm}(\Omega') \}$$

$= -(1/2) * r^{n} / [ (2n+1) ] * b_{nm}$ // which agrees with "my guessed result".

So this did not produce my error.

2. Is my result for $b_{mn}$ correct. In this book,