Mystery Question

This "mystery" involves trying to reconcile Stak Vol II page 107 equations A and B with the standard Fourier Series formalism. This is Chapter 6, Exercise 6.10.

Note added later: things below would probably be simpler if I did (-α,α) instead of (0,2α).

Suppose we have f(x) defined only on (0,2L). We make it periodic so f(x) = f(x+2L). The Fourier Series says this: (Schaum p 131 or Wolfram http://mathworld.wolfram.com/FourierSeries.html)

$$f(x') = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x'}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x'}{L}\right).$$

$$a_0 = \frac{1}{L}\int_0^{2L} f(x')\,dx'$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x')\cos\left(\frac{n\pi x'}{L}\right)dx'$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x')\sin\left(\frac{n\pi x'}{L}\right)dx'.$$

Someone hands you this series:

$f(\varphi) = \Sigma_{n=1}^{\infty} g_n \sin(n\pi\varphi/\alpha)$

What is the periodicity of this function? To find out, look at the longest wave n = 1:

$\sin(\pi\varphi/\alpha + \pi K/\alpha) = \sin(\pi\varphi/\alpha)$ $\quad \pi K/\alpha = 2\pi \quad K = 2\alpha$

So this is the source of my confusion! We have a function f(φ) which is defined only on the interval (0,α) which is the wedge, but the periodicity of f(φ) as the series shows is in fact 2α.

What can we say about f(φ) in the range (α,2α)? Well

$f(\varphi) = \Sigma_{n=1}^{\infty} g_n \sin(n\pi\varphi/\alpha)$ // φ in (0,α), f over (0,α)

$f(\varphi+\alpha) = \Sigma_{n=1}^{\infty} g_n \sin(n\pi\varphi/\alpha + n\pi) = \Sigma_{n=1}^{\infty} [g_n (-1)^n] \sin(n\pi\varphi/\alpha)$ // φ in (0,α), f over (α,2α)

Suppose we have some set of coefficients g_n that specify f(φ) in the range (0,α). That then forces the function to have some specific shape in the range (α,2α), you are not free to specify this as you like! In the (α,2α) range, all the odd coefficients change sign. So given a drawing of f(φ) in the (0,α) range, I cannot easily *draw* what it looks like in the (α,2α) range.

Now let's apply the FS correctly for a function having period 2α so that 2L = 2α and L = α. Then

$$f(\varphi) = a_0 + \Sigma_{n=1}^{\infty} \left(a_n \cos(n\pi\varphi/\alpha) + b_n \sin(n\pi\varphi/\alpha) \right)$$

$$b_n = (1/\alpha) \int_0^{2\alpha} d\varphi \, f(\varphi) \sin(n\pi\varphi/\alpha)$$

The big issue is why the two halves of this last integral give the same result!

So consider the upper integral half and use $\varphi = \varphi' + \alpha$

$$\int_{\alpha}^{2\alpha} d\varphi \, f(\varphi) \sin(n\pi\varphi/\alpha) = \int_0^{\alpha} d\varphi' \, f(\varphi'+\alpha) \sin(n\pi\varphi'/\alpha + n\pi)$$

$$= (-1)^n \int_0^{\alpha} d\varphi' \, f(\varphi'+\alpha) \sin(n\pi\varphi'/\alpha) = (-1)^n \int_0^{\alpha} d\varphi \, f(\varphi+\alpha) \sin(n\pi\varphi/\alpha)$$

$$= (-1)^n \int_0^{\alpha} d\varphi \, \{\Sigma_{m=1}^{\infty} [g_m (-1)^m] \sin(m\pi\varphi/\alpha)\} \sin(n\pi\varphi/\alpha)$$

$$= (-1)^n \{\Sigma_{m=1}^{\infty} [g_m (-1)^m]\} \int_0^{\alpha} d\varphi \sin(m\pi\varphi/\alpha) \sin(n\pi\varphi/\alpha)$$

So in the end, if comes down to this famous integral. Let $x = \pi\varphi/\alpha$ so that $d\varphi = (\alpha/\pi)\, dx$

$$\int_0^{\alpha} d\varphi \sin(m\pi\varphi/\alpha) \sin(n\pi\varphi/\alpha) = (\alpha/\pi) \int_0^{\pi} dx \sin(mx) \sin(nx) = (\alpha/\pi)(\pi/2)\, \delta_{n,m} \quad \text{// Schaum p 96}$$

$$= (\alpha/2)\, \delta_{n,m}$$

so we then get

$$= (-1)^n \{\Sigma_{m=1}^{\infty} [g_m (-1)^m] (\alpha/2)\, \delta_{n,m}$$

$$= (-1)^n [g_n (-1)^n] (\pi/2) = (\alpha/2)\, g_n$$

Now what about the base integral?

$$\int_0^{\alpha} d\varphi \, f(\varphi) \sin(n\pi\varphi/\alpha) = \int_0^{\alpha} d\varphi \, \{\Sigma_{m=1}^{\infty} g_m \sin(m\pi\varphi/\alpha)\} \sin(n\pi\varphi/\alpha)$$

$$= \Sigma_{m=1}^{\infty} g_m \int_0^{\alpha} d\varphi \sin(m\pi\varphi/\alpha) \sin(n\pi\varphi/\alpha)$$

$$= \Sigma_{m=1}^{\infty} g_m (\alpha/2)\, \delta_{n,m} = (\alpha/2)\, g_n$$

Only after doing all this messy work to we find that the upper integral is the same as the lower integral. Only then are we justified in going the last step:

$$b_n = (1/\alpha) \int_0^{2\alpha} d\varphi\, f(\varphi) \sin(n\pi\varphi/\alpha) = (2/\alpha) \int_0^{\alpha} d\varphi\, f(\varphi) \sin(n\pi\varphi/\alpha)$$

Conclusion: it was non-trivial to "fit" this problem into the standard FS framework. When we first see our function $f(\varphi)$ defined on interval $(0,\alpha)$, we are inclined to try to make it periodic with period α using just that data, or maybe to fill in the rest of the data with 0.