The Basics of Potential Theory and Heat Conduction Theory PhL 3.26.11

We examine here the Dirichlet, Neumann and Radiative (mixed) potential theory problems. The idea is that a quantity prescribed as f(x) on the closed bounding surface determines the potential everywhere inside the volume. The only ingredients are the Laplace BV problem statement, the Green's BV problem statement, and Green's Second Identity. Notice that each problem involves a different Green's Function! I don't think these have official names, but you could call them the Dirichlet Green's Function, and the Neumann one and the Radiative one.

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Part I: The Laplace Equation

1. The Dirichlet Boundary Value Problem of Potential Theory

$$-\nabla^2 u = q(x)$$
$$u = f(x) \qquad \text{for } x \text{ on } \sigma$$

And here is the corresponding Green's Function problem

$$-\nabla^2 g(x|\xi) = \delta(x-\xi)$$
$$g(x|\xi) = 0 \qquad \text{for } x \text{ on } \sigma$$

Next we have Green's 2nd Identity,

$$\int_V dV\,[\,\psi\,\nabla^2\varphi - \varphi\,\nabla^2\psi\,] = \int_S dS\,[\,\psi(\partial\varphi/\partial n) - \varphi(\partial\psi/\partial n)\,] \qquad (6) \qquad \text{Green \#2}$$

So apply this with $\psi(x) = u(x)$ and $\varphi(x) = g(\xi|x)$:

$$\int_V dV\,[u(x)\,\nabla^2 g(\xi|x) - g(\xi|x)\,\nabla^2 u(x)\,] = \int_S dS_x\,[u(x)\,(\,\partial\, g(\xi|x)/\partial n) - g(\xi|x)\,(\,\partial\, u(x)/\partial n)\,]$$

But we know that $\nabla^2 g(\xi|x) = -\,\delta(x-\xi)$ [g is symmetric] and we know that $\nabla^2 u(x) = -\,q(x)$, so

$$\int_V dV\,[u(x)\,\delta(x-\xi) - g(\xi|x)\,q(x)\,] = -\int_S dS_x\,[u(x)\,(\,\partial\, g(\xi|x)/\partial n) - g(\xi|x)\,(\,\partial\, u(x)/\partial n)\,]$$

$$u(\xi) = \int_V dx\; g(\xi|x)q(x) - \int_S dS\,[u(x)\,\partial_{nx}\, g(\xi|x) - g(\xi|x)\partial_n u(x)\,]$$

which we rewrite as

$$u(x) = \int_V d\xi\; g(x|\xi)q(\xi) - \int_S dS_\xi\,[u(\xi)\,\partial_{n\xi}\, g(x|\xi) - g(x|\xi)\partial_{n\xi} u(\xi)\,] \qquad (*)$$

But of course g = 0 on the boundary for our Dirichlet situation and u(x) = f, so we get

$$u(x) = \int_V d\xi\; g(x|\xi)q(\xi) - \int_S dS_\xi\,[f(\xi)\,\partial_{n\xi}\, g(x|\xi)\,] \qquad \text{// verify with Stak p 135 6.81}$$

and this is the very famous Dirichlet result for a closed boundary S around volume V. In general I think the quantity $\partial_{n\xi}\, g(x|\xi)$ on the boundary is called the Poisson Kernel.

2. The Neumann Boundary Value Problem of Potential Theory

$$-\nabla^2 u = q(x)$$
$$\partial_n u = f(x) \qquad \text{for x on } \sigma$$

And here is the corresponding Green's Function problem

$$-\nabla^2 g(x|\xi) = \delta(x-\xi)$$
$$\partial_n g(x|\xi) = 0 \qquad \text{for x on } \sigma$$

We follow the steps above in Section 1 down to the (*) result

$$u(x) = \int_V d\xi\; g(x|\xi)q(\xi) - \int_S dS_\xi\,[u(\xi)\,\partial_{n\xi}\, g(x|\xi) - g(x|\xi)\partial_{n\xi} u(\xi)\,] \qquad (*)$$

and this time we have $\partial_n g(x|\xi) = 0$ so we find

$$u(x) = \int_V d\xi\; g(x|\xi)q(\xi) + \int_S dS_\xi\; g(x|\xi)\partial_{n\xi} u(\xi)$$

or

$$u(x) = \int_V d\xi\, g(x|\xi)q(\xi) + \int_S dS_\xi\, g(x|\xi)\, f(\xi)$$

and this is the Neumann result for a closed boundary S around volume V. Inspection shows that we can interpret f(x) as a surface charge density σ(x) that is being prescribed on the surface.

3. The Radiative Boundary Value Problem of Potential Theory

$$-\nabla^2 u = q(x)$$
$$\partial_n u + \theta u = f(x) \qquad \text{for } x \text{ on } \sigma$$

And here is the corresponding Green's Function problem

$$-\nabla^2 g(x|\xi) = \delta(x-\xi)$$
$$\partial_n g(x|\xi) + \theta g(x|\xi) = 0 \qquad \text{for } x \text{ on } \sigma$$

Again we take it down to pre (*) in Section 1,

$$u(\xi) = \int_V dx\, g(\xi|x)q(x) - \int_S dS\, [u(x)\, \partial_{nx}\, g(\xi|x) - g(\xi|x)\partial_n u(x)\,]$$

Let's look just at the surface terms:

$$\int_S dS\, [u(x)\, \partial_{nx}\, g(\xi|x) - g(\xi|x)\, \partial_n u(x)\,]$$

$$= \int_S dS\, [u(x)\{\, \partial_{nx}\, g(\xi|x) + \theta\, g(\xi|x) - \theta\, g(\xi|x)\} \; - g(\xi|x)\, \{\partial_n u(x) + \theta u(x) - \theta u(x)\, \}]$$

$$= \int_S dS\, [u(x)\{\, \partial_{nx} g(\xi|x) + \theta\, g(\xi|x) - \theta\, g(x|\xi)\} \; - g(\xi|x)\, \{\partial_n u(x) + \theta u(x) - \theta u(x)\} \,]$$

$$= \int_S dS\, [u(x)\{\qquad\qquad\qquad - \theta\, g(\xi|x)\} \; - g(\xi|x)\, \{f(x) - \theta u(x)\, \}]$$

$$= \int_S dS\, [u(x)\{\qquad\qquad\qquad\qquad\} \; - g(\xi|x)\, \{f(x)\qquad\}]$$

$$= -\int_S dS\, g(\xi|x)f(x)$$

so we get

$$u(\xi) = \int_V dV\, g(\xi|x)q(x) + \int_S dS\, g(\xi|x)\, f(x)$$

or

$$u(x) = \int_V d\xi\, g(x|\xi)q(\xi) + \int_S dS_\xi\, g(x|\xi)\, f(\xi)$$

This result looks just like the Neumann result, but f(x) has a different meaning, and of course if we set θ = 0 this replicates the Neumann case. It is not obvious that we can recover the Dirichlet case by somehow taking θ→∞, but I have no need to do that.

Part II: The Heat Conduction Equation

The Boundary Value Problem of Heat Conduction and Feynman Diagrams

$\partial_t u - \nabla^2 u = q(x,t)$		$t > 0$	(7.3)
$u(x,t) = h(x,t)$	for x on σ	$t > 0$	
$u(x,0) = f(x)$	for x in R		

The entire problem above is only defined for $t > 0$. We have no interest in $t < 0$.

Here is the corresponding (Dirichlet) Green's Function Problem (in is understood that x_0 always in R)

$(\partial_t - \nabla_x^2)\, g(x,t\|x_0,t_0) = \delta^n(x-x_0)\delta(t-t_0)$		all t and t_0	(7.8)
$g(x,t\|x_0,t_0) = 0$	for x on σ	all t and t_0	
$g(x,t\|x_0,t_0) = 0$	$t < t_0$		

Discussion: 2nd line says $g(x,t|x_0,t_0) = 0$ for x on the boundary σ of R regardless of the values of the other three variables t, x_0 and t_0. 3rd line says that g is a propagator of the form g(past|future) = 0 so you only get propagation ≠ 0 when you have g(future|past), so think of this as g(←). The point x^μ must lie in the future of the point $x_0{}^\mu$ in spacetime, or there is no propagation amplitude. This forward time condition is why we call this thing a Causal Green's Function.

Unlike the case in Potential Theory for Laplace, we can write (7.8) in the following alternate manner, where this entire problem is defined only for $t > t_0$:

$(\partial_t - \nabla_x^2)\, g(x,t\|x_0,t_0) = 0$		positive t and t_0	(7.8a)
$g(x,t\|x_0,t_0) = 0$	for x on σ	positive t and t_0	
$g(x,t_0+\|x_0,t_0) = \delta^n(x-x_0)$			
$g(x,t\|x_0,t_0) = 0$	$t < t_0$		

The claim is that these two problems (7.8) and (7.8a) have the same solution $g(x,t|x_0,t_0)$! In both problems, the propagator g is 0 it you try to go backward in time, so still have g(←). If you solve the second problem for g, this g is defined only for positive times. So in the region of positive times only, we claim that this same g satisfies the first problem. We can agree to only think about positive times in all our work.

As a technical detail, Stak shows that we can rewrite (7.8) in this manner:

$(-\partial_{t0} - \nabla_{x0}^2)\, g(x,t\|x_0,t_0) = \delta^n(x-x_0)\delta(t-t_0)$		all t and t_0	(7.13pre)
$g(x,t\|x_0,t_0) = 0$	for x on σ	all t and t_0	
$g(x,t\|x_0,t_0) = 0$	$t < t_0$		

where I show changes in red. This new operator acts on the second arguments of g. If we then make the swap $x^\mu \leftrightarrow x_0^\mu$ we get (7.13), but I prefer it as stated above. We still have g(←) .

Now we come to Green's 2nd identity written for the heat operator L which is this

$$\int_0^T dt \int_{R_n} d^n x \; [\, v(t,\mathbf{x})\,(\partial_t - \nabla^2)u(t,\mathbf{x}) - u(t,\mathbf{x})\,(-\partial_t - \nabla^2)v(t,\mathbf{x})\,]$$

$$= \int d^n x\,[u(T,\mathbf{x})v(T,\mathbf{x}) - u(0,\mathbf{x})v(0,\mathbf{x})] + \int_0^T dt \int_{\sigma_n} dS_n\,[\, u(t,\mathbf{x})\partial_n v(t,\mathbf{x}) - v(t,\mathbf{x})\partial_n u(t,\mathbf{x})\,] \qquad (7.6)$$

I derived this in detail in a separate document. It is in n+1 dimensional space, but the time aspects are separated out carefully, and everything else is in n dimensional space, such as $d^n x$. The surface σ_n and its surface element dS_n are boundaries of the n-dimensional region R_n. Let's compare this thing with our simpler Laplace result:

$$\int_{R_n} d^n x\,[\, v(x)\,(-\nabla^2)u(x) - u(x)\,(-\nabla^2)v(x)\,] = \int_{\sigma_n} dS_n\,[u(x)\partial_n v(x) - v(x)\partial_n u(x)\,]$$

$$\int_0^T dt \int_{R_n} d^n x \; [\, v(t,\mathbf{x})\,(\partial_t - \nabla^2)u(t,\mathbf{x}) - u(t,\mathbf{x})\,(-\partial_t - \nabla^2)v(t,\mathbf{x})\,]$$

$$= \int d^n x\,[u(T,\mathbf{x})v(T,\mathbf{x}) - u(0,\mathbf{x})v(0,\mathbf{x})] + \int_0^T dt \int_{\sigma_n} dS_n\,[\, u(t,\mathbf{x})\partial_n v(t,\mathbf{x}) - v(t,\mathbf{x})\partial_n u(t,\mathbf{x})\,] \qquad (7.6)$$

Notice that the exact same "surface integral" appears in both results, but now there is a whole new term on the RHS, and we have ∂_t objects and we have two time integrals added. If in the second equation we happened to have u and v having no time dependence, the red first term vanishes, and both time integrals just generate a factor of T which then cancels out on both sides. And the ∂_t operators make 0, so in fact we do recover the Laplace version, something Stak did not mention.

Now as we did in the Laplace case, we set u and v as follows:

$u(t,\mathbf{x}) = u(t,\mathbf{x})$
$v(t,\mathbf{x}) = g(x_0,t_0|x,t)$

When we insert these objects for u and v we get

$$\int_0^T dt \int_{R_n} d^n x \; [g(x_0|x)\,(\partial_t - \nabla^2)u(t,\mathbf{x}) - u(t,\mathbf{x})\,(-\partial_t - \nabla^2)\,g(x_0|x)\,]$$

$$= \int d^n x\,[u(T,\mathbf{x})\,g(x_0|x) - u(0,\mathbf{x})\,g(x_0|x)|^{t=0}] + \int_0^T dt \int_{\sigma_n} dS_n\,[\, u(t,\mathbf{x})\partial_{n\mathbf{x}}\,g(x_0|x) - g(x_0|x)\partial_n u(t,\mathbf{x})\,]$$

Now if we use these results selected from (7.3) and (7.13)

$(\partial_t - \nabla^2)u(t,\mathbf{x}) = q(t,\mathbf{x})$ // from 7.3 line 1
$(-\partial_t - \nabla^2)\,g(x_0|x) = \delta(x-x_0)\delta(t-t_0)$ // from 7.11 line 1

we get

$$\int_0^T dt \int_{R_n} d^n x \; g(x_0|x)\, q(t,\mathbf{x}) \qquad - u(t_0,\mathbf{x}_0)$$

$$= \int d^n x \, [\{u(t,\mathbf{x})\, g(x_0|x)\}|^{t=T} - \{u(t,\mathbf{x})\, g(x_0|x)\}|^{t=0}]$$
$$+ \int_0^T dt \int_{\sigma_n} dS_n \, [\, u(t,\mathbf{x}) \partial_{n x} g(x_0|x) - g(x_0|x) \partial_n u(t,\mathbf{x}) \,]$$

At this point, we can knock out a term by noting that $g(x_0|x)\}|^{t=T} = 0$ for this reason: T is the most future time we ever consider, and $g(\leftarrow) = 0$ as we go backwards in time from this most future point to any other point. So with this term gone we have

$$\int_0^T dt \int_{R_n} d^n x \; g(x_0|x)\, q(t,\mathbf{x}) \qquad - u(t_0,\mathbf{x}_0)$$

$$= - \int d^n x \, [\, \{u(t,\mathbf{x})\, g(x_0|x)\}|^{t=0}]$$
$$+ \int_0^T dt \int_{\sigma_n} dS_n \, [\, u(t,\mathbf{x}) \partial_{n x} g(x_0|x) - g(x_0|x) \partial_n u(t,\mathbf{x}) \,]$$

I am maintaining both terms in the surface integral so I can apply this same equation to our three cases Dirichlet, Neumann and Radiative. The next step is to swap $x^\mu \rightarrow x_0^\mu$ everywhere so that we can expose the desired term $u(t,\mathbf{x})$:

$$\int_0^T dt_0 \int_{R_n} d^n x_0 \; g(x|x_0)\, q(t_0,\mathbf{x}_0) \qquad - u(t,\mathbf{x})$$

$$= - \int d^n x_0 \, \{u(t_0,\mathbf{x}_0)\, g(x|x_0)\}|^{t0=0}]$$
$$+ \int_0^T dt_0 \int_{\sigma_n} dS_{n0} \, [\, u(t_0,\mathbf{x}_0) \partial_{n x 0} g(x|x_0) - g(x|x_0) \partial_{n 0} u(t_0,\mathbf{x}_0) \,]$$

This forces us to add extra labels to show which variables ∂ act on. Rewrite this then as

$$u(t,\mathbf{x}) = \int_0^T dt_0 \int_{R_n} d^n x_0 \; g(x|x_0)\, q(t_0,\mathbf{x}_0)$$
$$+ \int_{R_n} d^n x_0 \; \{u(t_0,\mathbf{x}_0)\, g(x|x_0)\}|^{t0=0}]$$
$$- \int_0^T dt_0 \int_{\sigma_n} dS_{n0} \, [\, u(t_0,\mathbf{x}_0) \partial_{n x 0} g(x|x_0) - g(x|x_0) \partial_{n 0} u(t_0,\mathbf{x}_0) \,]$$

Now we know that $g(x|x_0) = g(x|x_0)H(t>t_0)$ and this implies $\partial_{n x 0} g(x|x_0) = \partial_{n x 0} g(x|x_0)H(t>t_0)$. Thus, everywhere we have a time integral in the above equation, we must have $t_0 < t$ so we replace our "large" endpoint T with t in two places and we then have (and now all T's are gone)

$$u(t,\mathbf{x}) = \int_0^t dt_0 \int_{\mathbf{R_n}} d^n x_0 \; g(x|x_0)\, q(t_0,\mathbf{x_0})$$

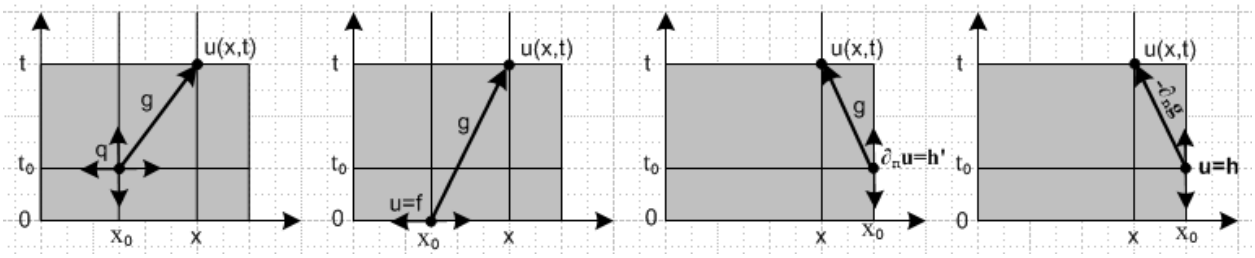
$$+ \int_{\mathbf{R_n}} d^n x_0 \; \{u(0,\mathbf{x_0})\, g(x|x_0)\}|^{t0=0}]$$

$$+ \int_0^t dt_0 \int_{\sigma_n} dS_{n0} \, [\, u(t_0,\mathbf{x_0})\{-\partial_{\mathbf{n}x0}\, g(x|x_0)\} + g(x|x_0)\partial_{n0} u(t_0,\mathbf{x_0})\,] \qquad (7.14\ \text{gen})$$

which we compare to the simpler Laplace result

$$u(x) = \int_{\mathbf{R_n}} d^n x_0 \; g(x_0|x) q(x_0) + \int_{\sigma_n} dS_{n0} \, [u(x_0)\, \partial_{\mathbf{n}x0} g(x_0|x) - g(x_0|x)\, \partial_{n0} u(x_0)\,]$$

Once again, note that the same surface integrals appear in both results. Here is my Feynman diagram for equation (7.14 gen)



In these pictures, the spatial region R is represented by a full horizontal line segment. You might imagine that the region R is really 2D but we cannot see the other dimension because it goes into the page. Each of these pictures is the Stakgold cylinder in the sense that it would be a cylinder if R were a 2D disk that we are viewing from the side.

The first diagram shows the contribution of sources inside our spacetime volume. The short arrows indicate that this integral must be done over the entire gray spacetime region, wherever there are sources q. The long arrow is the propagator g. The total contribution to u at x^μ from q at $x_0{}^\mu$ is the product of q and g. We might imagine that the longer the arrow g is, the more the effect of q at our u is attenuated. So we might think of g as a "heat influence attenuation propagator". This first term is only present if there are volume sources.

The second term represents the effect of the initial temperature distribution in the volume R. Each point in R at the initial time has its influence as shown by the arrow, and the horizontal double arrow means we integrate over R to get the influence at each point (x,t).

The remaining two terms account for the influence of activity on the boundary of R. In the third diagram we have a little Neumann surface heat source $\partial_n u$ which acts just the way q acts in the left picture. The fourth diagram is a little different and represents the Dirichlet boundary effect. The propagator here is not g, but is the Poisson Kernel Propagator $-\partial_n g$ where n is the normal n_0 at the source of the propagator. In both these pictures, the vertical arrows imply that we integrate over the entire boundary of R (that part is not shown very well) and we integrate over all time from 0 to t.

If we have only one of the rightmost two diagrams, we would refer to the prescribed u or $\partial_{nu}u$ value as h, but if both are present I have allowed h' (not a derivative!) and h. Similarly, the prescribed t=0 value for u is usually called f by Stakgold.

To summarize: the temperature u(x,t) has four contributions corresponding to the four diagrams (and all signs are positive so they just add). The first and third are very similar and account for internal and boundary heat sources. The second diagram shows contributions from the t = 0 "boundary" while the last term shows contributions from the σ boundary. You can therefore think of the last three terms as boundary contributions in the sense of the n+1 dimensional Green's Identity. We get no contribution from the top boundary or from regions above due to our causality assumption on g.

If we were to crush the four diagrams down to the t=0 axis, we would have the Laplace situation. In this case, the 2nd diagram gets cancelled by another diagram showing the effect of the t = t boundary (crushed down) which we threw out in our heat analysis due to causality, but there is no causality for Laplace. But the other three diagrams would still exist, with horizontal propagators g.

1.The Dirichlet Case.

Our two "problems" appear this way for Dirichlet

$$\begin{array}{llll} \partial_t u - \nabla^2 u = q(x,t) & & t > 0 & (7.3) \\ u(x,t) = h(x,t) & \text{for x on } \sigma & t > 0 & \\ u(x,0) = f(x) & \text{for x in R} & & \end{array}$$

$$\begin{array}{llll} (\partial_t - \nabla_x^2)\, g(x,t|x_0,t_0) = \delta^n(x-x_0)\delta(t-t_0) & & \text{all t and } t_0 & (7.8) \\ g(x,t|x_0,t_0) = 0 & \text{for x on } \sigma & \text{all t and } t_0 & \\ g(x,t|x_0,t_0) = 0 & t < t_0 & & \end{array}$$

and this lets us throw out one of the surface terms in our (7.14 gen) so we get (7.14):

$$\begin{aligned} u(t,\mathbf{x}) \;=\; & \int_0^t dt_0 \int_{R_n} d^n x_0 \; g(x|x_0)\, q(t_0,\mathbf{x}_0) \\ & + \int_{R_n} d^n x_0 \; \{u(0,\mathbf{x}_0)\, g(x|x_0)\}|^{t0=0}] \\ & + \int_0^t dt_0 \int_{\sigma_n} dS_{n0} \; [\, u(t_0,\mathbf{x}_0)\{-\partial_{nx0}\, g(x|x_0)\}\,] \end{aligned}$$

and we then install the prescribed function names f and h to get

$$\begin{aligned} u(t,\mathbf{x}) \;=\; & \int_0^t dt_0 \int_{R_n} d^n x_0 \; g(x|x_0)\, q(t_0,\mathbf{x}_0) \\ & + \int_{R_n} d^n x_0 \; \{f(\mathbf{x}_0)\, g(x|x_0)\}|^{t0=0}] \\ & + \int_0^t dt_0 \int_{\sigma_n} dS_{n0}\; h(t_0,\mathbf{x}_0)\{-\partial_{nx0}\, g(x|x_0)\} \qquad \text{(7.14 Dirichlet)} \end{aligned}$$

In this case, we cross out the third Feynman diagram.

2.The Neumann Case.

Our two "problems" appear this way for Neumann (notice that our t=0 initial condition is still on u, it is only the spatial boundary lines where $\partial_n u$ appears).

$$\begin{array}{llll} \partial_t u - \nabla^2 u = q(x,t) & & t > 0 & (7.3) \\ \partial_n u(x,t) = h(x,t) & \text{for x on } \sigma & t > 0 & \\ u(x,0) = f(x) & \text{for x in R} & & \end{array}$$

$$\begin{array}{llll} (\partial_t - \nabla_x^2)\, g(x,t|x_0,t_0) = \delta^n(x-x_0)\delta(t-t_0) & & \text{all t and } t_0 & (7.8) \\ \partial_n g(x,t|x_0,t_0) = 0 & & \text{for x on } \sigma & \text{all t and } t_0 \\ g(x,t|x_0,t_0) = 0 & & t < t_0 & \text{// and all derivatives like } \partial_n \end{array}$$

and this lets us throw the other surface term in our (7.14 gen) so we get (7.14):

$$\begin{aligned} u(t,\mathbf{x}) \;=\; & \int_0^t dt_0 \int_{R_n} d^n x_0 \;\; g(x|x_0)\, q(t_0,\mathbf{x}_0) \\ & + \int_{R_n} d^n x_0 \;\; \{f(\mathbf{x}_0)\, g(x|x_0)\}|^{t0=0}] \\ & + \int_0^t dt_0 \int_{\sigma_n} dS_{n0} \;\; g(x|x_0)\, h(t_0,\mathbf{x}_0) \qquad \text{(7.14 Neumann)} \end{aligned}$$

In this case, we cross out the fourth Feynman diagram.

3. The Radiative Case.

Our two "problems" appear this way for Neumann (notice that our t=0 initial condition is still on u, it is only the spatial boundary lines where $\partial_n u$ appears).

$$\begin{array}{llll} \partial_t u - \nabla^2 u = q(x,t) & & t > 0 & (7.3) \\ \partial_n u(x,t) + \theta u(x,t) = h(x,t) & \text{for x on } \sigma & t > 0 & \\ u(x,0) = f(x) & \text{for x in R} & & \end{array}$$

$$\begin{array}{llll} (\partial_t - \nabla_x^2)\, g(x,t|x_0,t_0) = \delta^n(x-x_0)\delta(t-t_0) & & \text{all t and } t_0 & (7.8) \\ \partial_n g(x,t|x_0,t_0) + \theta\, g(x,t|x_0,t_0) = 0 & & \text{for x on } \sigma & \text{all t and } t_0 \\ g(x,t|x_0,t_0) = 0 & & t < t_0 & \text{// and all derivatives like } \partial_n \end{array}$$

Since the surface terms of the heat problems are identical to those appearing in the Laplace problem, the radiative result is the same and "looks like" the Neumann result but the function h(x,t) has a different meaning, so we can just quote the Neumann result:

$$u(t,\mathbf{x}) = \int_0^t dt_0 \int_{R_n} d^n x_0 \; g(x|x_0)\, q(t_0,\mathbf{x}_0)$$
$$+ \int_{R_n} d^n x_0 \; \{f(\mathbf{x}_0)\, g(x|x_0)\}|^{t0=0}]$$
$$+ \int_0^t dt_0 \int_{\sigma_n} dS_{n0} \; g(x|x_0)\, h(t_0,\mathbf{x}_0) \qquad \text{(7.14 Radiative)}$$

The Feynman diagram aspect of things here is the same as for the Neumann case.

In passing we might note that the propagators shown above are **non-relativistic** in the sense that they don't have light delay times the way particle propagators do in a relativistic particle theory. Thus we don't worry about light cones here in our non-relativistic heat theory. We might imagine, however, that there is in fact a relativistic heat theory that accounts for time delays of the movement of macroscopic parts of the medium, such as in a stellar plasma. Perhaps I will run into that at some point, the heat equation will look different and will have a combination of heat and wave aspects.

Part III: The Wave Equation

The Boundary Value Problem of Wave Equation and Feynman Diagrams

$\partial_t^2 u - \nabla^2 u = q(x,t)$		$t > 0$	(7.109)
$u(x,t) = h(x,t)$	for x on σ	$t > 0$	
$u(x,0) = f_1(x)$	$\partial_t u(x,0) = f_2(x)$	for x in R	

The entire problem above is only defined for $t > 0$. We have no interest in $t < 0$.

Here is the corresponding (Dirichlet) Green's Function Problem (in is understood that x_0 always in R)

$(\partial_t^2 - \nabla_x^2)\, g(x,t\|x_0,t_0) = \delta^n(x-x_0)\delta(t-t_0)$		all t and t_0	(7.113)
$g(x,t\|x_0,t_0) = 0$	for x on σ	all t and t_0	
$g(x,t\|x_0,t_0) = 0$	$t < t_0$		

Discussion: 2nd line says $g(x,t|x_0,t_0) = 0$ for x on the boundary σ of R regardless of the values of the other three variables t, x_0 and t_0. 3rd line says that g is a propagator of the form g(past|future) = 0 so you only get propagation ≠ 0 when you have g(future|past), so think of this as g(←). The point x^μ must lie in the future of the point x_0^μ in spacetime, or there is no propagation amplitude. This forward time condition is why we call this thing a Causal Green's Function.

Unlike the case in Potential Theory for Laplace, we can write (7.113) in the following alternate manner, where this entire problem is defined only for $t > t_0$:

$(\partial_t^2 - \nabla_x^2)\, g(x,t\|x_0,t_0) = 0$		positive t and t_0	(7.114)
$g(x,t\|x_0,t_0) = 0$	for x on σ	positive t and t_0	
$\partial_t g(x,t_0+\|x_0,t_0) = \delta^n(x-x_0)$			

$$g(x,t|x_0,t_0) = 0 \qquad\qquad t \le t_0$$

The claim is that these two problems (7.113) and (7.114) have the same solution $g(x,t|x_0,t_0)$! In both problems, the propagator g is 0 it you try to go backward in time, so still have g(←). If you solve the second problem for g, this g is defined only for positive times. So in the region of positive times only, we claim that this same g satisfies the first problem. We can agree to only think about positive times in all our work.

Now we come to Green's 2nd identity written for the wave operator L. This is derived in a separate doc "N dim ...". It differs from the heat equation in that $uv \to v\partial_0 u - u\partial_0 v$, and we get

$$\int_0^T dt \int_{R_n} d^n x \; [\, v(t,\mathbf{x})\,(\partial_t^2 - \nabla^2)u(t,\mathbf{x}) - u(t,\mathbf{x})\,(\partial_t^2 - \nabla^2)v(t,\mathbf{x})\,]$$

$$= \int d^n x \; [\; \{\, v(T,\mathbf{x})\partial_t u(T,\mathbf{x}) - u(T,\mathbf{x})\partial_t v(T,\mathbf{x})\,\} \; - \{\, v(0,\mathbf{x})\partial_t u(0,\mathbf{x}) - u(0,\mathbf{x})\partial_t v(0,\mathbf{x})\,\} \;]$$

$$+ \int_0^T dt \int_{\sigma_n} dS_n \,[\, u(t,\mathbf{x})\partial_n v(t,\mathbf{x}) - v(t,\mathbf{x})\partial_n u(t,\mathbf{x})\,] \qquad (7.7)$$

It is in n+1 dimensional space, but the time aspects are separated out carefully, and everything else is in n dimensional space, such as $d^n x$. The surface σ_n and its surface element dS_n are boundaries of the n-dimensional region R_n. Let's compare this thing with our simpler Laplace and Heat results:

Laplace:

$$\int_{R_n} d^n x \,[\, v(x)\,(-\nabla^2)u(x) - u(x)\,(-\nabla^2)v(x)\,] = \int_{\sigma_n} dS_n \,[u(x)\partial_n v(x) \; - v(x)\partial_n u(x)\,]$$

Heat:

$$\int_0^T dt \int_{R_n} d^n x \; [\, v(t,\mathbf{x})\,(\partial_t - \nabla^2)u(t,\mathbf{x}) - u(t,\mathbf{x})\,(-\partial_t - \nabla^2)v(t,\mathbf{x})\,]$$

$$= \int d^n x \,[u(T,\mathbf{x})v(T,\mathbf{x}) - u(0,\mathbf{x})v(0,\mathbf{x})]$$

$$+ \int_0^T dt \int_{\sigma_n} dS_n \,[\, u(t,\mathbf{x})\partial_n v(t,\mathbf{x}) - v(t,\mathbf{x})\partial_n u(t,\mathbf{x})\,] \qquad (7.6)$$

Wave:

$$\int_0^T dt \int_{R_n} d^n x \; [\, v(t,\mathbf{x})\,(\partial_t^2 - \nabla^2)u(t,\mathbf{x}) - u(t,\mathbf{x})\,(\partial_t^2 - \nabla^2)v(t,\mathbf{x})\,]$$

$$= \int d^n x \; [\; \{\, v(T,\mathbf{x})\partial_t u(T,\mathbf{x}) - u(T,\mathbf{x})\partial_t v(T,\mathbf{x})\,\} \; - \{\, v(0,\mathbf{x})\partial_t u(0,\mathbf{x}) - u(0,\mathbf{x})\partial_t v(0,\mathbf{x})\,\} \;]$$

$$+ \int_0^T dt \int_{\sigma_n} dS_n \,[\, u(t,\mathbf{x})\partial_n v(t,\mathbf{x}) - v(t,\mathbf{x})\partial_n u(t,\mathbf{x})\,] \qquad (7.7)$$

Notice that the exact same "surface integral" appears in all three results, and comparing the heat and wave cases, the entire "boundary line" is exactly the same! The RHS first line for the wave case has four terms instead of two as in the Heat case, and each of the four terms has a time derivative on something. If in the Wave equation we happened to have u and v having no time dependence, the blue terms vanish, and both

time integrals just generate a factor of T which then cancels out on both sides. And the ∂_t^2 operators make 0, so in fact we do recover the Laplace version, something Stak did not mention.

Now as we did in the Laplace case, we set u and v as follows:

$u(t,\mathbf{x}) = u(t,\mathbf{x})$
$v(t,\mathbf{x}) = g(x_0,t_0|x,t)$

When we insert these objects for u and v we get

$$\int_0^T dt \int_{R_n} d^n x \; [g(x_0|x)\,(\partial_t^2 - \nabla^2)u(t,\mathbf{x}) - u(t,\mathbf{x})\,(\partial_t^2 - \nabla^2)\,g(x_0|x)\,]$$

$$= \int d^n x \; [\; \{g(x_0,t_0|x,T)\partial_t u(T,\mathbf{x}) - u(T,\mathbf{x})\partial_t g(x_0,t_0|x,T)\,\}$$
$$- \{g(x_0,t_0|x,0)\,\partial_t u(0,\mathbf{x}) - u(0,\mathbf{x})\partial_t g(x_0,t_0|x,0)\,\}\;]$$
$$+ \int_0^T dt \int_{\sigma_n} dS_n \; [\; u(t,\mathbf{x})\partial_n\, g(x_0,t_0|x,t) - g(x_0,t_0|x,t)\partial_n u(t,\mathbf{x})\;] \qquad (7.7)$$

where $\partial_t g(x_0,t_0|x,0)$ means $[\partial_t g(x_0,t_0|x,t)]|^{t=0}$ and similarly for $\partial_t u(0,\mathbf{x})$.

Now if we use these results

$(\partial_t^2 - \nabla^2)u(t,\mathbf{x}) = q(t,\mathbf{x})$ // from 7.109 line 1
$(\partial_t^2 - \nabla^2)\,g(x_0|x) = \delta(x-x_0)\delta(t-t_0)$ // from 7.113 line 1

we get this result (the LHS is exactly the same as in the heat case)

$$\int_0^T dt \int_{R_n} d^n x \; g(x_0|x)\, q(t,\mathbf{x}) \qquad - u(t_0,\mathbf{x}_0)$$

$$= \int d^n x \; [\; \{g(x_0,t_0|x,T)\partial_t u(T,\mathbf{x}) - u(T,\mathbf{x})\partial_t g(x_0,t_0|x,T)\,\}$$
$$- \{g(x_0,t_0|x,0)\partial_t u(0,\mathbf{x}) - u(0,\mathbf{x})\partial_t g(x_0,t_0|x,0)\,\}\;]$$
$$+ \int_0^T dt \int_{\sigma_n} dS_n \; [\; u(t,\mathbf{x})\partial_n\, g(x_0,t_0|x,t) - g(x_0,t_0|x,t)\partial_n u(t,\mathbf{x})\;] \qquad (7.7)$$

At this point, we can knock out two terms by noting that $g(x_0|x)\}|^{t=T} = 0$ for this reason: T is the most future time we ever consider, and $g(\leftarrow) = 0$ as we go backwards in time from this most future point to any other point. I think it follows that $\partial_t g(x_0,t_0|x,T) = 0$ for the same reason.

So with these two terms gone we have

$$\int_0^T dt \int_{R_n} d^n x \; g(x_0|x)\, q(t,\mathbf{x}) \qquad - u(t_0,\mathbf{x}_0)$$

$$= - \int d^n x \;\; \{g(x_0,t_0|x,0)\partial_t u(0,\mathbf{x}) - u(0,\mathbf{x})\partial_t g(x_0,t_0|x,0)\,\}$$

$$+ \int_0^T dt \int_{\sigma_n} dS_n \, [\, u(t,\mathbf{x})\partial_n g(x_0,t_0|x,t) - g(x_0,t_0|x,t)\partial_n u(t,\mathbf{x}) \,] \qquad (7.7)$$

The heat result was the same as the above except it had 2nd line = $-\int d^n x \, [\, \{u(t,\mathbf{x})\, g(x_0|x)\}|^{t=0}]$.

I am maintaining both terms in the surface integral so I can apply this same equation to our three cases Dirichlet, Neumann and Radiative. The next step is to swap $x^\mu \rightarrow x_0^\mu$ everywhere so that we can expose the desired term u(t,**x**):

$$\int_0^T dt_0 \int_{R_n} d^n x_0 \; g(x|x_0)\, q(t_0,\mathbf{x}_0) \qquad - u(t,\mathbf{x})$$

$$= -\int d^n x_0 \, \{g(x,t|x_0,0)\partial_{t_0} u(0,\mathbf{x}_0) - u(0,\mathbf{x}_0)\partial_{t_0} g(x,t|x_0,0)\,\}$$
$$+ \int_0^T dt_0 \int_{\sigma_n} dS_{n_0} \, [\, u(t_0,\mathbf{x}_0)\partial_{n_0 x_0} g(x|x_0) - g(x|x_0)\partial_{n_0} u(t_0,\mathbf{x}_0) \,]$$

This forces us to add extra labels to show which variables ∂ act on. Rewrite this then as

$$u(t,\mathbf{x}) = \int_0^T dt_0 \int_{R_n} d^n x_0 \; g(x|x_0)\, q(t_0,\mathbf{x}_0)$$
$$+ \int_{R_n} d^n x_0 \; \{g(x,t|x_0,0)\partial_{t_0} u(0,\mathbf{x}_0) - u(0,\mathbf{x}_0)\partial_{t_0} g(x,t|x_0,0)\,\}$$
$$- \int_0^T dt_0 \int_{\sigma_n} dS_{n_0} \, [\, u(t_0,\mathbf{x}_0)\partial_{n_0 x_0} g(x|x_0) - g(x|x_0)\partial_{n_0} u(t_0,\mathbf{x}_0) \,]$$

Now we know that $g(x|x_0) = g(x|x_0)H(t>t_0)$ and this implies $\partial_{n_0 x_0} g(x|x_0) = \partial_{n_0 x_0} g(x|x_0)H(t>t_0)$. Thus, everywhere we have a time integral in the above equation, we must have $t_0 < t$ so we replace our "large" endpoint T with t in two places and we then have (and now all T's are gone)

$$u(t,\mathbf{x}) = \int_0^t dt_0 \int_{R_n} d^n x_0 \; g(x|x_0)\, q(t_0,\mathbf{x}_0)$$
$$+ \int_{R_n} d^n x_0 \; \{g(x,t|x_0,0)\partial_{t_0} u(0,\mathbf{x}_0) - u(0,\mathbf{x}_0)\partial_{t_0} g(x,t|x_0,0)\,\}$$
$$- \int_0^t dt_0 \int_{\sigma_n} dS_{n_0} \, [\, u(t_0,\mathbf{x}_0)\partial_{n_0 x_0} g(x|x_0) - g(x|x_0)\partial_{n_0} u(t_0,\mathbf{x}_0) \,]$$

which we compare to the simpler Laplace result

$$u(x) = \int_{R_n} d^n x_0 \; g(x_0|x) q(x_0) + \int_{\sigma_n} dS_{n_0} \, [u(x_0)\, \partial_{n_0 x_0} g(x_0|x) - g(x_0|x)\, \partial_{n_0} u(x_0) \,]$$

Now, for the wave result Stak does not do the $x \leftrightarrow x_0$ business, so let's convert our result back

$$u(t_0,\mathbf{x_0}) = \int_0^{t_0} dt \int_{\mathbf{R_n}} d^n x \; g(x_0|x)\, q(t,\mathbf{x})$$

$$+ \int_{\mathbf{R_n}} d^n x \; \{ g(x_0,t_0|x,0)\partial_t u(0,\mathbf{x}) - u(0,\mathbf{x})\partial_t g(x_0,t_0|x,0) \}$$

$$- \int_0^{t_0} dt \int_{\sigma_n} dS_n \; [\, u(t,\mathbf{x})\partial_{\mathbf{nx}} g(x_0|x) - g(x_0|x)\partial_{\mathbf{n}} u(t,\mathbf{x}) \,]$$

Now we install our BC and IC's from above

$u(x,t) = h(x,t)$ for x on σ $t > 0$
$u(x,0) = f_1(x)$ $\partial_t u(x,0) = f_2(x)$ for x in R

and we then have

$$u(t_0,\mathbf{x_0}) = \int_0^{t_0} dt \int_{\mathbf{R_n}} d^n x \; g(x_0|x)\, q(t,\mathbf{x})$$

$$+ \int_{\mathbf{R_n}} d^n x \; \{ g(x_0,t_0|x,0)\, f_2(x) - f_1(x)\partial_t g(x_0,t_0|x,0) \}$$

$$- \int_0^{t_0} dt \int_{\sigma_n} dS_n \; [\, h(t,\mathbf{x})\partial_{\mathbf{nx}} g(x_0|x) - g(x_0|x)\partial_{\mathbf{n}} h(t,\mathbf{x}) \,]$$

One more change is to use 246 A to show (see pencil scribble in book) that $\partial_t g = -\partial_{t_0} g$ so we have

$$u(t_0,\mathbf{x_0}) = \int_0^{t_0} dt \int_{\mathbf{R_n}} d^n x \; g(x_0|x)\, q(t,\mathbf{x})$$

$$+ \int_{\mathbf{R_n}} d^n x \; \{ g(x_0,t_0|x,0)\, f_2(x) + f_1(x)\partial_{t_0} g(x_0,t_0|x,0) \}$$

$$- \int_0^{t_0} dt \int_{\sigma_n} dS_n \; [\, h(t,\mathbf{x})\partial_{\mathbf{nx}} g(x_0|x) - g(x_0|x)\partial_{\mathbf{n}} h(t,\mathbf{x}) \,]$$

$$u(t_0,\mathbf{x_0}) = \int_0^{t_0} dt \int_{\mathbf{R_n}} d^n x \; g(x_0|x)\, q(t,\mathbf{x})$$

$$+ \int_{\mathbf{R_n}} d^n x \; g(x_0,t_0|x,0)\, f_2(x) + \int_{\mathbf{R_n}} d^n x \; f_1(x)\partial_{t_0} g(x_0,t_0|x,0)$$

$$- \int_0^{t_0} dt \int_{\sigma_n} dS_n \; [\, h(t,\mathbf{x})\partial_{\mathbf{nx}} g(x_0|x) - g(x_0|x)\partial_{\mathbf{n}} h(t,\mathbf{x}) \,]$$

$$u(t_0,\mathbf{x_0}) = \int_0^{t_0} dt \int_{\mathbf{R_n}} d^n x \; g(x_0,t_0|x,t)\, q(t,\mathbf{x})$$

$$+ \int_{\mathbf{R_n}} d^n x \; g(x_0,t_0|x,0)\, f_2(x) + \partial_{t_0} \{ \int_{\mathbf{R_n}} d^n x \; f_1(x)\, g(x_0,t_0|x,0) \}$$

$$- \int_0^{t_0} dt \int_{\sigma_n} dS_n \; [\, h(t,\mathbf{x})\partial_{\mathbf{nx}} g(x_0|x) - g(x_0|x)\partial_{\mathbf{n}} h(t,\mathbf{x}) \,] \qquad (7.116 \text{ gen})$$

and when g=0 on σ (Dirichlet), this agrees with 7.116, hurray! Let's now compare the other version of this last Wave result with the Heat Result:

Wave:

$$u(t,\mathbf{x}) = \int_0^t dt_0 \int_{R_n} d^n x_0 \; g(x|x_0)\, q(t_0,\mathbf{x}_0)$$

$$+ \int_{R_n} d^n x_0 \; g(x,t|x_0,0) f_2(x_0) - \int_{R_n} d^n x_0 \; \partial_{t_0} g(x,t|x_0,0) f_1(x_0)$$

$$- \int_0^t dt_0 \int_{\sigma_n} dS_{n0} \left[h(t_0,\mathbf{x}_0) \partial_{\mathbf{n}\mathbf{x}_0} g(x|x_0) - g(x|x_0) \partial_{n_0} h(t_0,\mathbf{x}_0) \right] \qquad (7.116 \text{ gen})$$

Heat:

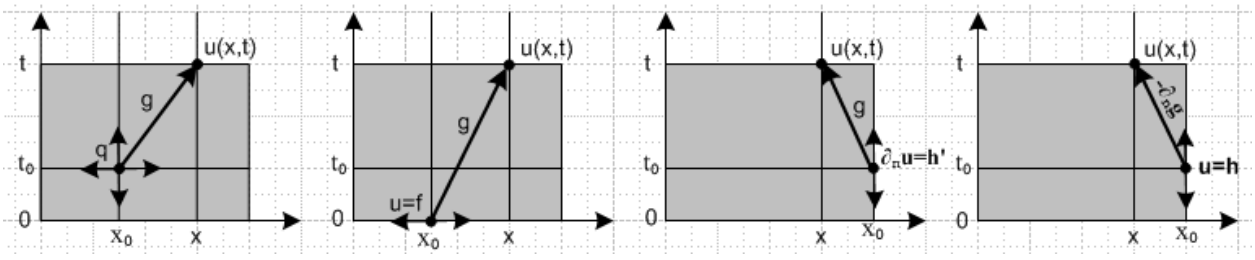
$$u(t,\mathbf{x}) = \int_0^t dt_0 \int_{R_n} d^n x_0 \; g(x|x_0)\, q(t_0,\mathbf{x}_0)$$

$$+ \int_{R_n} d^n x_0 \; g(x,t|x_0,0)\, f(\mathbf{x}_0)$$

$$- \int_0^t dt_0 \int_{\sigma_n} dS_{n0} \left[h(t_0,\mathbf{x}_0) \{ \partial_{\mathbf{n}\mathbf{x}_0} g(x|x_0) \} - g(x|x_0) \partial_{n_0} h(t_0,\mathbf{x}_0) \right] \qquad (7.14 \text{ gen})$$

We see that f_2 replaces f of the heat equation, and the f_1 term for Wave is new, and everything else is the same!! (of course the propagator g is different)

Once again, note that the same surface integrals appear in both results. Here is my Feynman diagram for equation (7.1116 gen) except: change f to f_2 in 2nd picture, and I have no picture for the f_1 term:



In these pictures, the spatial region R is represented by a horizontal line segment. You might imagine that the region R is really 2D but we cannot see the other dimension because it goes into the page. Each of these pictures is the Stakgold cylinder in the sense that it would be a cylinder if R were a 2D disk that we are viewing from the side.

The first diagram shows the contribution of sources inside our spacetime volume. The short arrows indicate that this integral must be done over the entire gray spacetime region, wherever there are sources q. The long arrow is the propagator g. The total contribution to u at x^μ from q at $x_0{}^\mu$ is the product of q and g. We might imagine that the longer the arrow g is, the more the effect of q at our u is attenuated. So we might think of g as a "wave influence attenuation propagator". This first term is only present if there are volume sources.

The second diagram represents the effect of the initial *velocity* field in the volume R, so f = f_2. Each point in R at the initial time has its influence as shown by the arrow, and the horizontal double arrow means we integrate over R to get the influence at each point (x,t).

I don't know how to draw a diagram for the f_1 term shown in 7.116 gen above. I could draw it but with a propagator that was $-\partial_{t0}\, g(x_0,t_0|x,0)$, then the diagram would be like the second diagram.

The remaining two terms account for the influence of activity on the boundary of R, and things are exactly the same for Wave and Heat. In the third diagram we have a little Neumann surface source $\partial_n u$ which acts just the way q acts in the left picture. The fourth diagram is a little different and represents the Dirichlet boundary effect. The propagator here is not g, but is the Poisson Kernel Propagator $-\partial_n g$ where n is the normal n_0 at the source of the propagator. In both these pictures, the vertical arrows imply that we integrate over the entire boundary of R (that part is not shown very well) and we integrate over all time from 0 to t.

If we have only one of the rightmost two diagrams, we would refer to the prescribed u or $\partial_{nu} u$ value as h, but if both are present I have allowed h' (not a derivative!) and h. Similarly, the prescribed t=0 value for u is usually called f_2 by Stakgold (f in picture).

To summarize: the displacement u(x,t) has five contributions corresponding to the four diagrams (and all signs are positive (except the Neumann σ term) so they just add) and the one diagram not shown. The first and third are very similar and account for internal and boundary sources. The second diagram shows contributions from the t = 0 "boundary" while the last term shows contributions from the σ boundary. You can therefore think of the last three terms as boundary contributions in the sense of the n+1 dimensional Green's Identity. We get no contribution from the top boundary or from regions above due to our causality assumption on g.

If we were to crush the four diagrams down to the t=0 axis, we would have the Laplace situation. In this case, the 2nd diagram gets cancelled by another diagram showing the effect of the t = t boundary (crushed down) which we threw out in our heat analysis due to causality, but there is no causality for Laplace. But the other three diagrams would still exist, with horizontal propagators g.

Since the "boundary line" of the Wave and Heat equations is *the same*, I won't comment separately here on the three cases Dirichlet, Neumann and Radiative. I think it is all pretty clear.