Chapter 7 page 275 Debug 0 PhL 7.9.11

This file contains notes "moved" from the original ch7raw2notes doc. The other two docs Debug 1 and 2 were further work I did while I was stumped trying to verify 7.183 arising from p 275 E. I updated the actual ch7raw2notes to be a coherent presentation of the correct solution, but I keep these three docs around since I did a lot of detail work that might come up again. Note that a lot of the stuff in these three debug docs is WRONG!

(i) Try to do the last steps at the end of the section to get 7.183

We already know how to write the E part of g as a sum, see 7.166. Thus, adjusting, we get

$$E = (1/2\pi)\,\Sigma_n\, e^{in(\varphi-\pi)} I_n(kr_<)K_n(kr_>) = (1/2\pi)\,\Sigma_n\,(-1)^n e^{in\varphi}\, I_n(kr_<)K_n(kr_>)$$

So our problem is just to somehow write v(r,φ) as a sum, where

$$v(r,\varphi) = -(1/\pi^2)\int_0^\infty d\gamma\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, ch[\gamma(\varphi-\pi)] / ch(\gamma\pi)$$

Now, looking at subsection (x) above, we can see that we have [wrong starts right here! The IK product shown has no disc at all in the γ plane!]

$$disc[I_{-i\gamma}(kr_<)\, K_{-i\gamma}(kr_>)] = (2i/\pi)sh(\pi\gamma)K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)$$

where the discontinuity is across the positive real axis, which is our integral range for g above.
So if we mult and divide by (2i/π)sh(πγ), we can write

$$v = -(1/\pi^2)\int_0^\infty d\gamma\, disc[I_{-i\gamma}(kr_<)\, K_{-i\gamma}(kr_>)]\;\; ch[\gamma(\varphi-\pi)] / ch(\gamma\pi) * (\pi/2i)\; 1/sh(\pi\gamma)$$

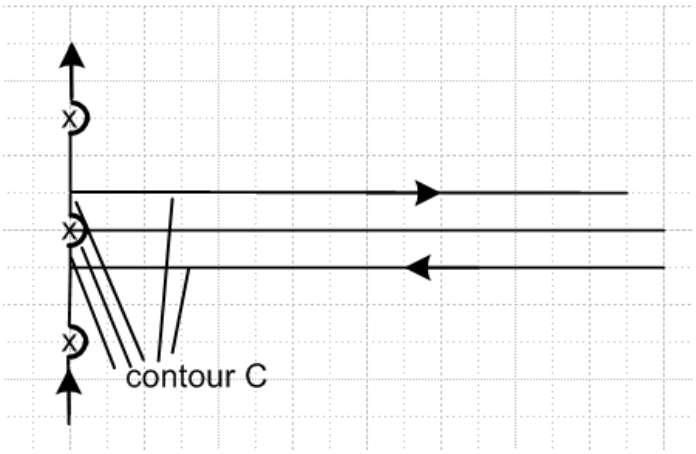
$$= -(1/\pi^2)\int_0^\infty d\gamma\, disc[I_{-i\gamma}(kr_<)\, K_{-i\gamma}(kr_>)]\;\; ch[\gamma(\varphi-\pi)] * (\pi/i)\; /sh(2\pi\gamma)$$

$$= -(1/i\pi)\int_0^\infty d\gamma\, disc[I_{-i\gamma}(kr_<)\, K_{-i\gamma}(kr_>)]\;\; ch[\gamma(\varphi-\pi)]\; /sh(2\pi\gamma)$$

If we now define C as a contour enveloping the positive real axis (top to right), we can write this as

$$= -(1/\pi i)\int_C d\gamma\, I_{-i\gamma}(kr_<)\, K_{-i\gamma}(kr_>)\, ch[\gamma(\varphi-\pi)]\; /sh(2\pi\gamma)$$

Out on the great circle it seems that we have good expo γ decay from the factors to the right of K, so let's assume for now that the GC integral vanishes. This means we can unwrap that contour and get it vertical and move it to the left and we get the pole residues of the function 1/ sh(2πγ) ! Here is a picture



The original contour is the limit of the pieces shown by lines, and this original contour already includes half the residue of the pole at the origin. When we unwrap the contour, we first make it roughly vertical as shown, then shift it to the left. When we do this shift, we wrap all the non-origin poles in the normal CCW sense, but we only add HALF the residue of the origin pole, so be careful with this!

The poles are located here

$$\text{ish}(2\pi\gamma_n) = \sin(i2\pi\gamma_n) = 0 \qquad \Rightarrow \qquad i2\pi\gamma_n = 0, \pm\pi, \pm 2\pi \quad = -n\pi \qquad n = \text{integers}$$

But we need to be very careful here. If you track the contour sense, we wrap the poles in either half plane in the correct sense. However, the pole at the origin is wrapped only half way and in the wrong sense.

So the poles are located at half integers *and* integers on the imaginary axis:

$$\gamma_n = -n\pi/(2\pi i) = ni/2 \qquad -i\gamma_n = n/2$$

Near one of these poles we have $\gamma = \gamma_n + \delta\gamma = ni/2 + \delta\gamma$ so

$$f(\gamma) = f(\gamma_n) + \delta\gamma\, f'(\gamma_n) \qquad f(\gamma) = \text{ish}(2\pi\gamma) \Rightarrow \qquad f'(\gamma) = 2\pi i\, \text{ch}(2\pi\gamma)$$

$$\text{ish}(2\pi\gamma) = \text{ish}(2\pi\gamma_n) + \delta\gamma\, 2\pi i\, \text{ch}(2\pi\gamma_n) = \delta\gamma\, 2\pi i\, \text{ch}(2\pi ni/2) = \delta\gamma\, 2\pi i \cos(2\pi n/2)$$

$$= \delta\gamma\, 2\pi i \cos(\pi n) = 2\pi i\, (-1)^n\, \delta\gamma$$

Therefore we have laboriously shown that

$$\text{sh}(2\pi\gamma) \approx 2\pi\, (-1)^n\, (\gamma - \gamma_n) \qquad \text{for } \gamma \text{ near the pole position } \gamma_n$$

So we can then say that

$$-v/2 = (1/2\pi i) \int_C d\gamma\, I_{-i\gamma}(kr_<)\, K_{-i\gamma}(kr_>)\, \text{ch}[\gamma(\varphi-\pi)] / \text{sh}(2\pi\gamma))$$

$$= \text{sum of residues} = \Sigma_n \; I_{-i\gamma n}(kr_<)\, K_{-i\gamma n}(kr_>) \; ch[\gamma_n(\varphi-\pi)] \,/\, [2\pi\,(-1)^n]$$

$$= (1/2\pi)\,\Sigma_n \; (-1)^n I_{-i\gamma n}(kr_<)\, K_{-i\gamma n}(kr_>) \; ch[ni/2(\varphi-\pi)]$$

$$= (1/2\pi)\,\Sigma_n \; (-1)^n I_{n/2}(kr_<)\, K_{n/2}(kr_>) \; \cos[n(\varphi-\pi)/2] \qquad\qquad n = 0, \pm1, \pm2 \ldots$$

EXCEPT, for the n = 0 pole we need to add a factor of which we can handle by adding $\varepsilon_n/2$ where ε_n is the Neumann factor which is 1 at 0 and 2 everywhere else.

So we then have shown that

$$v(r,\varphi|r_o,\pi) = -\,(1/\pi)\,\Sigma_n\,(-1)^n\, I_{n/2}(kr_<)\, K_{n/2}(kr_>) \; \cos[n(\varphi-\pi)/2]\;(\varepsilon_n/2)$$

Meanwhile, just as a reminder, we earlier found that

$$E(r,\varphi|r_o,\pi) = (1/2\pi)\,\Sigma_n\,(-1)^n e^{in\varphi}\; I_n(kr_<)K_n(kr_>)$$

and we then add these to get our candidate for g shown in 7.183. But we have to "fold" the negative parts of both these sums to get the type of form in 7.183. Again, from Vol I p 317 we have

$$K_{-\alpha} = K_\alpha \qquad\qquad I_{-\alpha} = I_\alpha \;+ (2/\pi)\sin(\pi\alpha)\, K_\alpha$$

so that

$$I_{-\alpha}K_{-\alpha} = [I_\alpha + (2/\pi)\sin(\pi\alpha)\, K_\alpha]K_\alpha \;= I_\alpha K_\alpha + (2/\pi)\sin(\pi\alpha)\, K_\alpha K_\alpha$$

Applying this to our case with n being integers,

$$I_{-n}K_{-n} = I_nK_n + (2/\pi)\sin(\pi n)\, K_nK_n \;=\; I_nK_n$$

and then the other case

$$I_{-n/2}K_{-n/2} = I_{n/2}K_{n/2} + (2/\pi)\sin(\pi n/2)\, K_{n/2}K_{n/2}$$

Now when n is even we just get $I_{-n/2}K_{-n/2} = I_{n/2}K_{n/2}$. When n is odd we get

$$\sin(\pi n/2) \;= (-1)^{n-1}(\pi/2)$$

so to summarize

$$I_{-n}K_{-n} = I_nK_n \qquad\qquad \text{n any integer}$$

$$I_{-n/2}K_{-n/2} = I_{n/2}K_{n/2} \qquad\qquad \text{n an even integer}$$

$$I_{-n/2}K_{-n/2} = I_{n/2}K_{n/2} + (-1)^{n-1}K_{n/2}K_{n/2} \qquad \text{n an odd integer}$$

So let's start by folding over the negative part of the E sum

$$E(r,\varphi|r_o,\pi)_{neg} = (1/2\pi)\,\Sigma_{n=-1}^{-\infty}\,(-1)^n\; e^{in\varphi}\; I_n(kr_<)K_n(kr_>)$$

$$= (1/2\pi)\,\Sigma_{n=1}^{\infty}\,(-1)^n\, e^{-in\varphi}\; I_n(kr_<)K_n(kr_>)$$

So we then have

$$E = (1/2\pi)\,\Sigma_{n=1}^{\infty}\,(-1)^n\,(e^{in\varphi} + e^{-in\varphi})\;\; I_n(kr_<)K_n(kr_>) \;+ (1/2\pi)\; I_0(kr_<)K_0(kr_>$$

$$= \;(1/2\pi)\,\Sigma_{n=1}^{\infty}\,(-1)^n\; 2\,\cos(n\varphi)\;\; I_n(kr_<)K_n(kr_>) \;+ (1/2\pi)\; I_0(kr_<)K_0(kr_>)$$

$$= \;(1/\pi)\,\Sigma_{n=1}^{\infty}\,(-1)^n\,\cos(n\varphi)\; I_n(kr_<)K_n(kr_>) \;+ (1/2\pi)\; I_0(kr_<)K_0(kr_>)$$

Now we turn to the v sum which was this

$$v(r,\varphi|r_o,\pi) = -\,(1/\pi)\,(\Sigma_{neven} + \Sigma_{nodd})\;\; (-1)^n\; I_{n/2}(kr_<)\; K_{n/2}(kr_>)\;\; \cos[n(\varphi-\pi)/2]\;(\varepsilon_n/2)$$

The even part is this, where in the second line we take $n \rightarrow 2n$

$$v(r,\varphi|r_o,\pi)_{even} = \; -\,(1/\pi)\,\Sigma_{neven}\,(-1)^n\; I_{n/2}(kr_<)\; K_{n/2}(kr_>)\;\; \cos[n(\varphi-\pi)/2]\;(\varepsilon_n/2)$$

$$= \; -\,(1/\pi)\,\Sigma_n\,(-1)^{2n}\; I_n(kr_<)\; K_n(kr_>)\;\; \cos[n(\varphi-\pi)]\;(\varepsilon_{2n}/2)$$

$$= \; -\,(1/\pi)\,\Sigma_n\; I_n(kr_<)\; K_n(kr_>)\;\; \cos[n(\varphi-\pi)]\;(\varepsilon_{2n}/2)$$

We can do the fold here by inspection and we then have

$$= \; -\,(2/\pi)\,\Sigma_{n=1}^{\infty}\; I_n(kr_<)\; K_n(kr_>)\;\; \cos[n(\varphi-\pi)] \;- (1/2\pi)\; I_0(kr_<)\; K_0(kr_>)$$

But of course we know that $\cos[n(\varphi-\pi)] = \cos(n\varphi - n\pi) = (-1)^n\cos(n\varphi)$ so we have then

$$= \; -\,(2/\pi)\,\Sigma_{n=1}^{\infty}(-1)^n\; I_n(kr_<)\; K_n(kr_>)\;\; \cos(n\varphi) \;- (1/2\pi)\; I_0(kr_<)\; K_0(kr_>)$$

So far then we have these results:

$$E(r,\varphi|r_o,\pi) \;\; = \;(1/\pi)\,\Sigma_{n=1}^{\infty}\,(-1)^n\,\cos(n\varphi)\; I_n(kr_<)K_n(kr_>) \;+ (1/2\pi)\; I_0(kr_<)K_0(kr_>)$$

$$v(r,\varphi|r_o,\pi)_{even} = -\,(2/\pi)\,\Sigma_{n=1}^{\infty}(-1)^n\; I_n(kr_<)\; K_n(kr_>)\;\; \cos(n\varphi) \;- (1/2\pi)\; I_0(kr_<)\; K_0(kr_>)$$

I am now very happy to see the n=0 terms cancel, since I know they are not in the final answer. Also, the two series can be added to give

$$E(r,\varphi|r_o,\pi) + v(r,\varphi|r_o,\pi)_{\mathbf{even}} = -(1/\pi)\,\Sigma_{n=1}^{\infty}(-1)^n\, I_n(kr_<)\, K_n(kr_>)\, \cos(n\varphi)$$

Now define n = m/2 and m = 2n and the sum is then m = 2,4,6…. We can then write

$$E(r,\varphi|r_o,\pi) + v(r,\varphi|r_o,\pi)_{\mathbf{even}} = -(1/\pi)\,\Sigma_{m=2,4,6..}(-1)^{m/2}\, I_{m/2}(kr_<)\, K_{m/2}(kr_>)\, \cos(m\varphi/2)$$

or

$$E(r,\varphi|r_o,\pi) + v(r,\varphi|r_o,\pi)_{\mathbf{even}} = -(1/\pi)\,\Sigma_{n=2,4,6..}(-1)^{n/2}\, I_{n/2}(kr_<)\, K_{n/2}(kr_>)\, \cos(n\varphi/2)$$

We now must deal with the odd terms in v. We have

$$v(r,\varphi|r_o,\pi)_{\mathbf{odd}} = -(1/\pi)\,\Sigma_{\mathbf{nodd}}\,(-1)^n\, I_{n/2}(kr_<)\, K_{n/2}(kr_>)\, \cos[n(\varphi-\pi)/2]\,(\varepsilon_n/2)$$

$$= -(1/\pi)\,\Sigma_{n=\pm1,\pm3..}\,(-1)^n\, I_{n/2}(kr_<)\, K_{n/2}(kr_>)\, \cos[n(\varphi-\pi)/2] \qquad \text{// since } \varepsilon_n = 2$$

$$= (1/\pi)\,\Sigma_{n=\pm1,\pm3..}\, I_{n/2}(kr_<)\, K_{n/2}(kr_>)\, \cos[n(\varphi-\pi)/2] \qquad \text{// since } (-1)^n = -1$$

Reflecting the negative terms here we get

$$v(r,\varphi|r_o,\pi)_{\mathbf{odd,neg}} = (1/\pi)\,\Sigma_{n\,=\,-1,-3..}\, I_{n/2}(kr_<)\, K_{n/2}(kr_>)\, \cos[n(\varphi-\pi)/2]$$

$$= (1/\pi)\,\Sigma_{n=1,3..}\, I_{-n/2}(kr_<)\, K_{-n/2}(kr_>)\, \cos[n(\varphi-\pi)/2]$$

$$= (1/\pi)\,\Sigma_{n=1,3..}\, [I_{n/2}K_{n/2} + (-1)^{n-1}K_{n/2}K_{n/2}]\, \cos[n(\varphi-\pi)/2]$$

Meanwhile we have

$$v(r,\varphi|r_o,\pi)_{\mathbf{odd,pos}} = (1/\pi)\,\Sigma_{n=1,3..}\, I_{n/2}(kr_<)\, K_{n/2}(kr_>)\, \cos[n(\varphi-\pi)/2]$$

Adding these together gives

$$v(r,\varphi|r_o,\pi)_{\mathbf{odd}} = (2/\pi)\,\Sigma_{n=1,3..}\, I_{n/2}(kr_<)\, K_{n/2}(kr_>)\, \cos[n(\varphi-\pi)/2]$$

$$+ (1/\pi)\,\Sigma_{n=1,3..}\, (-1)^{n-1}K_{n/2}K_{n/2}\, \cos[n(\varphi-\pi)/2]$$

We then want to add this to our previous stuff which was

$$E(r,\varphi|r_o,\pi) + v(r,\varphi|r_o,\pi)_{\mathbf{even}} = -(1/\pi)\,\Sigma_{n=2,4,6..}(-1)^{n/2}\, I_{n/2}(kr_<)\, K_{n/2}(kr_>)\, \cos(n\varphi/2)$$

Adding it all together gives this

$$g = E+v = -(1/\pi)\,\Sigma_{n=2,4,6..}(-1)^{n/2}\, I_{n/2}(kr_<)\, K_{n/2}(kr_>)\, \cos(n\varphi/2)$$

$$+ (2/\pi)\,\Sigma_{n=1,3..}\, I_{n/2}(kr_<)\, K_{n/2}(kr_>)\, \cos[n(\varphi-\pi)/2]$$

$$+ (1/\pi)\,\Sigma_{n=1,3..}\,(-1)^{n-1}K_{n/2}(kr_<)K_{n/2}(kr_>)]\cos[n(\varphi-\pi)/2]$$

This was supposed to come out being 7.183. It is dimly close, but still way off! Notice that

$$2\sin(n\varphi/2)\sin(n\pi/2) = \cos[n(\varphi-\pi)/2] - \cos[n(\varphi+\pi)/2]$$

$$= \cos[n\varphi/2-n\pi/2] - \cos[n\varphi/2+n\pi/2]$$

If n is even, then n/2 is an integer and we get

$$= (-1)^{n/2}\cos[n\varphi/2] - (-1)^{n/2}\cos[n\varphi/2] \quad = 0$$

which is obvious since $\sin(n\pi/2) = 0$ in this case. So really in 7.183 we only have odd terms! In this case we have

$$\sin(n\pi/2) = (-1)^{(n-1)/2}$$

so we then have

$$\sin(n\varphi/2)\sin(n\pi/2) = (-1)^{(n-1)/2}\sin(n\varphi/2)$$

and we can rewrite 7.183 this way (leave with sin)

$$g = (1/\pi)\,\Sigma_{n=1,3..}\,\sin(n\varphi/2)\sin(n\pi/2)\,I_{n/2}(kr_<)\,K_{n/2}(kr_>)$$

Now consider for n odd

$$\cos[n(\varphi-\pi)/2] = \cos[n\varphi/2-n\pi/2] = \cos(n\varphi/2)\cos(n\pi/2) + \sin(n\varphi/2)\sin(n\pi/2)$$

$$= \sin(n\varphi/2)\sin(n\pi/2)$$

So I can rewrite MY answer this way

$$g = E+v = -(1/\pi)\,\Sigma_{n=2,4,6..}(-1)^{n/2}\,I_{n/2}(kr_<)\,K_{n/2}(kr_>)\,\cos(n\varphi/2)$$

$$+ (2/\pi)\,\Sigma_{n=1,3..}\,I_{n/2}(kr_<)\,K_{n/2}(kr_>)\,\sin(n\varphi/2)\sin(n\pi/2)$$

$$+ (1/\pi)\,\Sigma_{n=1,3..}\,(-1)^{n-1}K_{n/2}(kr_<)K_{n/2}(kr_>)]\sin(n\varphi/2)\sin(n\pi/2)$$

where has HIS answer is this

$$g = (1/\pi)\,\Sigma_{n=1,3..}\,I_{n/2}(kr_<)\,K_{n/2}(kr_>)\sin(n\varphi/2)\sin(n\pi/2)$$

so my middle term is twice his answer. I guess it is time to move this off to a separate doc now.