Chapter 7 page 275 Debug 1 PhL 7.9.11

Overview of the Problem. On page 273 we consider the generic KL transform which involves the second-kind modified Bessel K function which Stak calls McDonald's function. In this transform the original variable is called r (polar coordinate r in our case), and the conjugate variable is called γ. Whereas in the Fourier Transform the conjugate variable appears in a simple exponential ($e^{iγr}$), in the KL transform it appears in the order of the K function in the form $K_{iγ}$, so we have purely imaginary order on the K function. Like many transforms, such as the Hankel one, there is another fixed parameter floating around in the KL transform, and in the KL it is called k and appears as $K_{iγ}(kr)$.

Stak uses this KL transform acting on a certain PDE and is able to convert that PDE into a simple ODE by replacing coordinate r by conjugate coordinate γ. The PDE has polar coordinates r and φ and has both ∂_r and $\partial_φ$ operators, whereas the ODE has only the $\partial_φ$ operator. The PDE turns out to be the PDE which defines the Green's Function for the Helmholtz equation in 2D in polar coordinates. The fixed KL transform parameter k comes from the fixed Helmholtz parameter $\lambda = -k^2$. Stak refers to the solution of this equation as E rather than g since this PDE is for a "fundamental solution" and he always uses E for such solutions. In any event, the resulting ODE in variable φ is so simple that Stak can obtain its solution by inspection and here is his result for the KL transform $\tilde{E}$:

$$\tilde{E}(\gamma, \varphi \mid r_0, \varphi_0) = [\, 2\gamma \sinh(\pi\gamma)\,]^{-1} \cosh[\gamma(\pi - |\varphi-\varphi_0|)]\, K_{i\gamma}(kr_0) \qquad (1)$$

Recall that the 0-subscripted coordinates are the location of the Green's point source. This equation is page 273C modified for an arbitrary source location. I am pretty sure it is correct, but this is something I need to check carefully as part of the debug. On page 273 C the solution stated there is valid for $|\varphi| < \pi$, and in my modication that condition becomes $|\varphi-\varphi_0| < \pi$. When we later set $\varphi_0 = \pi$, this condition is met for φ in the range (0,2π), so I think all is well here. If you install the above into 7.178's inverse, which has the fom 7.176, the γ sinh(γπ) factors cancel and you end up with the RHS of 7.180 which is then exactly the E function you want it to be, so this gives a direct method of checking the above result.

Now we come to the specific problem of interest. We place our point source at $\varphi_0 = \pi$ and we arrange to have our Green's boundary conditions E= 0 located on the limiting lines φ = 0 and φ = 2π. I am used to this kind of stuff in the Lebedev world, for example in my bowl solution. The picture is on page 274. We seek the Green's solution g in the presence of these "knife edge" boundary conditions. We make the ansatz that we can express g = E + v where is the fundy solution discussed above and where v is a homo solution of the Helmholtz equation. This leads to the homo PDE system 7. 182 with the BC's shown. We use the same KL transform to convert this PDE into an ODE involving $\partial_φ$ and of first task is to find the solution $\tilde{v}$ to this transformed ODE system. Although Stak and I agree on equations C and D (after I replace his sin by sinh), our first disagreement comes with result E which is the solution for $\tilde{v}$. So sadly we are failing to agree at this important "waypoint".

I know his result is wrong for this simple reason. If I set φ = 0, I get

$$\tilde{v} = (K/\pi^2)\,[\, \sinh(-2\pi\gamma) - \sinh(0)]/\sinh(2\pi\gamma) = -(K/\pi^2) \qquad \text{// evaluating Stak's wrong result}$$

But according to Stak's B and C, we should have $\tilde{v} = A+B = -K/[2\gamma\sinh(\pi\gamma)]$.

So the first issue is to find the correct version of p 275 E for solution $\tilde{v}$. When I do it myself, I obtain these results:

$$A = -K e^{\pi\gamma} / [2\gamma \, sh(2\pi\gamma)]$$

$$B = -K e^{-\pi\gamma} / [2\gamma \, sh(2\pi\gamma)]$$

$$\tilde{v}(\gamma,\varphi|r_0,\pi) = -K \, ch[\gamma(\varphi-\pi)] / [\gamma \, sh(2\pi\gamma)] \qquad (2)$$

When I evaluate my solution for $\tilde{v}$ at $\varphi = 0$ and 2π, I get the correct RHS's of p 275 C and D. When I transform back to get $v(r,\varphi|r_0,\pi)$ using the KL transform, I get

$$v(r,\varphi|r_0,\pi) = -(1/\pi^2) \int_0^\infty d\gamma \, K_{i\gamma}(kr) \, K_{i\gamma}(kr_0) \, ch[\gamma(\varphi-\pi)] / ch(\gamma\pi) \qquad (3)$$

I cannot prove instantly that this is the correct result, but when I evaluate it at $\varphi = 0$ and 2π, I get

$$v(r,\varphi=0) = v(r,\varphi=2\pi) = -(1/\pi^2) \int_0^\infty d\gamma \, K_{i\gamma}(kr) \, K_{i\gamma}(kr_0)$$

and using 7.180, I see that this is -E on the two boundaries, in agreement with system 7.182. So I am fairly confident of my solutions (2) and (3) given above.

Now the solution to our knife problem is $g = E + v$ where E is given by 7.180 with $\varphi_0 = \pi$, so here is my solution to the knife problem:

$$g(r,\varphi|r_0,\pi) = E + v$$

$$= (1/\pi^2) \int_0^\infty d\gamma \, K_{i\gamma}(kr) \, K_{i\gamma}(kr_0) \, ch[\gamma(\pi - |\varphi-\pi|)]$$

$$-(1/\pi^2) \int_0^\infty d\gamma \, K_{i\gamma}(kr) \, K_{i\gamma}(kr_0) \, ch[\gamma(\varphi-\pi)] / ch(\gamma\pi) \qquad (4)$$

Again, I have no waypoint verification for this result, but I think it is right. There are two terms E and v.

Our problem now is to convert this result from an integral to a sum and we want that final sum to agree with 7.183. This is where my trouble lies!

First of all, is 7.183 itself correct? We know that our "knife edge" problem is the same problem as the "wedge problem" where we set the wedge angle to $\psi = 2\pi$. The wedge solution is 7.174 which I think is correct, and if we set $\psi = 2\pi$ in that formula, the sines become $\sin(m\varphi/2)\sin(m\pi/2)$ and indeed this gives 7.183, so I do think 7.183 is in fact correct. It turns out that $\sin(m\pi/2) = 0$ when m = even integers, so in reality the sum on 7.183 is only over the values m = 1,3,5…. -- the sum has only odd terms.

Now, we already have a sum formula for the E term in g shown above, this is given in 7.166 and I write that here, where I have added the usual modification for $\varphi_0 = \pi$.

$$E(r,\varphi|r_0,\pi) = (1/2\pi) \Sigma_n \, e^{in(\varphi-\pi)} I_n(kr_<)K_n(kr_>) = (1/2\pi) \Sigma_n \, (-1)^n e^{in\varphi} \, I_n(kr_<)K_n(kr_>)$$

In this formula, the sum is over ALL integers n. Later on I show that, for general α,

$$I_{-\alpha}K_{-\alpha} = I_\alpha K_\alpha + (2/\pi) \sin(\pi\alpha) \, K_\alpha K_\alpha \qquad (5)$$

something to check carefully later in this debug perhaps. When α = integer, $I_{-\alpha}K_{-\alpha} = I_\alpha K_\alpha$ and we get a simple reflection. [Note that when $\alpha = i\gamma$, we do NOT get simple reflection like this.] For integers, we can use this simple reflection rule to fold the E sum above onto the positive side only, and we then get (I think)

$$E(r,\varphi|r_0,\pi) = (1/\pi) \sum_{n=1}^{\infty} (-1)^n \cos(n\varphi)\, I_n(kr_<)K_n(kr_>) + (1/2\pi)\, I_0(kr_<)K_0(kr_>) \qquad (6)$$

and this is in a "final form" as far as comparison to our target sum 7.183.

So all we need to do is convert the v integral shown in (3) above into a sum, and then add that sum to (6), and hope we end up with 7.183. Here is the integral I want to "convert" to a sum.

$$v(r,\varphi|r_0,\pi) = -(1/\pi^2) \int_0^{\infty} d\gamma\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, ch[\gamma(\varphi-\pi)] / ch(\gamma\pi) \qquad (3)$$

Digression to Volume I page 317 Exercise 4.30

This exercise seems to contain the key information I need to do the sum conversion above, but things are pretty confusing since the parameters have swapped names. If we stick with the page 317 names for now, we are supposed to find the Green's Function g as shown in C for our certain Bessel's equation. Now it turns out that we later will find this same Bessel equation arises in the r coordinate when we do separation on the Helmholtz equation, but let's suppress that fact for now.

So how do we show that p 317 C is the correct Green's Function for system B ? I never really did this in either the Ch4 or Ch7 notes. Recall that the Green's function is constructed from homo solutions, one on the left and one on the right. So our first order of business is to find the homo solutions of A, and this is just some ODE data.

Problem A: Consider this ODE

$$-(xw')' + k^2xw - \gamma^2 w/x = 0 \qquad \mu = k^2 \qquad \lambda = \gamma^2$$

Show that a solution to this equation is given by

$$w = K_{\pm i\gamma}(kx)$$

As a first step, let's define

$$\nu^2 = -\gamma^2 = -\lambda \qquad \Rightarrow \qquad \nu = \sqrt{-\gamma^2} = \pm i\gamma$$

So our problem now is to show that

$$w(x) = K_\nu(kx) \qquad \text{solves} \qquad -(xw')' + k^2xw + \nu^2 w/x = 0$$

As a first step, let's change from coordinate x, to coordinate z = kx. We then have

$$d/dx = d/dz * dz/dx = d/dz * k = k\, d/dz \qquad \Rightarrow \qquad \partial_x = k\, \partial_z$$

Let's also define

$W(z) \equiv w(x(z)) = w(z/k)$ // => $w(x) = W(kx)$

Then we compute all the little pieces as follows:

$w'(x) = dw(x)/dx = \partial_{\mathbf{x}} w(x) = k\partial_{\mathbf{z}} W(z) = k\, W'$

$xw' = (z/k)k\, W' = zW'$

$(xw')' = \partial_{\mathbf{x}} (zW') = k\, \partial_{\mathbf{z}}(zW') = k\, (zW')'$

We may then transform our ODE and the new ODE is:

$- k\, (zW')' + k^2\, (z/k)W + \nu^2\, W\, (k/z) = 0$
$- k\, (zW')' + k\, zW + \nu^2\, W\, (k/z) = 0$
$- (zW')' + zW + \nu^2\, W\, /z = 0$
$-zW'' - W' + zW + \nu^2 W/z = 0$
$-z^2W'' - zW' + z^2W + \nu^2W = 0$
$z^2W'' + zW' - z^2W - \nu^2W = 0$
$z^2W'' + zW' - (z^2 + \nu^2)W = 0$

and we see that k is completely removed. If we look at AS2010 p 248, we see this data

Modified Bessel Functions

10.25 Definitions

10.25(i) Modified Bessel's Equation

10.25.1 $$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} - (z^2 + \nu^2)w = 0.$$

10.25(ii) Standard Solutions

10.25.2 $$I_\nu(z) = (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!\,\Gamma(\nu + k + 1)}.$$

and similarly for $K_\nu(z)$

Therefore, we have shown that $K_\nu(z)$ and $I_\nu(z)$ are solutions to our ODE. We then convert backwards

$W(z) \equiv w(x(z)) = w(z/k)$ => $w(x) = W(kx) = K_\nu(kx)$

so we know that $K_\nu(kx)$ is a solution of our original ODE. If we then take $\nu^2 = -\gamma^2$ as above, we find that in fact $K_{\pm i\gamma}(kx)$ are solutions.

Problem B: Show that the correct Green's Function is that shown page 317 C.

Our only task is to show that I as shown is finite at x = 0, and K is finite at x = ∞. I notice that AS2010 have a special redefinition of the I and K functions when ν is imaginary, and this makes the reflection formula for the new I be the same as it is for the new K, but let's NOT use these specially defined functions and stick with the traditional I and K functions. Then at x = 0 we have

10.30 Limiting Forms

10.30(i) $z \to 0$

When ν is fixed and $z \to 0$,

10.30.1 $$I_\nu(z) \sim (\tfrac{1}{2}z)^\nu / \Gamma(\nu+1), \quad \nu \neq -1, -2, -3, \ldots,$$

10.30.2 $$K_\nu(z) \sim \tfrac{1}{2}\Gamma(\nu)(\tfrac{1}{2}z)^{-\nu}, \qquad \Re\nu > 0,$$

10.30.3 $$K_0(z) \sim -\ln z.$$

In the case that $\nu = \pm i\gamma$ we get $I_{i\gamma}(z) \to z^{\pm i\gamma} = e^{\pm i\gamma \ln(z)}$. This limit is undefined really land the same for the K limit, so we have to make some assumption about ν. Suppose we assume that

$\nu = \pm i\gamma + \varepsilon$ where ε is real and small and we will later take $\varepsilon \to 0$

Then we have

$$z^\nu = z^{\pm i\gamma} z^\varepsilon = e^{\pm i\gamma \ln(z)} e^{\varepsilon \ln(z)}$$

Now if we first take $z \to 0$ and THEN take $\varepsilon \to 0$, we get $z^\nu \to 0$ in the first step. So with this in mind, we can conclude that $I_{\pm i\gamma}(z) \to 0$ as $z \to 0$ and this is appropriate for the "left" homo solution. If we change signs, we can then say

$$z^{-\nu} = e^{\pm i(-\gamma)\ln(z)} e^{-\varepsilon \ln(z)}$$

and then when $z \to \infty$, $K_{\pm i\gamma}(z) \to 0$.

To summarize, we have found appropriate candidate functions for the "left" and "right" side of our Green's function. Our Green's ODE is this

$-(xw')' + k^2xw - \gamma^2 w/x = \delta(x-\xi)$ $\mu = k^2$ $\lambda = \gamma^2$ // Green's

$-(xw')' + k^2xw - \gamma^2 w/x = 0$ $\mu = k^2$ $\lambda = \gamma^2$ // homo

and these are our candidate solutions

$$g = A\ I_{\pm i\gamma}(kx_<)K_{\pm i\gamma}(kx_>)$$

with either choice of sign. In my notes section (c) in Ch7raw2 I show that A = 1 provided, and I also show that the orders on the K and K function must be the same to use the Wronskian to show A = 1. So it would seem that we have at least two choices here:

$$g_1 = I_{i\gamma}(kx_<)\ K_{i\gamma}(kx_>)$$
$$g_2 = I_{-i\gamma}(kx_<)\ K_{-i\gamma}(kx_>)$$

Moreover, we know that $K_\alpha(z) = K_{-\alpha}(z)$ so we can add two more solutions:

$$g_3 = I_{i\gamma}(kx_<)\ K_{-i\gamma}(kx_>)$$
$$g_4 = I_{-i\gamma}(kx_<)\ K_{+i\gamma}(kx_>)$$

Now here is my question: We can see that $g_1 = g_3$ and $g_2 = g_4$ from the symmetry just noted. But the question is: are g_1 and g_2 the same, or are they different? I argue that they are different as follows:

$$g_2 = g_4 = I_{-i\gamma}(kx_<)\ K_{+i\gamma}(kx_>) = [\ I_{i\gamma}(kx_<) + (2/\pi)\sin(i\gamma\pi)\ K_{i\gamma}(kx_<)\ ]\ K_{+i\gamma}(kx_>)$$

$$= g_1 + i(2/\pi)\mathrm{sh}(\gamma\pi)\ K_{i\gamma}(kx_<)\ K_{+i\gamma}(kx_>)$$

So g_1 and g_2 are not the same because they differ by this quantity:

$$g_2 - g_1 = i(2/\pi)\mathrm{sh}(\gamma\pi)\ K_{i\gamma}(kx_<)\ K_{+i\gamma}(kx_>)$$

So if these two g's are really different, how can that be since I thought the Green's function was unique? But maybe a fundy Green's function is NOT unique! For example, E = 1/4πr is the fundy for $\nabla^2 E = \delta(\mathbf{r})$ in 3D, but $E_1 = 1/4\pi r + 3.6$ is also a fundy solution. If we insist that E→0 on the boundary at r=∞, then perhaps 1/4πr is the unique fundy solution.

Now what about a solution like g_1 above? We know from AS that

The defining property of the second standard solution $K_\nu(z)$ of (10.25.1) is

10.25.3 $$K_\nu(z) \sim \sqrt{\pi/(2z)}e^{-z},$$

so this decay is independent of the value of ν. So both g_1 and g_2 decay this way at r→∞. The difference also decays since it is KK.

If we look at the Green's ODE $-(xw')' + k^2xw - \gamma^2 w/x = \delta(x-\xi)$, it seems pretty clear that if you find a solution $f(x|\xi; k,\gamma)$, then $f(x|\xi; k,-\gamma)$ must also be a solution. Consider $w''+k^2w = 0$. If $w = e^{ikx}$ is a solution, then e^{-ikx} is also a solution by this same argument. That does not mean the two solutions thus obtained are the SAME solution. Certainly $e^{ikx} \neq e^{-ikx}$ in the same sense that $g_1 \neq g_2$.

So I arrive at a different conclusion than the one I reached earlier in the raw notes. My conclusion now is that

$g_1 = I_{i\gamma}(kx_<) K_{i\gamma}(kx_>)$
$g_2 = I_{-i\gamma}(kx_<) K_{-i\gamma}(kx_>)$

are each viable fundy Green's function solutions to $-(xw')' + k^2xw - \gamma^2 w/x = \delta(x-\xi)$, though they are not the same solution. In those other notes I conjectured that these were the same solution.

Side Question: you might try to argue that since everything in the ODE is real, then g_1 and g_2 must each be real. But that is a bogus argument. Consider again $w''+k^2w = 0$, an equation where everything is real. we can consider solutions $e^{\pm ikx}$ which are not real. So solutions need not be real (though real solutions might be found).

I turns out that $K_\nu(x)$ is real (for real x) when ν is real or when ν is pure imaginary. This follows from the following integral representation from AS p252, claimed valid for Re(ν) < 1 and x > 0:

$$K_\nu(x) = \sec\left(\tfrac{1}{2}\nu\pi\right) \int_0^\infty \cos(x \sinh t) \cosh(\nu t)\, dt$$

When $\nu = i\gamma$, we have $\sec(\pi\nu/2)\cosh(\nu t) = \sec(\pi i\gamma/2)\cosh(i\gamma\nu t) = \operatorname{sech}(\pi\gamma/2)\cos(\gamma\nu t)$ = stays real. This is a strange integral in this sense: for large t, factor ch(νt) is blowing up as $e^{\nu t}$, but the other factor is $\cos(xe^t)$ and is oscillating exponentially fast! Convergence of the integral is not obvious but possible.

I think $I_\nu(x)$ is in general complex for real x and imaginary ν. This is why AS on page 261 define a modified $I_{i\gamma}$ function $\mathfrak{I}_\nu \equiv \mathrm{Re}(I_{i\nu})$ to go along with $\mathfrak{K}_\nu \equiv K_{i\nu}$.

So my conclusion is that g_1 and g_2 as shown above are both complex.

So back to Exercise 4.30. If we take $g = g_2$ as shown p 317 C, I have shown that you end up with 4.137 and the "usual form" of the KL transform. See section (c) in the Ch7raw2 notes.

Back to our problem of converting ν to a sum

$$v(r,\varphi|r_0,\pi) = -(1/\pi^2) \int_0^\infty d\gamma\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, \mathrm{ch}[\gamma(\varphi-\pi)] / \mathrm{ch}(\gamma\pi) \qquad (3)$$

In earlier notes I argued that I could write KK as disc[IK], but I now think that is incorrect. The disc in that sense was in λ, not in γ, so I will try not a different approach.

The integrand in (3) is manifestly even in γ, so we can write the integral instead as

$$v(r,\varphi|r_0,\pi) = -(1/2\pi^2) \int_{-\infty}^\infty d\gamma\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, \mathrm{ch}[\gamma(\varphi-\pi)] / \mathrm{ch}(\gamma\pi)$$

We would now like to close the contour up or down and pick up poles of ch(γπ) along the real axis. We can see that in general the ch/ch factor is oscillatory up and down, so we hope to get GC convergence from the K functions. I am not sure what these do, but I can quote at least something from AS

10.41 Asymptotic Expansions for Large Order

10.41(i) Asymptotic Forms

If $\nu \to \infty$ through positive real values with $z(\neq 0)$ fixed, then

10.41.1 $$I_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ez}{2\nu}\right)^\nu,$$

10.41.2 $$K_\nu(z) \sim \sqrt{\frac{\pi}{2\nu}} \left(\frac{ez}{2\nu}\right)^{-\nu}.$$

As refer to a 1990 paper by Dunster that I cannot access the paper. Let's just assume that the above are roughly correct, so we then have

$$K_{i\gamma}(z) \sim (i\gamma)^{-1/2}\ (ez/2i\gamma)^{-i\gamma} \quad \sim \quad \gamma^{-1/2}\,(-iez/2)^{-i\gamma}\,\gamma^{-\gamma}$$

If γ goes large *positive*, the $\gamma^{-\gamma}$ factor is pretty convergent! But not so if γ goes large and negative. Thus, we have to close our contour upwards! Also, this helps explain why integrals like 7.179 converge. And of course our expression for v shown above. So close it upwards and we then have to deal with the pole residues.

Zeros of ch(γπ)

The zeros are located here

$$ch(\pi\gamma_n) = \cos(i\pi\gamma_n) = 0 \qquad \Rightarrow \qquad i\pi\gamma_n = -n(\pi/2) \qquad n = \pm1, \pm3, \pm5 \ldots$$

Our contour is going to wrap the poles of 1/ch(πγ) in the upper half plane in the correct sense.

So the poles are located at half integers *and* integers on the imaginary axis:

$$\gamma_n = -n(\pi/2)/(\pi i) = ni/2 \qquad -i\gamma_n = n/2 \qquad i\gamma_n = -n/2$$

Near one of these poles we have $\gamma = \gamma_n + \delta\gamma = ni/2 + \delta\gamma$ so

$$f(\gamma) = f(\gamma_n) + \delta\gamma\, f'(\gamma_n) \qquad f(\gamma) = ch(\pi\gamma) \quad \Rightarrow \quad f'(\gamma) = \pi\, sh(\pi\gamma)$$

$$ch(\pi\gamma) = ch(\pi\gamma_n) + \delta\gamma\,\pi\, sh(\pi\gamma_n) = \delta\gamma\,\pi\, sh(\pi\gamma_n) = \delta\gamma\,\pi\, sh(\pi\, ni/2) = \delta\gamma\,\pi\, i\, \sin(\pi n/2)$$

$$= \delta\gamma\,\pi\, i\,(-1)^{(n-1)/2} = \delta\gamma\,\pi\, i / \sin(\pi n/2) \qquad n = \pm1, \pm3, \pm5 \ldots$$

Notice that with this restricted range of values of n, we can put the sin(nπ/2) up or down, it makes no difference. So let's put it down, knowing where we are headed.

Therefore we have laboriously shown that

$$\mathrm{ch}(\pi\gamma) \;\approx\; i\pi\,(\gamma - \gamma_n) / \sin(\pi n/2) \qquad \text{for } \gamma \text{ near the pole position } \gamma_n$$

Therefore, our residue will pick up a factor of $\sin(n\pi/2)/(i\pi)$ from this little coefficient. So here we go:

$$v(r,\varphi|r_0,\pi) = -(1/2\pi^2) \int_{-\infty}^{\infty} d\gamma\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, \mathrm{ch}[\gamma(\varphi-\pi)] / \mathrm{ch}(\gamma\pi)$$

$$= -(1/2\pi^2)\, 2\pi i * (2\pi i)^{-1} \int_C d\gamma\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, \mathrm{ch}[\gamma(\varphi-\pi)] / \mathrm{ch}(\gamma\pi)$$

where C captures all the upper half plane poles just mentioned in the correct sense

$$= -(1/2\pi^2)\, 2\pi i \sum_{n=1,3,5..}^{\infty} K_{-n/2}(kr)\, K_{-n/2}(kr_0)\, \mathrm{ch}[in(\varphi-\pi)/2\,] * \sin(n\pi/2)/(i\pi)$$

Now we use the symmetry of the K in order, and change ch to cos to get

$$= -(1/\pi^2) \sum_{n=1,3,5..}^{\infty} K_{n/2}(kr)\, K_{n/2}(kr_0) \cos[n(\varphi-\pi)/2\,] * \sin(n\pi/2)$$

Review to this point

From earlier above we found that

$$E(r,\varphi|r_0,\pi) = (1/\pi) \sum_{n=1}^{\infty} (-1)^n \cos(n\varphi)\, I_n(kr_<)K_n(kr_>) + (1/2\pi)\, I_0(kr_<)K_0(kr_>) \qquad (6)$$

In this sum let's replace n = m/2 or m = 2n and write this as

$$E(r,\varphi|r_0,\pi) = (1/\pi) \sum_{m=2,4..}^{\infty} (-1)^{m/2} \cos(m\varphi/2)\, I_{m/2}(kr_<)K_{m/2}(kr_>) + (1/2\pi)\, I_0(kr_<)K_0(kr_>)$$

or

$$E(r,\varphi|r_0,\pi) = (1/\pi) \sum_{n=2,4..}^{\infty} (-1)^{n/2} \cos(n\varphi/2)\, I_{n/2}(kr_<)K_{n/2}(kr_>) + (1/2\pi)\, I_0(kr_<)K_0(kr_>)$$

And we have now just purportedly shown that

$$v(r,\varphi|r_0,\pi) = -(1/\pi^2) \sum_{n=1,3,5..}^{\infty} K_{n/2}(kr)\, K_{n/2}(kr_0) \cos[n(\varphi-\pi)/2\,] * \sin(n\pi/2)$$

If we add these two animals, we are *supposed* to get 7.183. Obviously I have failed now a second time to get anywhere close, another day is now consumed. The most glaring problem is that this v thing has KK instead of IK. So maybe we are not allowed to "close up" as I assumed.

Can we somehow replace the first K? Consider again this fact from Stak I p 317 D

$$I_{-\alpha} - I_\alpha = (2/\pi) \sin(\pi\alpha)\, K_\alpha$$

$$I_{-n/2} - I_{n/2} = (2/\pi) \sin(n\pi/2) K_{n/2}$$

If we were to replace the first K from this formula and keep only the $I_{n/2}$ term for the moment, we would get

$$v(r,\varphi|r_0,\pi)_{part} = -(1/\pi^2)\ \Sigma_{n=1,3,5..}^{\infty}\ [-I_{n/2}(kr)\ (\pi/2\ \sin(n\pi/2)]K_{n/2}(kr_0)\ \cos[n(\varphi-\pi)/2\ ]\ *\ \sin(n\pi/2)$$

$$= +(1/2\pi)\ \Sigma_{n=1,3,5..}^{\infty}\ I_{n/2}(kr)K_{n/2}(kr_0)\ \cos[n(\varphi-\pi)/2\ ]$$

Meanwhile we have (remember that n = 1,3,5….only)

$$\cos[n(\varphi-\pi)/2\ ] = \cos(n\varphi/2)\cos(n\pi/2) + \sin(n\varphi/2)\sin(n\pi/2) = \sin(n\varphi/2)\sin(n\pi/2)$$

so we then have

$$v(r,\varphi|r_0,\pi)_{part} = +(1/2\pi)\ \Sigma_{n=1,3,5..}^{\infty}\ I_{n/2}(kr)K_{n/2}(kr_0)\ \sin(n\varphi/2)\sin(n\pi/2)$$

THIS is much closer to our desired result. Apart from a factor of 2, this IS 7.183 ! So we are dancing around the result somehow, but I have goofed up and have extra pieces that make the result wrong.

Plan C. Consider again the fact that

$$I_{-\alpha} - I_{\alpha} = (2/\pi)\ \sin(\pi\alpha)\ K_{\alpha}$$

and apply this with $\alpha = -i\gamma$ (and use the symmetry of K in order)

$$I_{i\gamma} - I_{-i\gamma} = (2/\pi)\ \sin(-\pi i\gamma)K_{i\gamma} = -(2i/\pi)\ \mathrm{sh}(\pi\gamma)K_{i\gamma}$$

Resume Sunday 7.10.11

Let's <u>try changing from the γ plane to the λ plane</u> where, say,

$$\gamma = -\sqrt{\lambda} \qquad\qquad i\gamma = -i\sqrt{\lambda}$$

and we go back to our friend

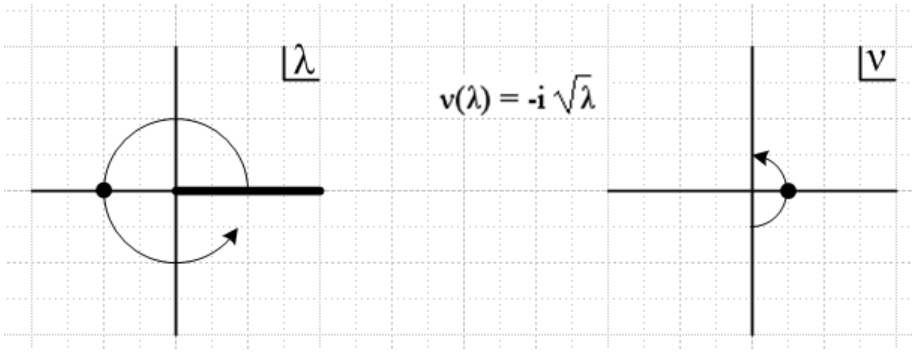
$$v(r,\varphi|r_0,\pi) = -(1/\pi^2)\ \int_0^{\infty}\ d\gamma\ K_{i\gamma}(kr)\ K_{i\gamma}(kr_0)\ \mathrm{ch}[\gamma(\varphi-\pi)]\ /\ \mathrm{ch}(\gamma\pi) \qquad (3)$$

Now looking at previous ch7raw2 notes, we know that if we define

$$h \equiv I_{-i\sqrt{\lambda}}(kx_<)\ K_{-i\sqrt{\lambda}}(kx_>)$$

Then if we then define $\nu(\lambda) = -i\sqrt{\lambda}$, I quote:

"in order to have $\nu > 0$ we need $\sqrt{\lambda} = i$ and that means we need $\lambda = -1$, and that means we need to take the cut on the *positive* λ axis. Here then is our picture set: "



In other words, in order for $\nu > 0$ to be in a clean region of ν space, we need $\lambda = -1$ to be in a clean region of λ space, so the cut cannot be on the negative λ axis which is where we usually put it, hence the above picture. Continuing with those notes: [we are defining the angle of λ not to be in the $(0,2\pi$ range), and we are talking about the discontinuity across the positive real λ axis]

$$\text{disc}_{\lambda}[h] = h(\lambda = \lambda e^{i0}) - h(\lambda = \lambda e^{2i\pi})$$

$$= h(\sqrt{\lambda} = \sqrt{\lambda}) - h(\sqrt{\lambda} = -\sqrt{\lambda})$$

$$= I_{-i\sqrt{\lambda}}(\sqrt{\mu}\, x_<)\; K_{-i\sqrt{\lambda}}(\sqrt{\mu}\, x_>) - I_{i\sqrt{\lambda}}(\sqrt{\mu}\, x_<)\; K_{i\sqrt{\lambda}}(\sqrt{\mu}\, x_>) \qquad (**)$$

Now for shorthand, let $\alpha = i\sqrt{\lambda}$ so we have

$$= I_{-\alpha} K_{-\alpha} - I_{\alpha} K_{\alpha}$$

$$= [I_{\alpha} + (2/\pi)\sin(\pi\alpha)K_{\alpha}]\, K_{\alpha} - I_{\alpha} K_{\alpha} \qquad \text{// using p 317 D which I assume is correct}$$

$$= I_{\alpha} K_{\alpha} + (2/\pi)\sin(\pi\alpha)K_{\alpha}K_{\alpha} - I_{\alpha} K_{\alpha}$$

$$= (2/\pi)\sin(\pi\alpha)K_{\alpha}K_{\alpha}$$

So we have then shown that

$$\text{disc}_{\lambda}[h] = (2/\pi)\sin(\pi\alpha)K_{\alpha}K_{\alpha} \quad \text{where} \quad \alpha = i\sqrt{\lambda} = -i\gamma \quad \text{since} \quad i\gamma = -i\sqrt{\lambda}$$

$$= (2/\pi)\sin(\pi[-i\gamma])K_{-i\gamma}K_{-i\gamma}$$

$$= -(2/\pi)\sin(i\pi\gamma)\, K_{i\gamma}K_{i\gamma}$$

$$= -(i2/\pi)\, \text{sh}(\pi\gamma)\, K_{i\gamma}K_{i\gamma}$$

$$= (2/i\pi)\ \mathrm{sh}(\pi\gamma)\ K_{i\gamma}K_{i\gamma}$$

So let's restate this result in full detail

$$\mathrm{disk}_{\lambda}[I_{i\gamma}(kx_<)K_{i\gamma}(kx_>)\ ] = (2/i\pi)\ \mathrm{sh}(\pi\gamma)\ K_{i\gamma}(kx_<)K_{i\gamma}(kx_>) \qquad \gamma = -\sqrt{\lambda} \qquad \gamma^2 = \lambda$$

So this holds out the Big Hope of replacing KK with IK. Back again to Old Friend,

$$v(r,\varphi|r_0,\pi) = -(1/\pi^2)\ \int_0^\infty d\gamma\ K_{i\gamma}(kr)\ K_{i\gamma}(kr_0)\ \mathrm{ch}[\gamma(\varphi-\pi)] / \mathrm{ch}(\gamma\pi) \qquad (3)$$

Now try to change this integration from γ to λ where $\gamma = -\sqrt{\lambda}$ so $\sqrt{\lambda} = -\gamma$. We can see first of all that the range of γ being $(0,\infty)$ becomes what? The 0 end becomes 0. Then we need $\sqrt{\lambda} = -\infty$. It we take $\lambda = +\infty$, the two possible roots are $\sqrt{\lambda} = \pm\infty$, so we are on one of two branches of the $\sqrt{\lambda}$ function. This makes sense since we have shown the cut being to the right above in the λ plane. So our range keeps the same numbers. Then we have $\gamma^2 = \lambda$ so $2\gamma d\gamma = d\lambda$ so we then have

$$v(r,\varphi|r_0,\pi) = -(1/\pi^2)\ \int_0^\infty d\lambda / [2\gamma]\ K_{i\gamma}(kr)\ K_{i\gamma}(kr_0)\ \mathrm{ch}[\gamma(\varphi-\pi)] / \mathrm{ch}(\gamma\pi) \qquad \gamma = -\sqrt{\lambda}$$

$$= -(1/2\pi^2)\ \int_0^\infty d\lambda\ K_{i\gamma}(kr)\ K_{i\gamma}(kr_0)\ \mathrm{ch}[\gamma(\varphi-\pi)] / [\gamma\ \mathrm{ch}(\gamma\pi)]$$

where γ is just a shorthand for $-\sqrt{\lambda}$. But now I see a problem brewing. I would like to think that I can replace KK by disk[h], more or less, but that γ sitting in the denominator also has a cut! I will then want to say that

$$K_{i\gamma}(kr)\ K_{i\gamma}(kr_0)/\gamma = \mathrm{disc}_{\lambda}[\text{something}]$$

What could that "something" be? We could try something = h/γ and see what happens. But I can see that this removes the relative minus sign between our two terms in (**) above, and then things don't work!! If I could come up with an extra numerator γ factor in v, things might be better.

Pause to check on the boundary conditions starting with 7.182.

I have been assuming that C and D are correct, but perhaps this needs a closer look, since we are now going on 3-4 days on this problem, we need to turn over more stones. Here is what the BC's are

$$v(r,\varphi=0|r_0,\pi) = -E(r,\varphi=0|r_0,\pi)$$
$$v(r,\varphi=2\pi|r_0,\pi) = -E(r,\varphi=2\pi|r_0,\pi)$$

Now what do these BC's look like in the KL γ space. I have naively assumed we just add twiddles, but maybe that is wrong. We know from the KL (such as 7.178) that

$$\tilde{v}(\gamma,\varphi=0|r_0,\pi) = \int_0^\infty dr\, K_{i\gamma}(kr)\, v(\gamma,\varphi=0|r_0,\pi)/r$$

$$= - \int_0^\infty dr\, K_{i\gamma}(kr)\, E(r,\varphi=0|r_0,\pi)/r$$

$$= - \tilde{E}(\gamma,\varphi=0|r_0,\pi)$$

So this confirms that the BC's in the two spaces have the same form, you just add twiddles. In other words, this confirms the left equalities in C and D.

Pause to recheck p 273 C.

Assume C is correct, let's show that it solves the ODE and the BC's shown in A and B. First, stay away from $\varphi = 0$ and just make sure we solve the homo equation

$$\tilde{E}(\gamma,\varphi=0|r_0,0) = [\, 2\gamma\, sh(\gamma\pi)]^{-1} ch[\gamma(\pi-|\varphi|)]\, K$$

$$\partial_\varphi \tilde{E}(\gamma,\varphi=0|r_0,0) = [\, 2\gamma\, sh(\gamma\pi)]^{-1} \partial_\varphi \{ ch[\gamma(\pi-|\varphi|)] \}\, K$$

$$\partial_\varphi \{ ch[\gamma(\pi-|\varphi|)] \} = sh[\gamma(\pi-|\varphi|)]\, (-\gamma) \partial_\varphi |\varphi| = -\gamma\, sh[\gamma(\pi-|\varphi|)]\, sign(\varphi)$$

$$\partial_\varphi^2 \{ ch[\gamma(\pi-|\varphi|)] \} = \partial_\varphi [-\gamma\, sh[\gamma(\pi-|\varphi|)]\, sign(\varphi)]$$

$$= sign(\varphi)\, (-\gamma)\, \partial_\varphi \{ sh[\gamma(\pi-|\varphi|)] \}$$

$$= sign(\varphi)\, (-\gamma)\, ch[\gamma(\pi-|\varphi|)]\, (-\gamma) \partial_\varphi |\varphi|$$

$$= sign(\varphi)\, (-\gamma)\, ch[\gamma(\pi-|\varphi|)]\, (-\gamma)\, sign(\varphi)$$

$$= \gamma^2\, ch[\gamma(\pi-|\varphi|)]$$

Then we have shown that

$$\partial_\varphi^2 \tilde{E}(\gamma,\varphi=0|r_0,0) = [\, 2\gamma\, sh(\gamma\pi)]^{-1}\, \partial_\varphi^2 \{ ch[\gamma(\pi-|\varphi|)] \}\, K$$

$$= [\, 2\gamma\, sh(\gamma\pi)]^{-1}\, \gamma^2\, ch[\gamma(\pi-|\varphi|)]\, K$$

$$= \gamma^2\, \tilde{E}(\gamma,\varphi=0|r_0,0)$$

and so the homo equation is correctly solved by C (as modified by me in pencil).

Second, we have to verify the jump condition. If we integrate A from $\varphi = -\varepsilon$ to $+\varphi$ we get

$(-\partial_\varphi \widetilde{E})|^{\varepsilon}_{-\varepsilon} = K$ or $(\partial_\varphi \widetilde{E})|^{\varepsilon}_{-\varepsilon} = -K$

From above we know that

$$\partial_\varphi \widetilde{E} = [\, 2\gamma\, sh(\gamma\pi)]^{-1} \partial_\varphi \{ ch[\gamma(\pi - |\varphi|)] \}\, K = [\, 2\gamma\, sh(\gamma\pi)]^{-1} \{ -\gamma\, sh[\gamma(\pi - |\varphi|)]\, sign(\varphi) \}\, K$$

Therefore we have

$$(\partial_\varphi \widetilde{E})|^{\varepsilon}_{-\varepsilon} = [\, 2\gamma\, sh(\gamma\pi)]^{-1} \{ -\gamma\, sh[\gamma(\pi - |0|)]\, (+1) \}\, K$$

$$- [\, 2\gamma\, sh(\gamma\pi)]^{-1} \{ -\gamma\, sh[\gamma(\pi - |0|)]\, (-1) \}\, K$$

$$= -(1/2)K - (1/2)K = -K$$

so all aspects of p 273 seem correct (point source is at $\varphi = 0$).

The modification $\varphi \rightarrow \varphi - \varphi_0$ seems correct, so we should then have

$$\widetilde{E}(\gamma, \varphi \,|r_0, \varphi_0) = [\, 2\gamma\, sh(\gamma\pi)]^{-1} ch[\gamma(\pi - |\varphi - \varphi_0|)]\, K$$

and then finally for our application

$$\widetilde{E}(\gamma, \varphi | r_0, \pi) = [\, 2\gamma\, sh(\gamma\pi)]^{-1} ch[\gamma(\pi - |\varphi - \pi|)]\, K$$

We can then evaluate our two boundary conditions using this result

$$\tilde{v}(\gamma, \varphi = 0 | r_0, \pi) = -\, \widetilde{E}(\gamma, \varphi = 0 | r_0, \pi) = -[\, 2\gamma\, sh(\gamma\pi)]^{-1} ch[\gamma(\pi - |0 - \pi|)]\, K = -[\, 2\gamma\, sh(\gamma\pi)]^{-1} K$$

$$\tilde{v}(\gamma, \varphi = 2\pi | r_0, \pi) = -\, \widetilde{E}(\gamma, \varphi = 2\pi | r_0, \pi) = -[\, 2\gamma\, sh(\gamma\pi)]^{-1} ch[\gamma(\pi - |2\pi - \pi|)]\, K = -[\, 2\gamma\, sh(\gamma\pi)]^{-1} K$$

and thus we have verified the two right equalities in C and D.

Pause to recheck the solutions for A and B

$$A + B = -[\, 2\gamma\, sh(\gamma\pi)]^{-1} K$$

$$A e^{-2\gamma\pi} + B e^{2\gamma\pi} = -[\, 2\gamma\, sh(\gamma\pi)]^{-1} K$$

Multiply first equation by $e^{-2\gamma\pi}$

$$A\, e^{-2\gamma\pi} + B\, e^{-2\gamma\pi} = -[\, 2\gamma\, sh(\gamma\pi)]^{-1}\, e^{-2\gamma\pi} K$$

$$Ae^{-2\gamma\pi} + Be^{2\gamma\pi} = -[2\gamma\, sh(\gamma\pi)]^{-1} K$$

Subtract these last two equations:

$$B(e^{-2\gamma\pi} - e^{2\gamma\pi}) = -[2\gamma\, sh(\gamma\pi)]^{-1}K(e^{-2\gamma\pi} - 1) = -[2\gamma\, sh(\gamma\pi)]^{-1}K\, e^{-\gamma\pi}(e^{-\gamma\pi} - e^{\gamma\pi})$$

Convert to sinh functions

$$B[-2sh(2\gamma\pi)] = -[2\gamma\, sh(\gamma\pi)]^{-1}K(e^{-2\gamma\pi} - 1) = -[2\gamma\, sh(\gamma\pi)]^{-1}K\, e^{-\gamma\pi}[-2sh(\gamma\pi)]$$

$$= [\gamma]^{-1}K\, e^{-\gamma\pi} = K\, e^{-\gamma\pi}/\gamma$$

$$B = [K\, e^{-\gamma\pi}/\gamma] / [-2sh(2\gamma\pi)] = -K\, e^{-\gamma\pi} / [2\gamma\, sh(2\gamma\pi)]$$ // agrees with earlier

Now doing A↔B is the same as doing γ↔-γ, so we know that

$$A = -K\, e^{\gamma\pi} / [2\gamma\, sh(2\gamma\pi)]$$

and therefore we obtain our previous results,

$$A = -K\, e^{+\gamma\pi} / [2\gamma\, sh(2\gamma\pi)]$$
$$B = -K\, e^{-\gamma\pi} / [2\gamma\, sh(2\gamma\pi)]$$

Now install these into p 275 B to get

$$\tilde{v}(\gamma,\varphi|r_0,\pi) = A\, e^{-\gamma\varphi} + B\, e^{+\gamma\varphi}$$

$$= -[e^{+\gamma\pi}\, e^{-\gamma\varphi} + e^{-\gamma\pi}\, e^{+\gamma\varphi}]\, K/[2\gamma\, sh(2\gamma\pi)]$$

$$= -[2\, ch[\gamma(\pi-\varphi)]]\, K/[2\gamma\, sh(2\gamma\pi)] = -ch[\gamma(\pi-\varphi)]\, K/[\gamma\, sh(2\gamma\pi)]$$

which is to say

$$\tilde{v}(\gamma,\varphi|r_0,\pi) = -ch[\gamma(\pi-\varphi)]\, K/[\gamma\, sh(2\gamma\pi)]$$ // also agrees with (2) above

So I have now explicitly double checked all three entries in our result above which I now quote:

$$A = -K\, e^{\pi\gamma}/[2\gamma\, sh(2\pi\gamma)]$$
$$B = -Ke^{-\pi\gamma}/[2\gamma\, sh(2\pi\gamma)]$$
$$\tilde{v}(\gamma,\varphi|r_0,\pi) = -K_{i\gamma}(kr_0)\, ch[\gamma(\varphi-\pi)] / [\gamma\, sh(2\pi\gamma)] \quad (2)$$

This last result is the corrected p 275 E and again I conclude that the Stak result is completely wrong. You just have to wonder, though, where he got that strange result?

Pause to recheck the resulting integral for v

We first quote 7.176

$$f(r) = (2/\pi^2) \int_0^\infty d\gamma \ \gamma \, sh(\pi\gamma) \, K_{i\gamma}(kr) \, \tilde{f}(\gamma)$$

which then tells us that

$$v(\gamma,\varphi|r_0,\pi) = (2/\pi^2) \int_0^\infty d\gamma \ \gamma \, sh(\pi\gamma) \, K_{i\gamma}(kr) \, \tilde{v}(\gamma,\varphi|r_0,\pi)$$

$$= (2/\pi^2) \int_0^\infty d\gamma \ \gamma \, sh(\pi\gamma) \, K_{i\gamma}(kr) \, \{- K_{i\gamma}(kr_0) \, ch[\gamma(\varphi-\pi)] / [\gamma \, sh(2\pi\gamma)]\}$$

$$= - (2/\pi^2) \int_0^\infty d\gamma \ sh(\pi\gamma) \, K_{i\gamma}(kr) \, \{K_{i\gamma}(kr_0) \, ch[\gamma(\varphi-\pi)] / [sh(2\pi\gamma)]\}$$

$$= - (2/\pi^2) \int_0^\infty d\gamma \ sh(\pi\gamma) \, K_{i\gamma}(kr) \, K_{i\gamma}(kr_0) \, ch[\gamma(\varphi-\pi)] / [2 \, sh(\pi\gamma) \, ch(\pi\gamma)]$$

$$= - (1/\pi^2) \int_0^\infty d\gamma \ K_{i\gamma}(kr) \, K_{i\gamma}(kr_0) \, ch[\gamma(\varphi-\pi)] / ch(\pi\gamma)$$

and this agrees exactly with (3) above. So at least I am sure my starting point is correct. We also know from 7.180 that

$$E(\gamma,\varphi|r_0,\pi) = (1/\pi^2) \int_0^\infty d\gamma \ K_{i\gamma}(kr) \, K_{i\gamma}(kr_0) \, ch[\gamma(|\varphi-\pi|-\pi)]$$

We can then summarize our results

$$E(\gamma,\varphi|r_0,\pi) = (1/\pi^2) \int_0^\infty d\gamma \ K_{i\gamma}(kr) \, K_{i\gamma}(kr_0) \, ch[\gamma(|\varphi-\pi|-\pi)]$$

$$v(\gamma,\varphi|r_0,\pi) = - (1/\pi^2) \int_0^\infty d\gamma \ K_{i\gamma}(kr) \, K_{i\gamma}(kr_0) \, ch[\gamma(\varphi-\pi)] / ch(\pi\gamma)$$

$$g(r,\varphi|r_0,\pi) = E(\gamma,\varphi|r_0,\pi) + v(\gamma,\varphi|r_0,\pi)$$

Thus we can write g as a single line this way

$$g(r,\varphi|r_0,\pi) = (1/\pi^2) \int_0^\infty d\gamma \ K_{i\gamma}(kr) \, K_{i\gamma}(kr_0) \ \{ \ ch[\gamma(|\varphi-\pi|-\pi)] \ - \ ch[\gamma(\varphi-\pi)] / ch(\pi\gamma) \ \}$$

It is this result that we somehow want to transform in to the sum 7/183, and I still have no idea how to do that! Stak says we do it "by contour integration". I am now going to start a separate debug document to get a clean start.