Math Problem PhL 7.10.11

The problem is to start with this

$$g(r,\varphi|r_0,\pi) = (1/\pi^2) \int_0^\infty d\gamma \; K_{i\gamma}(kr)\, K_{i\gamma}(kr_0) \; \{ \; ch[\gamma(|\varphi-\pi|-\pi)] \; - \; ch[\gamma(\varphi-\pi)] / ch(\pi\gamma) \; \}$$

and transform it into this

$$g(r,\varphi|r_0,\pi) = (1/\pi) \Sigma_{m=1}^{\infty} \sin(m\varphi/2)\sin(m\pi/2)\, I_{m/2}(kr_<)\, K_{m/2}(kr_>)$$

$$= (1/\pi) \Sigma_{m=1,3,5}^{\infty} \sin(m\varphi/2)\sin(m\pi/2)\, I_{m/2}(kr_<)\, K_{m/2}(kr_>)$$

This is a completely self-contained problem so we can forget about everything else on Stak p 275. We do have these basic facts concerning I and K for general complex α ,

$$I_{-\alpha} - I_\alpha = (2/\pi) \sin(\pi\alpha)\, K_\alpha \qquad\qquad K_{-\alpha} = K_\alpha$$

We can clearly see that the big factor {.... } is analytic in γ except for the poles of $1/ch(\gamma\pi)$.

I think $K_\nu(x)$ is analytic in ν everywhere, but perhaps that needs another look. Let's start with this series

$$I_\nu(z) = (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!\,\Gamma(\nu+k+1)}.$$

We can write $(z/2)^\nu = e^{\nu \ln(z/2)} = e^{a\nu}$ which function is analytic everywhere in ν !! The gamma function has some poles in ν, but they just become zeros in I. My conclusion is this:

$I_\nu(z)$ is analytic in ν everywhere

Now consider,

$$K_\nu(z) = \tfrac{1}{2}\pi \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}.$$

We conclude at once that $K_\nu(z)$ has no cuts in ν, but could have poles in ν at ν = integers, though the numerator might cancel these poles. We do know that $I_n = I_{-n}$ so indeed the poles are in fact cancelled since the formula above says K = 0/0 at these poles. I then conclude that

$K_\nu(z)$ is analytic in ν everywhere

I cannot find confirmation of these claims on the web or in my books or PDF's, but I have just proven that it is true.

Plan A. Can I write one of the K functions as a contour integral of an I function? That would at least cause a contour integral to appear in the problem. Did not pan out.

Status: I am starting with this integral

$$g(r,\varphi|r_0,\pi) = (1/\pi^2) \int_0^\infty d\gamma \; K_{i\gamma}(kr)\, K_{i\gamma}(kr_0) \; \{ \; ch[\gamma(|\varphi-\pi|-\pi)] \; - \; ch[\gamma(\varphi-\pi)] / ch(\pi\gamma) \; \}$$

and I know that the entire integrand is analytic in γ except for the poles due to $ch(\pi\gamma)$.

Plan B. Write the above in this way

$$g(r,\varphi|r_0,\pi) = (1/2\pi^2) \int_{-\infty}^\infty d\gamma \; K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, f(\gamma)$$

where

$$f(\gamma) = ch[\gamma(|\varphi-\pi|-\pi)] \; - \; ch[\gamma(\varphi-\pi)] / ch(\pi\gamma) \quad = \text{a function even in } \gamma.$$

Of course KK is also even in γ, that is why we can alter the integrand.

NOW, the next step is to replace one of the K's like this, using the AS fact quoted above

$$K_{i\gamma}(kr) = (\pi/2) [I_{-i\gamma}(kr) - I_{i\gamma}(kr)] / \sin(i\gamma\pi) = (\pi/2i) [I_{-i\gamma}(kr) - I_{i\gamma}(kr)] / sh(\gamma\pi)$$

$$= (i\pi/2) [I_{i\gamma}(kr) - I_{-i\gamma}(kr)] / sh(\gamma\pi)$$

We than have

$$g(r,\varphi|r_0,\pi) = (1/2\pi^2) \int_{-\infty}^\infty d\gamma \; \{(i\pi/2) [I_{i\gamma}(kr) - I_{-i\gamma}(kr)] / sh(\gamma\pi)\}\, K_{i\gamma}(kr_0)\, f(\gamma)$$

$$= (i/4\pi) \int_{-\infty}^\infty d\gamma \; \{ [I_{i\gamma}(kr) - I_{-i\gamma}(kr)] / sh(\gamma\pi)\}\, K_{i\gamma}(kr_0)\, f(\gamma)$$

$$= (i/4\pi) \int_{-\infty}^\infty d\gamma \; [I_{i\gamma}(kr) - I_{-i\gamma}(kr)] \; K_{i\gamma}(kr_0)\, f(\gamma)/ sh(\gamma\pi)$$

We know write this as the sum of two integrals so that $g = g_1+g_2$, where we use $K_{-i\gamma} = K_{i\gamma}$,

$$g_1 = (i/4\pi) \int_{-\infty}^\infty d\gamma \; I_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, f(\gamma)/ sh(\gamma\pi)$$

$$g_2 = - (i/4\pi) \int_{-\infty}^\infty d\gamma \; I_{-i\gamma}(kr)\, K_{-i\gamma}(kr_0)\, f(\gamma)/ sh(\gamma\pi)$$

Now in the second integral, replace $\gamma \rightarrow -\gamma$, use the fact that $f(\gamma)$ is even in γ, and we get

$$g_2 = (i/4\pi) \int_{-\infty}^{\infty} d\gamma\, I_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, f(\gamma)/\, sh(\gamma\pi) = g_1$$

So these two integrals are exactly the same !!!! So that is an easy way to "change a K into an I". We then have shown that either of the following is true:

$$g(r,\varphi|r_0,\pi) = (i/2\pi) \int_{-\infty}^{\infty} d\gamma\, I_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, f(\gamma)/\, sh(\gamma\pi)$$

$$g(r,\varphi|r_0,\pi) = -(i/2\pi) \int_{-\infty}^{\infty} d\gamma\, I_{-i\gamma}(kr)\, K_{-i\gamma}(kr_0)\, f(\gamma)/\, sh(\gamma\pi)$$

and we have taken finally a giant leap forward.

Note: At the very start, I could have written

$$K_{i\gamma}(kr)\, K_{i\gamma}(kr_0) = K_{i\gamma}(kr_<)\, K_{i\gamma}(kr_>)$$

Then I would have ended up with these results which are what we really will need below,

$$g(r,\varphi|r_0,\pi) = (i/2\pi) \int_{-\infty}^{\infty} d\gamma\, I_{i\gamma}(kr_<)\, K_{i\gamma}(kr_>)\, f(\gamma)/\, sh(\gamma\pi)$$

$$g(r,\varphi|r_0,\pi) = -(i/2\pi) \int_{-\infty}^{\infty} d\gamma\, I_{-i\gamma}(kr_<)\, K_{-i\gamma}(kr_>)\, f(\gamma)/\, sh(\gamma\pi)$$

NOW maybe we can close top or bottom! First, if we go high up or down in the γ plane, $f(\gamma)$ becomes oscillatory with some poles, so does not help convergence. The same is true with our $sh(\gamma\pi)$ factor. So it all hinges on the IK product. I requote earlier results

10.41 Asymptotic Expansions for Large Order

10.41(i) Asymptotic Forms

If $\nu \rightarrow \infty$ through positive real values with $z(\neq 0)$ fixed, then

10.41.1 $$I_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ez}{2\nu}\right)^\nu,$$

10.41.2 $$K_\nu(z) \sim \sqrt{\frac{\pi}{2\nu}} \left(\frac{ez}{2\nu}\right)^{-\nu}.$$

Maybe this result can be trusted for the IK product and we would then have

$$I_\nu K_\nu \rightarrow (1/2)\, \nu^{-1}$$

$$I_{i\gamma}(kr)\, K_{i\gamma}(kr_0) \;\rightarrow\; (1/2)\,(i\gamma)^{-1}$$

Then as we to up or down, we are looking at an integrand which is roughly $1/\gamma$ * oscillatory, and *maybe* this causes the GC to vanish. It at least has a chance. It seems that the chance is the same up or down, so let's close "up" and then we get the right sense on our pole residues.

Now write

$$f(\gamma) = ch[\gamma(|\varphi-\pi|-\pi)] \;-\; ch[\gamma(\varphi-\pi)] / ch(\pi\gamma) \quad = \text{a function even in } \gamma.$$

$$= \quad f_E(\gamma) \qquad + \qquad f_v(\gamma)$$

where

$$f_v(\gamma) = \; - \; ch[\gamma(\varphi-\pi)] / ch(\pi\gamma)$$

and this is the part that arises from our v contribution to g = g + E. We are then interested in

$$v(r,\varphi|r_0,\pi) = (i/2\pi) \int_{-\infty}^{\infty} d\gamma\; I_{i\gamma}(kr_<)\, K_{i\gamma}(kr_>)\, f_v(\gamma)/\, sh(\gamma\pi)$$

$$= (i/2\pi) \int_{-\infty}^{\infty} d\gamma\; I_{i\gamma}(kr_<)\, K_{i\gamma}(kr_>)\, \{-\; ch[\gamma(\varphi-\pi)] / ch(\pi\gamma)\}/\, sh(\gamma\pi)$$

$$= -(i/2\pi) \int_{-\infty}^{\infty} d\gamma\; I_{i\gamma}(kr_<)\, K_{i\gamma}(kr_>)\, ch[\gamma(\varphi-\pi)] / ch(\pi\gamma)\}/\, sh(\gamma\pi)$$

$$= -(i/\pi) \int_{-\infty}^{\infty} d\gamma\; I_{i\gamma}(kr_<)\, K_{i\gamma}(kr_>)\, ch[\gamma(\varphi-\pi)] / sh(2\pi\gamma)$$

$$= (1/i\pi) \int_{up_ccw} d\gamma\; I_{i\gamma}(kr_<)\, K_{i\gamma}(kr_>)\, ch[\gamma(\varphi-\pi)] / sh(2\pi\gamma) \qquad \text{// I moved the i downstairs}$$

I now want to take "the other form" for the I and K subscripts. Change them and put an overall minus sign, then we have

$$v(r,\varphi|r_0,\pi) = \; - (1/i\pi) \int_{up_ccw} d\gamma\; I_{-i\gamma}(kr_<)\, K_{-i\gamma}(kr_>) ch[\gamma(\varphi-\pi)] / sh(2\pi\gamma)$$

Now back in my raw2 notes I already dealt with the poles here and I showed there that

$$sh(2\pi\gamma) \;\approx\; 2\pi\,(-1)^n\,(\gamma - \gamma_n) \qquad \text{for } \gamma \text{ near the pole position } \gamma_n = ni/2, \quad -i\gamma_n = n/2$$

There is the same issue with how much of the pole at $\varphi = 0$ we "get". But looking now at the full $f(\gamma)$, we can see that when $\gamma = 0$, we have $f(0) = 0$ and I think things cancel between the two terms. So let's just ignore the pole at 0 for both the v and E terms (I will later remove it from the known E sum). Then we get

$$v(r,\varphi|r_0,\pi) = -(1/i\pi)\,(2\pi i)\,1/(2\pi i)\int_{up_ccw} d\gamma\, I_{-i\gamma}(kr_<)\,K_{-i\gamma}(kr_>)\,ch[\gamma(\varphi-\pi)] / sh(2\pi\gamma)$$

$$= -\,2\,\Sigma_{n=1,2,3..}\; I_{n/2}(kr_<)\,K_{n/2}(kr_>)\,ch[\gamma_n(\varphi-\pi)]\;/\,[\,2\pi(-1)^n\,]$$

$$= -\;(1/\pi)\,\Sigma_{n=1,2,3..}\,(-1)^n\;I_{n/2}(kr_<)\,K_{n/2}(kr_>)\,ch[\gamma_n(\varphi-\pi)]$$

and we can write

$$ch[\gamma_n(\varphi-\pi)] = ch[ni(\varphi-\pi)/2] = \cos[n(\varphi-\pi)/2]$$

so our final v result is then

$$v(r,\varphi|r_0,\pi) = -\;(1/\pi)\,\Sigma_{n=1,2,3..}\,(-1)^n\;I_{n/2}(kr_<)\,K_{n/2}(kr_>)\,\cos[n(\varphi-\pi)/2]$$

This is similar to -- but not the same as -- a result I obtained in the raw2 notes, because here the sum is only over positive integers. Now lets quote our E term sum, but exclude the n=0 term as noted above,

$$E(r,\varphi|r_o,\pi) = (1/\pi)\,\Sigma_{n=1}^{\infty}\,(-1)^n\cos(n\varphi)\,I_n(kr_<)K_n(kr_>)$$

In order to attempt addition of these, change to m = 2n or n = m/2 in this last sum

$$E(r,\varphi|r_o,\pi) = (1/\pi)\,\Sigma_{m=2,4,6..}^{\infty}\,(-1)^{m/2}\cos(m\varphi/2)\,I_{m/2}(kr_<)K_{m/2}(kr_>)$$

Then we have these results prior to addition

$$v(r,\varphi|r_0,\pi) = -\;(1/\pi)\,\Sigma_{n=1,2,3..}^{\infty}\,(-1)^n\;I_{n/2}(kr_<)\,K_{n/2}(kr_>)\,\cos[n(\varphi-\pi)/2]$$
$$E(r,\varphi|r_o,\pi) = (1/\pi)\,\Sigma_{n=2,4,6..}^{\infty}\,(-1)^{n/2}\,I_{n/2}(kr_<)\,K_{n/2}(kr_>)\,\cos(n\varphi/2)$$

But we are not yet ready to add. First note that

$$\cos[n(\varphi-\pi)/2] = \cos(n\varphi/2)\cos(n\pi/2) + \sin(n\varphi/2)\sin(n\pi/2)$$

$$= \cos(n\varphi/2)\cos(n\pi/2) \qquad \text{when n = even integers}$$
$$= \sin(n\varphi/2)\,\sin(n\pi/2) \qquad \text{when n = odd integers}$$

Let's now separate the v series into two series, odds and evens, so we then want to add three series

$$v(r,\varphi|r_0,\pi)_o = +\,(1/\pi)\,\Sigma_{n=1,3,5..}^{\infty}\;I_{n/2}(kr_<)\,K_{n/2}(kr_>)\;\sin(n\varphi/2)\,\sin(n\pi/2)$$
$$v(r,\varphi|r_0,\pi)_e = -\;(1/\pi)\,\Sigma_{n=2,4,6..}^{\infty}\;I_{n/2}(kr_<)\,K_{n/2}(kr_>)\,\cos(n\varphi/2)\cos(n\pi/2)$$
$$E(r,\varphi|r_o,\pi) = (1/\pi)\,\Sigma_{n=2,4,6..}^{\infty}\,(-1)^{n/2}\,I_{n/2}(kr_<)\,K_{n/2}(kr_>)\,\cos(n\varphi/2)$$

Now observe that for n even

$$\cos(n\pi/2) = (-1)^{n/2}$$

so we then have

$$v(r,\varphi|r_0,\pi)_o = +\,(1/\pi)\,\Sigma_{n=1,3,5..}^{\infty} \quad I_{n/2}(kr_<)\,K_{n/2}(kr_>)\,\sin(n\varphi/2)\sin(n\pi/2)$$
$$v(r,\varphi|r_0,\pi)_e = -\,(1/\pi)\,\Sigma_{n=2,4,6..}^{\infty} \quad I_{n/2}(kr_<)\,K_{n/2}(kr_>)\cos(n\varphi/2)\cos(n\pi/2)$$
$$E(r,\varphi|r_o,\pi) = (1/\pi)\,\Sigma_{n=2,4,6..}^{\infty} \quad I_{n/2}(kr_<)\,K_{n/2}(kr_>)\cos(n\varphi/2)\cos(n\pi/2)$$

Finally after 2-3 days, we now get cancellation of the two even series! We are then left with

$$g(r,\varphi|r_0,\pi) = (1/\pi)\,\Sigma_{n=1,3,5..}^{\infty}\; I_{n/2}(kr_<)\,K_{n/2}(kr_>)\,\sin(n\varphi/2)\sin(n\pi/2)$$

We can then add the even indices since $\sin(n\pi/2) = 0$ and we have

$$g(r,\varphi|r_0,\pi) = (1/\pi)\,\Sigma_{n=1}^{\infty}\; I_{n/2}(kr_<)\,K_{n/2}(kr_>)\,\sin(n\varphi/2)\sin(n\pi/2)$$

which finally agrees with (7.183) !!!