Failure of Laplace Method on the ω=0 rod problem PhL 4.12.11

Overview: I consider the simple frozen rod problem with heat A applied at t=0 and show that the Laplace method cannot solve this problem in s-space. This same conclusion is shown in other docs. There are various other obscure side details in the few pages below, so I will keep this thing around, but it really does not have any generally useful Laplace info.

Can we track things in the case ω→0 ? Lots of bad things happen in this limit. The four poles all collide at the origin, the contour is pinched, and so on. But let's try anyway to track it through using the Summary section.

A Summary of the spatial Laplace solution

(1) Set up the problem in x and s spaces. In regular x space, the problem is this:

$(\partial_t - \partial_x^2)u(x,t) = 0$ $\quad 0 \le x < \infty \quad t \ge 0$

$u(x,0) = 0$ boundary condition BC (a.e. , true that u(0,0) = A by next line)
$u(0,t) = A$ initial condition IC

In other words, we take a frozen half rod and start applying heat "A" at the left end at t = 0.

Question: How does this relate to 7.8a ? Answer: the $\delta(x-x_0)$ business shows up in 7.8a, that is true, but that is for a g function. The problem above is really 7.3 and has no δ functions.

Side note #1: I know the solution to this problem right away based on p 205 7.27. It is this

$u(x,t) = A[1 - \text{erf}(x/2\sqrt{t})] = A\,\text{erfc}(x/2\sqrt{t})$

Each term satisfies the PDE, and clearly we have u(0,t) = A from the first term, and u(x,0+) = 0. Notice that u(x,t→∞) = A, which just says eventually the heat works its way down the entire rod.

Side note #2: Looking at p 242 C, as ω→0, the first term becomes A, so we expect that the second term should be the error function. That is, we expect that

$(1/\pi) \int_0^\infty d\rho\, e^{-\rho t} \sin(x\sqrt{\rho})/\rho = \text{erf}(x/2\sqrt{t})$

The first integral has the form of a Laplace transform, but I cannot find it in Schaum or Bateman or GR7. However, Maple can do it,

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(1/Pi)*laplace(sin(x*sqrt(rho))/rho, rho,t);
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$$\text{erf}\left(\frac{1}{2}\frac{x}{\sqrt{t}}\right)$$

This proves additional evidence that Stak's result p 242 C is correct [ie, we have shown it is correct for the case ω = 0.]

Resume: When the PDE is transformed to s-space by the usual spatial Laplace transform, the PDE becomes

$$\partial_t U(s,t) - [\, s^2 U(s,t) - s\, u(0,t) - \partial_x u(0,t)] = 0$$

Since $u(x,0) = 0$ a.e., we have $\partial_x u(0,t) = 0$ a.e. as well. [a totally wrong claim!]

Comment: To have success using Laplace on $\partial_x^2 u(x,t)$, we need to know $u(0,t)$ *and* $\partial_x u(0,t)$. In our current problem as stated above, we know $u(0,t) = A$ but we don't know $\partial_x u(0,t)$ and we cannot just arbitrarily set it to 0. The blue text above is totally illogical by the way.

Start Over and Try Using a Heaviside Step Gizmo

[I thought that this might be the silver bullet which would allow a Laplace spatial transform solution.]

(1) Set up the problem in x and s spaces. In regular x space, the problem is this:

$$(\partial_t - \partial_x^2)u(x,t) = 0 \qquad 0 \le x < \infty \qquad t \ge 0$$

$u(x,0) = 0$ boundary condition BC (a.e. , true that $u(0,0) = A$ by next line)
$u(0,t) = A$ initial condition IC

Now, let's recognize that we have a Heaviside step at x = 0 and write

$$u(x,0) = A\,\theta(-x)$$

and we write this understanding that our only interest is $x > 0$. Notice now that

$$u'(x,0) = -A\delta(x)$$

and we can think of half this delta function extending into our $x>0$ region. While we're at it

$$u''(x,0) = -A\delta'(x)$$

Two Digressions:

Let's compare these items to the known solution,

where now I extend our known solution to all x, not just $x > 0$. This says that for $u < 0$, we have $u = A$ for $t>0$, and this is my give heat reservoir on the left side which forces $u = A$ for all $x< 0$ all the time after $t = 0$. So our known solution is:

$$u(x,t) = \theta(x)\,[1 - \mathrm{erf}(x/2\sqrt{t}\,)] + \theta(-x)$$

Maple tells us that

$$\partial_x[1 - \text{erf}(x/2\sqrt{t})] = -\exp(-x^2/(4t))/(\sqrt{\pi t})$$

Thus we have

$$\begin{aligned} u'(x,t) &= -\theta(x)\exp(-x^2/(4t))/(\sqrt{\pi t}) + \delta(x)[1 - \text{erf}(x/2\sqrt{t})] - \delta(x) \\ &= -\theta(x)\exp(-x^2/(4t))/(\sqrt{\pi t}) + \delta(x)[1 - \text{erf}(0/2\sqrt{t})] - \delta(x) \\ &= -\theta(x)\exp(-x^2/(4t))/(\sqrt{\pi t}) + \delta(x)[1 - 0] - \delta(x) \\ &= -\theta(x)\exp(-x^2/(4t))/(\sqrt{\pi t}) \end{aligned}$$

In Appendix A below I show that

$$\lim_{t\to 0}[\exp(-x^2/(4t))/(\sqrt{\pi t})] = 2\delta(x)$$

so we then have

$$u'(x,0) = -\theta(x)2\delta(x) = -\delta(x)$$

The fact that $2\theta(x)\delta(x) = \delta(x)$ is then the way we snow ANY delta function result:

$$\int_{-a}^{a} dx\,[2\theta(x)\delta(x)] = \int_{0}^{a} dx\,[2\delta(x)] = (1/2)\int_{-a}^{a} dx\,[2\delta(x)] = \int_{-a}^{a} dx\,\delta(x) = 1$$

and since this result is independent of a, we conclude that the integrands must be equal.

So the upshot is that our known solution to this problem does indeed satisfy my conditions as stated above in the current problem (of course when I set those conditions, I did not know the answer).

Resume. Now let's ponder the Laplace transform a bit. We start with

$$U(s,t) = \int_0^\infty dx\, e^{-sx} u(x,t)$$

We then apply s and take the usual parts integration to get

$$sU(s,t) = +\int_0^\infty dx\, e^{-sx}\partial_x u(x,t) + u(0,t)$$

What is the meaning of a parts integration if a function is discontinuous at an endpoint?

We want to ponder

$$\int_a^b \partial_x(fg)dx = fg|_a^b = \int_a^b [\, f\,\partial_x g + g\,\partial_x f\,]dx$$

Normally we assume that both f and g are differentiable on the closed interval in order to write this equation in the first place. But suppose there is a discontinuity in one of the functions right AT one of the endpoints. As an example, suppose a = 0 and suppose g(x) = θ(x). Then we are pondering:

$$f(x)\theta(x)|^{b}_{0} = \int_0^b [\, f(x)\,\delta(x) + \theta(x)\,\partial_x f \,]dx = \int_0^b dx\, f(x)\,\delta(x) + \int_0^b dx\,\partial_x f$$

If we think of the θ(x) being "just to the left of x=0" then our δ function is also to the left, and then we would say $\theta(x+\varepsilon)|^{x=0} = 1$ and so

$$f(x)\theta(x+\varepsilon)|^{b}_{0} = \int_0^b [\, f(x)\,\delta(x+\varepsilon) + \theta(x+\varepsilon)\,\partial_x f \,]dx = \int_0^b dx\, f(x)\,\delta(x+\varepsilon) + \int_0^b dx\,\partial_x f$$

$$f(b) - f(0) = 0 + \int_0^b dx\,\partial_x f \qquad \text{// Method 1}$$

which is certainly a correct answer. Another possibility is to thing of the θ being just to the right. Then we get

$$f(x)\theta(x-\varepsilon)|^{b}_{0} = \int_0^b [\, f(x)\,\delta(x-\varepsilon) + \theta(x-\varepsilon)\,\partial_x f \,]dx = \int_0^b dx\, f(x)\,\delta(x-\varepsilon) + \int_0^b dx\,\partial_x f$$

$$f(b) - 0 = f(0) + \int_0^b dx\,\partial_x f \qquad \text{// Method 2}$$

which is again the correct result. A third possibility is to think of the θ(x) being centered at x = 0, but having a slight slope, so that θ(0) = 1/2. At the same time, we think of the δ function as being half on each side. In this case we get

$$f(x)\theta(x)|^{b}_{0} = \int_0^b [\, f(x)\,\delta(x) + \theta(x)\,\partial_x f \,]dx = \int_0^b dx\, f(x)\,\delta(x) + \int_0^b dx\,\partial_x f$$

$$f(b) - (1/2)\, f(0) = (1/2)\, f(0) + \int_0^b dx\,\partial_x f \qquad \text{// Method 3}$$

All three methods give the same correct result. You just have to be consistent in your application of one of these three methods.

At general t, everything is smooth and this presents no problem. But at t = 0 we do have a problem to think about.

$$sU(s,0) = \lim_{t\to 0+} \{ \int_0^\infty dx\, e^{-sx}\,\partial_x u(x,t) + u(0,t) \}$$

Now we have to deal with a conflict of interest. On the one hand, we want to say u(0,t) = A for $t \geq 0$ because this is our applied heat at the left end. On the other hand, we want to say u(x,0) = 0 for $x \geq 0$ because this is our initial condition. The conflict is that we then get u(0,0) = A and u(0,0) = 0. However, if

we adopt Method 2 above, so our θ(x) steps at x = +ε, then u(x=ε/2,0) = A and our conflict goes away. So this puts our delta function fully on the right and we then get

$$sU(s,0) = \lim_{t\to 0+} \{ \int_0^\infty dx\, e^{-sx}\, \partial_x u(x,t) + u(0,t) \} = \int_0^\infty dx\, e^{-sx}\, \partial_x u(x,0) + u(0,0)$$

$$= \int_0^\infty dx\, e^{-sx}\, [-A\delta(x-\varepsilon)] + u(0,0) = -A + A = 0 \qquad \Rightarrow \qquad U(s,0) = 0$$

This is consistent with the idea that $U(s,0) = \int_0^\infty dx\, e^{-sx}\, u(x,0) = \int_0^\infty dx\, e^{-sx}\, A\, \theta(-x) = 0$. And I have always gotten U(s,0) = 0 in all approaches to this problem, so happy with that.

What about the second Laplace transform level? We found above that

$$sU(s,t) = + \int_0^\infty dx\, e^{-sx}\, \partial_x u(x,t) + u(0,t)$$

Now apply s again

$$s^2U(s,t) = + \int_0^\infty dx\, e^{-sx}\, \partial^2_x\, u(x,t) + \partial_x u(0,t) + su(0,t)$$

where the parts has generated a new term. In the above, we applied this integral to the second term of our ODE

$$\int_0^\infty dx\, e^{-sx}\, [\partial^2_x\, u(x,t)] = s^2U(s,t) - \partial_x u(0,t) - su(0,t)$$

and this gave us our transformed PDE

$$(\partial_t - \partial_x^2)u(x,t) = 0$$

$$\partial_t U(s,t) - [\, s^2U(s,t) - s\, u(0,t) - \partial_x u(0,t)] = 0$$

But what exactly *is* $\partial_x u(0,t)$? Thinking of the solution picture, at x=0+ this is perhaps something we don't know without knowing the solution. Recall Maple saying $\partial_x[1 - \mathrm{erf}(x/2\sqrt{t}\,)] = -\exp(-x^2/(4t))/(\sqrt{\pi t}\,)$ which tells us that $\partial_x u(0,t) = -1/\sqrt{\pi t}$.

We are Dead in the Water. Since I don't know $[\partial_x u(x,t)]^{x=0}$, I cannot continue with the Laplace method!

(1) Well, what happens if we just carry the unknown quantity along like some extra baggage? we then have

$$(\partial_t - s^2)U(s,t) = -s\,u(t,0) - \partial_x u(0,t) = -sA - [\,\partial_x u(x,t)\,]^{x=0}$$

But how are we supposed to solve this ODE without knowing the driving function $[\,\partial_x u(x,t)\,]^{x=0}$?