Snippets from failed Exercise 7.11 attempts PhL 4.5.11

If we can show that our double integral converges, then it converges to something which is independent of r. Assuming we can show this, we then have

$$|\partial_t \langle u(x,t)\rangle| \le (r^{n-1}/V_n(r))\ K(t)$$

But we know that $V_n(r) \sim r^n$ so this then says

$$|\partial_t \langle u(x,t)\rangle| \le (1/r)\ K'(t)$$

We then take our limit $r \to \infty$ and we get $|\partial_t \langle u(x,t)\rangle| = 0$ which says $\partial_t \langle u(x,t)\rangle = 0$ which says the spatial average of temperature remains a constant. So we "make it by one power".

Exercise 7.11 Average Temperature is a constant in time.

In this problem we are asked about the nature of a solution u(x,t) of the heat equation where our region of interest is all space, that is, $R = R_n$ in Stak notation. Since there are no boundaries and no sources, equation (7.14) has only the $f(x_0)$ term. This would be true for (7.14 gen) where we had different spatial BC's, because both surface integrals will be zero at $r = \infty$. [Is this justified for the surface integrals? I guess if we assume g = C as we do later below, we could show these surface integrals vanish.]

So we start off then with

$$u(x,t) = \int dx_0\ g(x,t|x_0,0)\ f(x_0) \qquad t>0$$

Now I think I can create a contradiction as follows. Given the above, we get

$$\partial_t\ u(x,t) = \int dx_0\ \partial_t\ g(x,t|x_0,0)\ f(x_0) \qquad t>0$$

Now if we adopt the (7.8a) "picture" for g, we can replace ∂_t above with ∇_x^2 and we then get

$$\partial_t\ u(x,t) = \int dx_0\ \nabla_x^2 g(x,t|x_0,0)\ f(x_0)$$

We cannot transfer ∂_x to ∂_{x0} as we might for a convolution-form kernel. Thus, this does not fit the form of any divergence theorem. So I guess I cannot make a contradiction here as I thought. Now, let's repeat the above doing a spatial average first. To do this, assume a sphere of radius r (in n dimensions) and then we have

$$\langle u(x,t)\rangle = (1/V_n(r)) \int dx_0\ f(x_0) \int dx\ g(x,t|x_0,0)$$

Now take the time derivative to get

$$\partial_t \langle u(x,t) \rangle = (1/V(r)) \int dx_0 \, f(x_0) \int dx \, \partial_t g(x,t|x_0,0)$$

Again we adopt the (7.8a) "picture" for g and replace ∂_t with ∇_x^2 to get

$$\partial_t \langle u(x,t) \rangle = (1/V(r)) \int dx_0 \, f(x_0) \int dx \, \nabla_x^2 g(x,t|x_0,0)$$

Now we CAN apply our simple divergence theorem $\int_V dV \, \nabla^2 \varphi = \int_S dS \, \partial_n \varphi$ to write this as

$$\partial_t \langle u(x,t) \rangle = (1/V(r)) \int dx_0 \, f(x_0) \int_S dS \, \partial_n \, g(x,t|x_0,0) \qquad \text{x on sphere radius r}$$

Now we know that $g(x,t|x_0,0)$ is the temperature resulting from $f(x) = \delta(x-x_0)$ in our (7.8a) picture. In fact, we know it is g = C as in 7.10. We can see from 7.10 that C decays strongly far away (gaussian decay). Specifically,

$$\int_S dS \, \partial_n \, g(x,t|x_0,0) = (4\pi t)^{-n/2} \int_S dS \, \partial_n \exp(-|x-x_0|^2/4t) \qquad \text{x is moving point on the surface}$$

Now we have sort of the crucial issue I think. We integrate over a sphere that we will make large soon, but the point x_0 is offset from its center. No matter how large we make r, we always have to worry about the point x_0 being close to the surface of the sphere. That is to say, $f(x_0)$ could extent to infinity in our initial condition statement. So just keep this in mind please. I will formalize our fixed sphere to write

$$\int_S dS \, \partial_n \, g(x,t|x_0,0) = (4\pi t)^{-n/2} \int_S dS \, \partial_n \exp(-|x-x_0|^2/4t)$$

$$= (4\pi t)^{-n/2} \, r^{n-1} \int d\Omega_n \; \partial_r \exp(-|x-x_0|)^2/4t)$$

We then have

$$\partial_t \langle u(x,t) \rangle = (4\pi t)^{-n/2} \, (r^{n-1}/V(r)) \int dx_0 \, f(x_0) \int d\Omega_n \; \partial_r \exp(-|x-x_0|)^2/4t)$$

and now we have things looking good! The next step is to say

$$|\partial_t \langle u(x,t) \rangle| = (4\pi t)^{-n/2} \, (r^{n-1}/V(r)) \int dx_0 \, |f(x_0)| \int d\Omega_n \; \partial_r \exp(-|x-x_0|^2/4t)$$

$$\leq (4\pi t)^{-n/2} \, (r^{n-1}/V(r)) \; M \int_{all} dy \int d\Omega_n \; \partial_r \exp(-|x-y|^2/4t)$$

where r = |x| and $d\Omega_{\mathbf{n}}$ refers to a solid angle pencil passing from center of sphere to point x. I argue now that since nothing blows up anywhere, we are free to interchange orders of our various operators. Then

$$I \equiv \int_{\mathbf{all}} dy \int d\Omega_{\mathbf{n}} \; \partial_{\mathbf{r}} \exp(-|x-y|^2/4t) \; = \int d\Omega_{\mathbf{n}} \; \partial_{\mathbf{r}} \; \{ \int_{\mathbf{all}} dy \exp(-|y-x|^2/4t) \}$$

The bracketed integral is just this, where we define y' = y-x and dy' = dy

$$\int_{\mathbf{all}} dy \exp(-|x-y|)^2/4t) \; = \int_{\mathbf{all}} dy \exp(-|y'|)^2/4t) \; = \int_{\mathbf{all}} dy' \exp(-|y'|)^2/4t)$$

$$= S_{\mathbf{n}}(1) \int_{\mathbf{0}}^{\infty} ds \exp(-s^2/4t) = S_{\mathbf{n}}(1)\sqrt{\pi t} \qquad \text{// Maple}$$

where for our full-space dy' integral we use s as the radial coordinate. What we learn here is that our internal integral $J = \int_{\mathbf{all}} dy \exp(-|x-y|)^2/4t)$ appears at first to depend on x (and on r = |x|), but since it is an all space integral

$$\int_{\mathbf{all}} dy \exp(-|x-y|^2/4t)$$

Consider this double integral.

$$I = \int dx_{\mathbf{0}} \int d\Omega_{\mathbf{xn}} \; \partial_{\mathbf{n}} \exp(-|x-x_{\mathbf{0}}|^2/4t) \; = \int_{\mathbf{all}} dy \; \int d\Omega_{\mathbf{x}} \; \partial_{\mathbf{nx}} \exp(-|x-y|^2/4t)$$

This is a slightly tricky situation to understand.(1) $d\Omega_{\mathbf{x}}$ is a piece of solid angle passing through the sphere on which point **x** lies; (2) $\partial_{\mathbf{nx}}$ is really $\partial_{\mathbf{r}}$ for this sphere; (3) r = |**x**| of course. (4) I don't really know much about $d\Omega_{\mathbf{x}}$ in n dimensions, but I do know there is always a polar angle. Not sure whether this detailed information is needed here. So write again as

$$I = \int_{\mathbf{all}} dy \int d\Omega_{\mathbf{x}} \; \partial_{\mathbf{r}} \exp(-|x-y|^2/4t)$$

Plan A: suppose we try to slide the dy integral to the right, we then get

$$I = \int_{\mathbf{all}} dy \int d\Omega \; \partial_{\mathbf{r}} \int_{\mathbf{all}} dy \exp(-|x-y|^2/4t)$$

which at least has a well defined meaning. But at this point, why can we not define y' = y-x and replace

$$\int_{\mathbf{all}} dy\, \exp(-|x-y|)^2/4t) \;=\; \int_{\mathbf{all}} dy\, \exp(-|y'|)^2/4t) \;=\; \int_{\mathbf{all}} dy'\, \exp(-|y'|)^2/4t)$$

$$= S_{\mathbf{n}}(1)\; \int_{\mathbf{0}}^{\infty} ds\, \exp(-s^2/4t) \;= S_{\mathbf{n}}(1)\sqrt{\pi t} \qquad \text{// Maple}$$

Then we find that $\partial_{\mathbf{r}}\, S_{\mathbf{n}}(1)\sqrt{\pi t} \;= 0$ and therefore I = 0. This just does not "smell good". Let's try it some other way and see if we still get 0.

$$|x-y|^2 = x^2 + y^2 - 2x.y \;= r^2 + y^2 - 2ry\cos\gamma$$

$$\partial_{\mathbf{r}}|x-y|^2 = 2r - 2y\cos\gamma$$

$$\partial_{\mathbf{r}}\, \exp(-|x-y|^2/4t) \;=\; \exp(-|x-y|^2/4t)\; (-1/4t)\; (2r - 2y\cos\gamma)$$

Then we have

$$I = \int_{\mathbf{all}} dy \int d\Omega_{\mathbf{x}}\; \partial_{\mathbf{r}}\, \exp(-|x-y|^2/4t) = \; (-1/2t)\; \int_{\mathbf{all}} dy \int d\Omega_{\mathbf{x}}\, \exp(-|x-y|^2/4t)\; (r - y\cos\gamma)$$

and we have disposed of our $\partial_{\mathbf{n}}$ derivative at the cost of picking up the factor shown. If we NOW move the y integral to the right, what happens? What do we do with this integral

$$\int_{\mathbf{all}} dy\; y\, \exp(-|x-y|^2/4t)$$

$$\int_{\mathbf{all}} dy$$

$$\partial_{\mathbf{nx}}\, \exp(-|x-y|)^2/4t) \;= n \bullet \nabla_{\mathbf{x}}\, \exp(-|x-y|)^2/4t) \;= \partial_{\mathbf{r}}\, \exp(-|x-y|)^2/4t)$$

$$\partial_{\mathbf{r}} \;= \partial_{\mathbf{r}} x_{\mathbf{i}}\; \partial_{\mathbf{i}}$$

and now I am more concerned that this is a function of r since x lies on the sphere of radius r. Since r is finite at this point, I could define y' = y-x to get

$$I = -\int_{\mathbf{all}} dy'\; \int d\Omega_{\mathbf{x}}\; \partial_{\mathbf{ny'}}\, \exp(-|y'|)^2/4t)$$

If we center a sphere on our point x_0 then $C \sim \exp(-r^2/4t)$ and $\partial_r C \sim \exp(-r^2/4t)\,(-2r/4t)$ which still has this decay. Now since $dS_n = r^{n-1} d\Omega_n$, the integral

$$\int_S dS\, \partial_n g(x,t|x_0,0) \sim r^{n-1} S_n(1) \exp(-r^2/4t)\,(-2r/4t) = \sim r^n \exp(-r^2/4t)$$

where I ignore constants and powers of t (think of t constant for the moment). f we move our origin back to x_0 we get

$$\int_S dS\, \partial_n g(x,t|x_0,0) \sim |x-x_0|^n \exp(-|x-x_0|^2/4t) \qquad x \text{ on sphere of radius } r$$

where x is a point on the sphere of radius r, and then we have

$$\partial_t \langle u(x,t)\rangle \sim (1/V_n(r)) \int dx_0\, f(x_0)\{\, |x-x_0|^n \exp(-|x-x_0|^2/4t)\,\}$$

Now we can use the fact that $f(x_0)$ is "bounded" by M, say, so

$$|\partial_t \langle u(x,t)\rangle| \sim (1/V_n(r)) \int dx_0\, |f(x_0)|\,\{\, |x-x_0|^n \exp(-|x-x_0|^2/4t)\,\}$$

$$\leq (1/V_n(r))\; K(t)\, M \int dx_0 \{\, |x-x_0|^n \exp(-|x-x_0|^2/4t)\,\} \qquad (*)$$

Now for this spatial integral, put the origin back at the center and use $dx_0 = r^n dr\, d\Omega_n$ to get

$$\int dx_0 \{\, |x-x_0|^n \exp(-|x-x_0|^2/4t)\,\} = S_n(1) \int dr\, r^{2n} \exp(-r^2/4t) \qquad \text{// Maple}$$

$$= S_n(1)(1/2)\,\Gamma(n+1/2)\,(4t)^{n+1/2} = \text{independent of radius } r \equiv K'(t)$$

We have then shown that

$$|\partial_t \langle u(x,t)\rangle| \leq (1/V_n(r))\; K(t)\, M\, K'(t) = (1/r^n)\, f(n,t)$$

Now as we let $r \to \infty$, we get $|\partial_t \langle u(x,t)\rangle| = 0$ which says $\partial_t \langle u(x,t)\rangle = 0$ which says the spatial average of temperature remains a constant. This argument could have been applied at equation (*) since the only r dependence is in V.

$$\langle u(t) \rangle = \int dx \int dx_0\, g(x,t|x_0,0)\, f(x_0) = \int dx_0\, f(x_0) \int dx\, g(x,t|x_0,0)$$

$$\partial_t \langle u(t) \rangle = \int dx_0\, f(x_0) \int dx\, \partial_t g(x,t|x_0,0) = \int dx_0\, f(x_0) \int dx\, \nabla^2 g(x,t|x_0,0)$$

$$= \int dx_0\, f(x_0) \int_S dS\, \partial_n g(x,t|x_0,0) = \int dx_0\, f(x_0)\ \{ 0 \}$$

because g and all its derivatives → 0 as we go to r→∞. There is no boundary. But by this same argument I could show that

$$u(x,t) = \int dx_0\, g(x,t|x_0,0)\, f(x_0)$$

$$\partial_t\, u(x,t) = \int dx_0\, \partial_t\, g(x,t|x_0,0)\, f(x_0) = \int dx_0 \nabla^2 g(x,t|x_0,0)\, f(x_0) = 0$$

which we know is wrong. Remember that $(\partial_t - \nabla^2)\, g(x,t|x_0,0) = \delta(x-x_0)\delta(t)$ in the (7.8) "picture" and not 0. So redoing the above line we get

$$\partial_t\, u(x,t) = \int dx_0\, \partial_t\, g(x,t|x_0,0)\, f(x_0) = \int dx_0\, \delta(x-x_0)\delta(t)\, f(x_0) = \delta(t) f(x)$$

But this is also obviously wrong.

Suppose we take u(x,t) as given in p 214 (7.56) which applies for the u = 0 on σ BC.

$$u(x,t) = \Sigma_i\, f_i \varphi_i(x) \exp(-\lambda_i t)$$

The spatial average would then be

$$\langle u(t) \rangle = \int dx\, u(x,t) = f_i \exp(-\lambda_i t) \int dx\, \varphi_i(x)$$

It is not obvious what to do with $\int dx\, \varphi_i(x)$ since we don't know that 1 is an EF. So let's try a time derivative

$$\partial_t \langle u(t) \rangle = \Sigma_i \int dx\, \varphi_i(x)\, \partial_t \exp(-\lambda_i t) = \Sigma_i \exp(-\lambda_i t) \int dx\, \varphi_i(x)\, (-\lambda_i)$$

$$= \Sigma_i \exp(-\lambda_i t) \int dx\, \nabla^2 \varphi_i(x) = \Sigma_i \exp(-\lambda_i t) \int_S dS\, \partial_n \varphi_i$$

but I am not getting anywhere, nothing to do next.

Exercise 7.11 Average Temperature is a constant in time.

In order to even talk about "average temperature" we have to restrict to a sphere of radius r which at least has some measureable finite volume. Then in the end we might take $r \to \infty$. With such a finite sphere, we do have spatial boundaries to worry about. Our generalized equation (7.14) says

$$\begin{aligned} u(t,\mathbf{x}) &= \int_0^t dt_0 \int_{R_n} d^n x_0 \; g(x|x_0)\, q(t_0,\mathbf{x}_0) \\ &+ \int_{R_n} d^n x_0 \; \{u(0,\mathbf{x}_0)\, g(x|x_0)\}|^{t0=0}] \\ &+ \int_0^t dt_0 \int_{\sigma_n} dS_{n0} \left[u(t_0,\mathbf{x}_0)\{-\partial_{nx0}\, g(x|x_0)\} + g(x|x_0)\partial_{n0} u(t_0,\mathbf{x}_0) \right] \end{aligned} \qquad (7.14 \text{ gen})$$

Now we are allowed to exclude internal sources in this problem, meaning q = 0, so we get

$$\begin{aligned} u(t,\mathbf{x}) &= \int_{R_n} d^n x_0 \; \{u(0,\mathbf{x}_0)\, g(x|x_0)\}|^{t0=0}] \\ &+ \int_0^t dt_0 \int_{\sigma_n} dS_{n0} \left[u(t_0,\mathbf{x}_0)\{-\partial_{nx0}\, g(x|x_0)\} + g(x|x_0)\partial_{n0} u(t_0,\mathbf{x}_0) \right] \end{aligned}$$

where σ_n is a spherical shell or radius r in n dimensions.