Stakgold Chapter 7 Exercise 7.16 PhL 4.6-17.11

I spent 11 days trying to do this problem. It was just exercise, I got most of it done. Part (c) I got into accidentally, Stak never asked for it. Part (d) he did ask for and I feel his question is illogical.

This thing ballooned out so much that this new doc was needed.

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Preliminary Comments on the Half Rod

The problem is that of a 1D semi-infinite rod which is driven at its left end by a sinusoidal source. We are supposed to solve this problem in three different ways below.

How does this problem fit with our "cylinder picture"? The space R is the half line, boundary is just the left end, so think of that as the positive x axis. Then translate this thing "up" in the t direction to create the cylinder, which in this case is just a semi-infinite vertical sheet running x = 0,∞ and t = 0,T say. The "initial condition" would be to specify u(x,0) on the bottom edge of the sheet, and the "boundary condition" would be to specify u(0,t) along the left edge of this sheet.

What is the causal Green's function here? It is what propagates you from one point on the sheet to another point above it on the sheet. If we require g = 0 on the cylinder base and walls (meaning on the bottom of the sheet and on the left edge of the sheet) , the Green's Function is (7.23) which Stak derived from the image method. Given that g function, we should be able to solve any heat problem using 7.14. We know that the dS_0 integral becomes just a point, so this (7.14) reads (no inside sources)

$$u(x,t) = \int_0^\infty dx_0\, g(x,t|x_0,0)\, u(x_0,0) \;-\; \int_0^t dt_0\, \partial_{n_0} g(x,t|x_0,t_0)\, u(0,t_0)$$

$$u(x,t) = \int_0^\infty dx_0\, g(x,t|x_0,0) f(x_0) \;-\; \int_0^t dt_0\, \partial_{n_0} g(x,t|x_0,t_0)\, h(0,t_0)$$

The "boundary" is the left end of the rod, and the "initial condition" if any would be u(x,0) from x = 0 to ∞. So our famous "cylinder picture" is just a vertical line going from t = 0 up to t = T. An "initial condition" would refer to

(a) Solve the Problem using XT separation of variables.

First, what exactly is "the problem" here? In his first statement of the problem on page 241, Stak assumes we have a specific q and h function: $q(x,t) = q(x)e^{-i\omega t}$ and $h(0,t) = A\, e^{-i\omega t}$. There is no f(x) specified. The problem is presented as 7.109. This system is presented in the (7.8) modality with time in (-∞,∞). Our task in part (a) here is to try a separated solution of the form u = XT where $T = e^{-i\omega t}$. We then get from the PDE

$$-i\omega\, XT - \partial_x^2 XT = q(x)T \qquad \Rightarrow \qquad (-\partial_x^2 - i\omega)X(x) = q(x)$$

which appears to be a 1D Helmholtz ODE $(\partial_x^2 + i\omega)X(x) = -q(x)$ so $k^2 = i\omega$. Thus we are left with a purely x-space equation for X(x) which might then have BC's X(0) = a and X(∞) = b. Our interval is in fact (0,∞) and so x = ∞ is a singular endpoint, and we will use X(∞) = 0 to get a finite s-norm solution.

(a1) Derive p 242 A

If q(x) = 0, the solution to this equation $(\partial_x^2 + i\omega)X(x) = 0$ is given by $X(x) = \exp(\pm\sqrt{-i\omega}\, x)$. Write $\sqrt{-i} = e^{-i\pi/2} = (1-i)/\sqrt{2}$. Then our solution would be $X(x) = \exp(\pm x\sqrt{\omega}\,(1-i)/\sqrt{2}) = \exp[\pm\sqrt{\omega/2}\,(1-i)x]$. Our solution at this point would be (we select constant A to match the $h(0,t) = A\, e^{-i\omega t}$ BC)

$$u(x,t) = A \exp[\pm\sqrt{\omega/2}\,(1-i)x]\, e^{-i\omega t} = A \exp[\pm\sqrt{\omega/2}\, x]\, \exp[\pm\sqrt{\omega/2}\,(-ix) - i\omega t)]$$

// for later use, we know that this also = A $\exp(\pm\sqrt{-i\omega}\, x)\, e^{-i\omega t}$

but now we have to select the - sign to get an L^2 solution on the half rod. So

$X(x) = A \exp[-\sqrt{\omega/2}\,(1-i)x]$ $\qquad$ $T(t) = e^{-i\omega t}$

$u(x,t) = A \exp[-\sqrt{\omega/2}\,(1-i)x]\, e^{-i\omega t} = A \exp[-\sqrt{\omega/2}\, x]\ \exp[+i\ \{\sqrt{\omega/2}\,(x)-\omega t)\}]$ $\qquad$ (*)

// for later use, we know that this also = A $\exp(-\sqrt{-i\omega}\, x)\, e^{-i\omega t}$

The left expression exactly matches **page 242 A.** The right expression shows a familiar complex wave moving to the right, but with a decaying envelope function. Our solution above was completely determined by the boundary conditions X(0) = A and X(∞) = 0, just as discussed above.

Question: What happened to condition f(x) ??

Going back to the original problem statement, suppose we had specified an f(x), and supposed we followed the usual EV method of solution of Stak page 213. We have - $\nabla^2 X = \lambda X$. In that method, we are solving problems which have u(x,t) = 0 on the boundary and that becomes X = 0 and then we have a real EV problem. However, in our present problem our boundary σ is the point x = 0, and we do not have the condition that u(0,t) = 0 there. Rather, our condition is that $u(0,t) = A\, e^{-i\omega t}$ there. Therefore the entire XT separation + EV method does not apply to our present problem, although we still seek a separated solution here of the form XT.

We have successfully solved our q=0 problem and have found a complex solution u(x,t) of the XT form. No further boundary conditions can be specified since we have our solution. It is true that at t = 0 we have $u(x,0) = X(x) = X(x) = A \exp[-\sqrt{\omega/2}\,(1-i)x]$. This "is what it is" and you cannot impose f(x) = 0 on this problem!

So how does this problem fit into our past framework? We had the notion of a cylinder where we specified h on σ and f at t=0, and that determined the solution u(x,t). Here, we do have h specified on σ still, namely we have $h(0,t) = A\, e^{-i\omega t}$. Why can't we specify an f(x) at t = 0 and seek a solution by our usual methods? Our generic half rod solution is this:

$$u(t,x) = \int_0^t dt_0 \int_0^\infty dx_0\ g(x|x_0)\, q(t_0,x_0)$$
$$+ \int_0^\infty dx_0\ f(x_0)\ \{\, g(x|x_0)\}|^{t0=0}]$$
$$+ \int_0^t dt_0\ [h(0,t_0)\{-\partial_{nx0}\, g(x|x_0)\} + g(x|x_0)\partial_{n0} u(t_{x0},t_0)\ ] \qquad \text{(7.14 gen, half rod)}$$

and if we go with the Dirichlet g function with g = 0 on the boundary, and we take q = 0, we get this problem solution

$$u(x,t) = \int_0^\infty dx_0\ f(x_0)\ \{\, g(x|x_0)\}|^{t0=0}] + \int_0^t dt_0\ [h(0,t_0)\{-\partial_{nx0}\, g(x|x_0)\}|^{x0=0}]$$

We already considered a half rod driven by h(t) at the left end on page 207, an and we found an explicit form in fact for $\partial_{nx0}\, g(x|x_0)$ in page 207B with g itself given in 7.23 (the half rod propagator), to wit,

$$g(x|x_0) = H(t-t_0) / \sqrt{4\pi(t-t_0)}\ \{ \exp[-(x-x_0)^2/ 4(t-t_0) - \exp[-(x+x_0)^2/ 4(t-t_0) \qquad 7.23$$

$$\partial_{nx0}\, g(x|x_0)|^{x0=0} = H(t-t_0) / \sqrt{4\pi(t-t_0)}\ * (x/(t-t_0)) * \exp[- x^2/4(t-t_0)] \qquad \text{p 207 B}$$

We can then set $h(0,t_0) = A\exp(-i\omega t_0)$ and then for whatever f(x) we select, we get a solution u(x,t). I presume that were we do impose $f(x) = A\ \exp[-\sqrt{\omega/2}\,(1-i)x]$, we would arrive at the damped wave solution we obtained above as (*).

The upshot is that the our "present problem" is in a different class of problems. Instead of specifying the three functions q,f and h to get a solution, here we assume that $XT = Xe^{-i\omega t}$ and we just see where that leads. It leads to an ODE for X with X(0) = A and X(∞) = 0 and we get the damped wave solution and f(x) comes out as it may. OK, fair enough.

(a2) Derive p 242 B.

Question: does the real part of u(x,t) solve our q(x) = 0 equation?

$$(\partial_t - \partial_x^2)\, u(x,t)= 0 \qquad \Rightarrow \qquad (\partial_t - \partial_x^2)\, \mathrm{Re}\{u(x,t)\} + i\,(\partial_t - \partial_x^2)\, \mathrm{Im}\{u(x,t)\} = 0$$

Therefore the two pieces must separately solve the PDE. So define v(x,t) = Re{u(x,t)} and we find

$$(\partial_t - \partial_x^2)\, v(x,t) = 0 \qquad \text{and } v(0,t) = A\cos(\omega,t)$$

so we have found a solution to this problem which is then the following

$$v(x,t) = \mathrm{Re}\ \{ A\ \exp[-\sqrt{\omega/2}\, x]\ \exp[+i\sqrt{\omega/2}\,(x)-i\omega t)]\}$$

$$= A\ \exp[-\sqrt{\omega/2}\, x]\ \cos\ [\sqrt{\omega/2}\, x - \omega t)\ ] \qquad \text{// agrees with } \textbf{p 242 B}$$

I agree that this represents a (real) wave moving to the right (as t increases, x increases to maintain the same phase) at velocity determined by $\sqrt{\omega/2}\,\Delta x - \omega\Delta t = 0$ or $v_{phase} = \Delta x/\Delta t = \omega/\sqrt{\omega/2} = \sqrt{2\omega}$. But the amplitude of the wave has a drop-off envelope as shown. I just created my **first-ever Maple animated plot** and it was great, added it to the user guide plot section. You see the sine thing running to the right with a fixed decay envelope.

So this is a very interesting problem. Now why is there no initial condition (IC) f(x) specified [I explained that above now, but keep this OK older notes] ? For our solution above, you see that

$$u(x,t=0) = A\ \exp[-\sqrt{\omega/2}\,(1-i)x] \qquad \text{f(x) for the complex problem}$$
$$v(x,t=0) = A\ \exp[-\sqrt{\omega/2}\, x]\ \cos\ [\sqrt{\omega/2}\, x\ )\ ] \qquad \text{f(x) for the real part problem}$$

Thus, it is already determined, you cannot impose it. Here is another argument I came up with for now f(x): This problem is specified on t in (-∞,∞) so you could argue that there IS no initial time. Again, we are in the (7.8) modality of specifying a heat problem where there is no initial condition (go look, p 198).

So our solution is one that "goes on forever" like my animation does. You wiggle the left end of the half rod at a fixed frequency, and a damped wave travels down the rod, just as you would expect. What is a little surprising is that we are doing the heat equation, not the wave equation. A great Stak problem.

(b) Solve the Problem using a *time* Laplace Transform: that is, derive p 242 C.

Warning: This is not the same problem as that in part (a). Stak calls it an "initial value problem" but he fails to state the initial value u(x,0) !! In fact the initial value is f(x) = 0! Thus, when we start our driving term at t = 0, there will be a transition period of time before the "steady state" solution of the part (a) problem sets in!!! We can regard the integral second term in p 242 C as providing "transients" for small t, and we can see that for large t this integral term → 0 and we then get just our wave solution.

(b1) Now we reconsider the same problem in the (7.8a) modality. There is no q, t>0 only, same BC as in part a (the real problem). The solution found here is the same, except there is a mysterious extra term that seems to be some kind of transient thing. I just updated my Laplace notes, and I think yes we can interpret this second term as a continuum of transients that are part of the solution. This is something I have never seen before.

Why do we have this transients term in Part (b) and not in Part (a) ? *******

We start out as usual and I will call U(x,s) the time Laplace transform of u(x,t) so we then have

$\int_0^\infty ds\, e^{-st} f(t) = F(s)$ // Schaum page 162

$\int_0^\infty ds\, e^{-st} \partial_t f(t) = sF(s)-f(0)$ // these are just general formulas ...

and here then is our time-Laplace transformed equation:

$$sU(x,s)-u(x,0) - \partial_x^2 U(x,s) = 0$$

The boundary condition is u(0,t) = Acos(ωt) so we can Laplace this to get

$$U(0,s) = As/(s^2+\omega^2)$$

So here then is a statement of our problem in terms of U(x,s)

$(\partial_x^2 - s)\, U(x,s) = -u(x,0)$ $\qquad$ $U(0,s) = As/(s^2+\omega^2)$

What are we going to do about u(x,0) which we don't know?? It is sitting there due to the f(0) in the Laplace transform. This f(0) means u(x,0). Since we are in the (7.8a) model, you need to specify this. But we are told that u(0,t) = Acos(ωt) so u(0,0) = A. Thus, it does not seem logical to assume that u(x,0) = 0

since we know u(0,0) = A . Nevertheless, I will go ahead and assume u(x,0) = 0 (almost everywhere, let's say) and see what happens. So then

$(\partial_x^2 - s)\, U(x,s) = 0 \qquad U(0,s) = As/(s^2+\omega^2)$

where s is regarded for now as a fixed parameter. The candidate ODE solutions are $\exp(\pm \sqrt{s}\, x)$, but since we are on our half-rod, we take $U(x,s) = K \exp(-\sqrt{s}\, x)$. Then $U(0,s) = K = As/(s^2+\omega^2)$. So our proposed solution is this

$U(x,s) = A \exp(-\sqrt{s}\, x)\, s/(s^2+\omega^2)$

Our next problem is to do the inverse Laplace transform which we see has two poles and a cut. *I could do it*, but let's try to look this up instead. My Schaum table does not have such a messy thing. Bateman has a table of Laplace and another table of Inverse Laplace, very excellent, and here is an entry on page 246 of ET 1

(11)	$p\,(p^2+\beta^2)^{-1} e^{-\alpha^{1/2} p^{1/2}}$ $\operatorname{Re} a \geq 0, \quad \operatorname{Re} \beta \geq 0$	$\exp[-(\tfrac{1}{2} a\beta)^{1/2}] \cos[\beta t - (\tfrac{1}{2} a\beta)^{1/2}]$ $-\frac{1}{\pi} \int_0^\infty e^{-ut} \sin(\alpha^{1/2} u^{1/2}) \frac{u}{u^2+\beta^2} du$

Scale Check on the above result. If α = 0 we get

$p/(p^2+\beta^2) \qquad \leftrightarrow \qquad \cos(\beta t)$ which is correct, Schaum p 164

This applies to my case if I take β = ω and $\alpha^{1/2} = x$ and of course p = s. We will have

$(\alpha\beta/2)^{1/2} = \sqrt{\omega/2 * x^2} = x\sqrt{\omega/2}$

The inverse is then

$A\exp(-x\sqrt{\omega/2}) \cos[\, \omega t - x\sqrt{\omega/2}\,]$

$-(A/\pi) \int_0^\infty du\, e^{-ut} \sin(x\sqrt{u})\, (u/(u^2+\omega^2))$

$= A\exp(-x\sqrt{\omega/2}) \cos[\, \omega t - x\sqrt{\omega/2}\,] - (A/\pi) \int_0^\infty d\rho\, e^{-\rho t} \sin(x\sqrt{\rho})\, (\rho/(\rho^2+\omega^2))$

and hurray, this **agrees exactly with p 242 C.**

Notice how Bateman is providing a whole *catalog* of GR scope for both Laplace and Inverse Laplace!

[Maple gives a much different result, which I just note here and then ignore.]

```
assume(a>0,b>0);
invlaplace( exp(-sqrt(s*a))*s/(s^2 + b^2),s,t);
```

$$\int_0^t \frac{1}{2}\frac{\sqrt{a}\, e^{\left(-\frac{1}{4}\frac{a}{_Ul}\right)} \cos(b\,(t-_Ul))}{\sqrt{\pi}\,_Ul^{\left(\frac{3}{2}\right)}}\, d_Ul$$

Comments about the Solution 242 C.

As noted above, the first term can be written this way

$$A\ \mathrm{Re}\{\ \exp(-\sqrt{-i\omega}\ x)\ e^{-i\omega t}\ \}$$

Notice that

$$(\partial_t - \partial_x^2)\ [\ \exp(-\sqrt{-i\omega}\ x)\ e^{-i\omega t}\ ]$$
$$= (-i\omega)[\ \exp(-\sqrt{-i\omega}\ x)\ e^{-i\omega t}] \ -\ (-)^2(-i\omega)\ [\ \exp(-\sqrt{-i\omega}\ x)\ e^{-i\omega t}] \ = 0$$

so it is obvious that the first term solves the homo equation. But this is also obvious for the second term (the integral term) since

$$(\partial_t - \partial_x^2)\ [\ e^{-\rho t} \sin(\rho^{1/2} x)\ ] = \ (-\rho)\ [\ e^{-\rho t} \sin(\rho^{1/2} x)\ ] - (-)\rho\ [\ e^{-\rho t} \sin(\rho^{1/2} x)\ ] \ = 0$$

So I just want to make the point that both terms trivially are solutions of our homo heat equation. We shall show below, with some pain, that at t = 0, the integral term exactly cancels the first term causing us to have f(x) = 0, and that is why we have the integrand shown which wraps [$e^{-\rho t} \sin(\rho^{1/2} x)$] in this second term.

Although I still don't know any theory of PDE's, it would seem that the two terms in p 242 C are in fact linearly independent solutions. I don't think Stak ever claimed that a generic PDE of order n has n independent solutions, but he may have.

(b1) Show that the solution p 242 C is consistent with our assumption that u(x,0) = 0.

Plan A: fails, and for a good reason

NOW, in the above I assumed that u(x,0) = 0. Does the solution show this to be true? The first term is easy to evaluate at t = 0 and is clearly non-zero, but the integral term takes more work. We start with

$$I \equiv \int_0^\infty d\rho\, \sin(x\sqrt{\rho}\,)\ (\rho/(\rho^2 + \omega^2))$$

which Maple cannot do. Change to

$$y = \sqrt{\rho} \qquad y^2 = \rho \quad 2ydy = d\rho$$

$$I = \int_0^\infty 2ydy \sin(xy)\, y^2/(y^4+\omega^2) = 2 \int_0^\infty dy \sin(xy)\, y^3/(y^4+\omega^2)$$

which Maple still cannot do, but I recognize this as a Fourier Sine transform as listed off in ET 1. If you look at the high end, integral is like $\int_0^\infty dy \sin(xy)/y = (\pi/2)\text{sign}(x)$ so things seem convergent.

My first step is to write I as two terms

$$I = 2 \int_0^\infty dy \sin(xy)\ (1/2)\ [\ y/(y^2+i\omega) + y/(y^2-i\omega)\ ] = \int_0^\infty dy \sin(xy)\, y/(y^2+i\omega) + CC$$

Let's rewrite this to fit with the Bateman table by swapping x and y , and add two minus signs:

$$I = -\int_0^\infty dx \sin(xy)\, x/(i\omega-x^2) + CC$$

Now Bateman says on page 65 ET 1

(19)	$x(a^2-x^2)^{-1}$	$a > 0$	$-\frac{1}{2}\pi \cos(ay)$ The integral is a Cauchy Principal Value

so I have to interpret $a^2 = i\omega = e^{i\pi/2}\,\omega$ => $a = e^{i\pi/4}\sqrt{\omega}$. So if I blindly apply this transform, I get

$$\begin{aligned}
I &= (\pi/2) \cos[e^{i\pi/4}\sqrt{\omega}\, y\ ] + cc \\
&= (\pi/2) \cos[\ (1+i)/\sqrt{2}\ \sqrt{\omega}\, y\ ] + cc \\
&= (\pi/2) \cos[\ (1+i)\, y\ ] + cc \\
&= (\pi/2)\{\ \cos[\sqrt{\omega/2}\, y\ ] \cos[i\sqrt{\omega/2}\, y\ ] - \sin[\sqrt{\omega/2}\, y\ ] \sin[i\sqrt{\omega/2}\, y\ ]\ \} + cc \\
&= (\pi/2)\{\ \cos[\sqrt{\omega/2}\, y\ ] \cosh[\sqrt{\omega/2}\, y\ ] - i \sin[\sqrt{\omega/2}\, y\ ] \sinh[\ \sqrt{\omega/2}\, y\ ]\ \} + cc \\
&= (\pi/2)\, 2\{\ \cos[\sqrt{\omega/2}\, y\ ] \cosh[\sqrt{\omega/2}\, y\ ]\ \} \\
&= \pi\{\ \cos[\sqrt{\omega/2}\, y\ ] \cosh[\sqrt{\omega/2}\, y\ ]\ \}
\end{aligned}$$

Let's go back now to our solution

$$u(x,t) = A\exp(-x\sqrt{\omega/2}\ ) \cos[\ \omega t - x\sqrt{\omega/2}\ ] - (A/\pi) \int_0^\infty d\rho\, e^{-\rho t} \sin(x\sqrt{\rho}\ )\, (\rho/(\rho^2+\omega^2))$$

We have shown that

$$u(x,0) = A\exp(-x\sqrt{\omega/2})\cos[x\sqrt{\omega/2}\ ] - (A/\pi)\ \{\pi\{\ \cos[\sqrt{\omega/2}\,x\ ]\cosh[\sqrt{\omega/2}\,x\ ]\ \}$$
$$= A\exp(-x\sqrt{\omega/2})\cos[x\sqrt{\omega/2}\ ] - A\cos[\sqrt{\omega/2}\,x\ ]\cosh[\sqrt{\omega/2}\,x\ ]$$
$$= A\cos[x\sqrt{\omega/2}\ ]\ \{\ \exp(-x\sqrt{\omega/2}) - \cosh[\sqrt{\omega/2}\,x\ \}$$

and this is NOT coming out zero.

Now we think about Bateman's remark. This integrand with a real has a pole right along the integration, and that is why he has done the CPV thing. But I don't have this situation, so I think the result simply does not apply.

Plan B: succeeds at showing that solution p 242 C has property u(x,0) = 0

Let's try the integral a different way. go back to

$$I = \int_0^\infty dx\ \sin(xy)\ x/(x^2+i\omega) + CC$$

Now define $\alpha^2 = -i\omega$ so we then have

$$I = \int_0^\infty dx\ \sin(xy)\ x/(x^2-\alpha^2) + CC$$

$$= \int_0^\infty dx\ \sin(xy)\ (1/2)\ [\ 1/(x+\alpha) + 1/(x-\alpha)\ ] + cc$$

Now define

$$\int_0^\infty dx\ \sin(xy)/\ (x+\alpha) \equiv J(\alpha)$$

Then our final result will be

$$I = (1/2)[J(\alpha) + J(-\alpha)] + cc$$

Where is α located in the complex plane?

$\alpha^2 = -i\omega = e^{-i\pi/2}\,\omega$ $\qquad$ $\alpha = e^{-i\pi/4}\sqrt{\omega}$ $\qquad$ so α in Q IV !

To get from α to -α, I would propose rotating CCW by π, so at all times we have $|\arg\alpha| < \pi$. So not on the real axis, then I am happy to use this Bateman result for complex α page 64.

(10)	$(x+a)^{-1}$	$\lvert\arg a\rvert < \pi$	$\mathrm{Ci}(ay)\sin(ay) - \mathrm{si}(ay)\cos(ay)$

Then it would seem that

J(α) = Ci(αy)sin(αy) - si(αy) cos(αy)

which is a fairly complicated result! Here are those functions defined

$$\mathrm{si}(x) = -\int_x^\infty \frac{\sin t}{t}\,dt = \frac{1}{2i}\left[\mathrm{Ei}(ix) - \mathrm{Ei}(-ix)\right]$$

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt = \tfrac{1}{2}\pi + \mathrm{si}(x)$$

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}\,dt = -\,\mathrm{ci}(x) = \tfrac{1}{2}\left[\mathrm{Ei}(ix) + \mathrm{Ei}(-ix)\right]$$

$$-\mathrm{Ei}(-x) = E_1(x) = \int_x^\infty e^{-t}\frac{dt}{t} = \Gamma(0, x) \qquad -\pi < \arg x < \pi$$

We can see that Ci is even in its argument whereas si is odd [wrong, see below]. So look again at

J(α) = Ci(αy)sin(αy) - si(αy) cos(αy)

We see that J(α) is odd in its argument (!), so therefore J(α) + J(-α) = 0. Therefore I = 0 and our entire integral second term vanishes. But then we still have the first term sitting around!

Let's look "more carefully" at the supposed symmetry rules just quoted. AS page 232 disagree and claim that

Ci(-z) = Ci(z) -iπ if argz in (0,π)

In my case, suppose -z = αy. Then since α is in Q4 and y is real, -z is in Q2 which meets the conditions. So I would then say

Ci(αy) = Ci(-αy) -iπ if arg(αy) in (0,π)

This detail is not quite clear if you just look at the naive expression for Ci(x) above. Now what about si you ask? From AS 232 we have

si(z) = Si(z) - π/2 and Si(-z) = - Si(z) without restrictions

Therefore

si(-z) = Si(-z) - π/2 = - Si(z) - π/2 = - [Si(z) - π/2] - π = - si(z) -π

si(-z) = - si(z) -π

Using these rules let's see what happens:

J(α) = Ci(αy)sin(αy) - si(αy) cos(αy)

$$J(-\alpha) = Ci(-\alpha y)\sin(-\alpha y) - si(-\alpha y)\cos(-\alpha y)$$
$$= - Ci(-\alpha y)\sin(\alpha y) - si(-\alpha y)\cos(\alpha y) \qquad \text{// trig only}$$
$$= - [Ci(\alpha y)+i\pi]\sin(\alpha y) - [- si(\alpha y) -\pi]\cos(\alpha y)$$
$$= - Ci(\alpha y)\sin(\alpha y) - i\pi\sin(\alpha y) + si(\alpha y)\cos(\alpha y) + \pi\cos(\alpha y)$$
$$= - Ci(\alpha y)\sin(\alpha y) + si(\alpha y)\cos(\alpha y) + \pi [\cos(\alpha y) - i\sin(\alpha y)]$$
$$= - Ci(\alpha y)\sin(\alpha y) + si(\alpha y)\cos(\alpha y) + \pi e^{-i\alpha y}$$
$$= - J(\alpha) + \pi e^{-i\alpha y}$$

So this seems to tell us that

$$J(\alpha) + J(-\alpha) = \pi e^{-i\alpha y}$$

Then we get for I (I change y back to x now)

$$I = (1/2)[J(\alpha) + J(-\alpha)] + cc$$

$$= (1/2)[\pi e^{-i\alpha x}] + cc = (\pi/2)[e^{-i\alpha x} + cc] \qquad \alpha = e^{-i\pi/4}\sqrt{\omega}$$

So here we go:

$$\alpha = e^{-i\pi/4}\sqrt{\omega} = (1-i)/\sqrt{2} * \sqrt{\omega} = (1-i)\sqrt{\omega/2}$$
$$e^{-i\alpha x} = \exp(-i\alpha x) = \exp(-i(1-i)\sqrt{\omega/2}\, x) = \exp((-i-1)\sqrt{\omega/2}\, x) = \exp(-i\sqrt{\omega/2}\, x)\exp(-\sqrt{\omega/2}\, x)$$

Then we have

$$I = (\pi/2)[e^{-i\alpha x} + cc]$$
$$= (\pi/2)[\exp(-i\sqrt{\omega/2}\, x)\exp(-\sqrt{\omega/2}\, x) + \exp(i\sqrt{\omega/2}\, x)\exp(-\sqrt{\omega/2}\, x)]$$
$$= (\pi/2)\exp(-\sqrt{\omega/2}\, x)[\exp(-i\sqrt{\omega/2}\, x) + \exp(i\sqrt{\omega/2}\, x)]$$
$$= \pi\exp(-\sqrt{\omega/2}\, x)\cos(\sqrt{\omega/2}\, x)$$

So this then is our claimed limit of the integral in the second term of p 242 C when t = 0. So here we can add both terms with t = 0 to get

$$u(x,0) = A\exp(-\sqrt{\omega/2}\, x)\cos(\sqrt{\omega/2}\, x) - (A/\pi)[\pi\exp(-\sqrt{\omega/2}\, x)\cos(\sqrt{\omega/2}\, x)]$$
$$= A\exp(-\sqrt{\omega/2}\, x)\cos(\sqrt{\omega/2}\, x) - A\exp(\sqrt{\omega/2}\, x)\cos(\sqrt{\omega/2}\, x)]$$
$$= 0$$

Well THAT didn't take long! So I assumed u(x,0) = 0 at the start (despite the left end point being at A), and this fact is recovered from the solution!)

(b3) Find the leading term of the transient part of the solution for large t.

You wonder that maybe for very small ρ the transient might go on forever giving a constant offset of some sort. But the rest of the integrand is 0 near ρ = 0 so probably not. This is of course exactly the subject of Stak Appendix B. I guess I need to wander off now on that Appendix and nail it down before continuing here. DONE.

The second term in p 242 C is of the form of a normal Laplace transform and we have

$$f(\rho) = \sin(\sqrt{\rho}\, x)\rho/(\rho^2+\omega^2)$$

For large t, the integral (without the -A/π factor) is equal to my corrected form of B.5 which is this

$$F(\rho) = \text{the integral} = \Sigma_{n=0}^{\infty} f^{(n)}(0)\, t^{-n-1}$$

so it is just question of doing the derivatives? The n = 0 term vanishes, so probably n = 1 will be the leading term. Using Maple I find that

$$f(0) = 0$$
$$f'(0) = 0$$
$$f''(\rho) \approx (3/4)\, x/(\omega^2\sqrt{\rho}\,) \text{ as } \rho \to 0.$$

so this is problematic! Obvious if you say $f(\rho) \approx x\, \rho^{3/2}$ for small ρ, then $f''(\rho) \sim \rho^{-1/2}$. So we have to roll our own here:

$$\int_0^{\infty} d\rho\, e^{-\rho t} \sin(x\sqrt{\rho}\,)\, (\rho/(\rho^2+\omega^2))$$

$$\approx (x/\omega^2) \int_0^{\infty} d\rho\, e^{-\rho t}\, \rho^{3/2} = (x/\omega^2)\, (3/4)\, \sqrt{\pi}\; t^{-5/2} \qquad \text{// Maple}$$

I would perhaps be a little more accurate by allowing that x might be large, then we have

$$(1/\omega^2) \int_0^{\infty} d\rho\, e^{-\rho t} \sin(x\sqrt{\rho}\,)\, \rho$$

Maple says this for the integral.

$$\frac{3}{4}\frac{\sqrt{\pi}\, x\, e^{\left(-\frac{1}{4}\frac{x^2}{t}\right)}}{t^{\left(\frac{5}{2}\right)}} - \frac{1}{8}\frac{\sqrt{\pi}\, x^3\, e^{\left(-\frac{1}{4}\frac{x^2}{t}\right)}}{t^{\left(\frac{7}{2}\right)}}$$

so this is indeed a much better result because we now see more info. So the answer to the question asked here (large t behavior of the integral in 242C) is this:

$$- (A/\pi)\,(x/\omega^2)\,(3/4)\,\sqrt{\pi}\;\exp(-x^2/4t)\;t^{-5/2} \qquad \text{// leading term for large t}$$

and this applies even if x is very large. You could of course write

$$\exp(-x^2/4t) \;\approx\; 1 - x^2/4t \;= 1 - (x/2)^2\,t^{-1}$$

and you could argue that this really contributes to the $t^{-7/2}$ second term, so the "leading term" is this think with the exp set to 1. This shows how fast the transient component of our solution drops off.

(c) Redo this same problem using the *spatial* Laplace Transform.

This is not what Stak asked us to do! He said use the spatial Fourier Cosine. But since I have started this Laplace thing, I will finish it. [this decision cost me about 5 days!] By "this same problem" we mean the problem with f(x) = 0 whose solution is p 242C. We do not mean the part (a) problem which just gives the steady state solution.

(c1) State the equation in s-space and find its solution there.

$$(\partial_t - \partial_x^2)u(x,t) = 0$$

$$\int_0^\infty ds\, e^{-sx}\;\partial_x^2\, f(x) = s^2F(s) - sf(0) - f'(0)$$

$$\int_0^\infty ds\, e^{-sx}\;\partial_x^2\, u(x,t) = s^2U(s,t) - s\,u(0,t) - \partial_x u(0,t)$$

We know that u(0,t) = A cos(ωt). But we do NOT know what $\partial_x u(x,t)|^{x=0}$ is. As in the last section, I will simply assume it is zero, and hope for the best.

Note: This assumption is completely wrong. For any moderate t > 0, there will be some slope u'(0,t) of the temperature profile at the left end of the rod which is NOT zero. I showed this explicitly in the ω = 0 case in a separate doc "Using the Laplace Transform.doc" section 5 . Thus, all the work below is invalid. But somehow, even with this wrong assumption about u'(0,t), and also dealing with "illegal" f(s) functions in s-space, we end up with a result which is exactly 1/2 the correct answer! This same thing happened in the ω = 0 case (section 9 of doc just mentioned), not surprising. This result is so strange, that I will not delete the erroneous work below, in the hope that someday I might learn why we get half the right answer despite taking glaringly wrong steps in the analysis.

Then we have

$$\partial_t\, U(s,t) - [s^2U(s,t) - s\,A\cos(\omega t)] = 0$$

Now in the previous solution I successfully assumed u(x,0) = 0 which suggests that U(s,0) = 0 as well. So here then is our little system:

$(\partial_t - s^2)\, U(s,t) = A\, s \cos(\omega t)$ $\qquad U(s,0) = 0$ $\qquad (*)$

PAUSE 3: The RHS should be - Ascos(ωt) ! So when all is said and done, I will take A→-A.

The homogeneous solution is $U(s,t) = B \exp(s^2 t)$. As a particular solution we could try

$U(s,t) = \alpha \sin(\omega t) + \beta \cos(\omega t)$
$\partial_t U(s,t) = \alpha\omega \cos(\omega t) - \beta\omega \sin(\omega t)$

$\alpha\omega \cos(\omega t) - \beta\omega \sin(\omega t) - \alpha s^2 \sin(\omega t) - \beta s^2 \cos(\omega t) = A\, s \cos(\omega t)$
$[\, \alpha\omega - \beta s^2 - As] \cos(\omega t) - [\, \beta\omega + \alpha s^2] \sin(\omega t) = 0$
$\alpha\omega - \beta s^2 - As = 0$
$\beta\omega + \alpha s^2 = 0 \qquad \Rightarrow\ \beta\omega = -\alpha s^2 \quad \Rightarrow \quad \beta = -\alpha\,(s^2/\omega)$

Then $\alpha\omega = As + \beta s^2 = As - \alpha\,(s^4/\omega)$
$\alpha\,[\, \omega + s^4/\omega\,] = As \quad \Rightarrow \alpha\,[\, \omega^2 + s^4\,] = As\omega \quad \Rightarrow \quad \alpha = As\omega/(\omega^2+s^4)$
Then $\beta = -\alpha\,(s^2/\omega) = -As\omega/(\omega^2+s^4) * (s^2/\omega) = -As^3/(\omega^2+s^4) \quad \Rightarrow \quad \beta = -As^3/(\omega^2+s^4)$

So here is our general solution to the equation of (*) above

$U(s,t) = \alpha \sin(\omega t) + \beta \cos(\omega t) + B \exp(s^2 t)$

Setting t = 0 gives

$U(s,0) = \beta + B = 0 \qquad \Rightarrow \qquad B = -\beta$

So **our final solution is then**

$U(s,t) = \alpha \sin(\omega t) + \beta \cos(\omega t) - \beta \exp(s^2 t) \qquad \alpha = As\omega/(\omega^2+s^4) \qquad \beta = -As^3/(\omega^2+s^4)$

Let's test this solution to make sure it is correct:

```
alpha := A*s*w/(w^2+s^4);
```

$$\alpha := \frac{A\,s\,w}{w^2+s^4}$$

```
beta := -A*s^3/(w^2+s^4);
```

$$\beta := -\frac{A\,s^3}{w^2+s^4}$$

```
U := alpha*sin(w*t) + beta*cos(w*t) - beta*exp(s^2*t);
```

$$U := \frac{A\,s\,w\,\sin(w\,t)}{w^2+s^4} - \frac{A\,s^3\cos(w\,t)}{w^2+s^4} + \frac{A\,s^3\,e^{(s^2\,t)}}{w^2+s^4}$$

```
t := 0;
```

$$t := 0$$

```
U;
```

$$0$$

```
unassign('t');t;
```

$$t$$

```
diff(U,t) - s^2*U;
```

$$\frac{A\,s\,w^2\cos(w\,t)}{w^2+s^4} + \frac{A\,s^3\sin(w\,t)\,w}{w^2+s^4} + \frac{A\,s^5\,e^{(s^2\,t)}}{w^2+s^4} - s^2\left(\frac{A\,s\,w\,\sin(w\,t)}{w^2+s^4} - \frac{A\,s^3\cos(w\,t)}{w^2+s^4} + \frac{A\,s^3\,e^{(s^2\,t)}}{w^2+s^4}\right)$$

```
simplify(%);
```

$$\cos(w\,t)\,A\,s$$

.

This we have verified both that U solves the PDE, but also that U=0 at t = 0.

(c2) Calculate the three required inverse Laplace Transforms

So now we have to inverse Laplace each of these terms to construct u(x,t). Here are the three cases we will need to study

$g_1(s) = s/(s^4+\omega^2)$ $g_2(s) = s^3/(s^4+\omega^2)$ $g_3(s) = \exp(s^2t)\, s^3/(s^4+\omega^2)$

The first two are standard, the third is "illegal", but I did not know that at the time. I realized that for this third one, you cannot pull the contour to the left and pretend it is just the four pole residues, because the residual contour does not vanish. In the next section, I realize that the vertical contour integral of g_3 is nevertheless finite so we can blindly compute it as if it were an ILT.

I think I know what went wrong

Recall that I was considering three Laplace inversion cases:

function of s		inverse Laplace Transform (t now replaced with x)	
$g_1(s) = s/(s^4-\omega^2)$	→	$(1/2a^2)$ { -cos(ax) + cosh(ax) }	
$g_2(s) = s^3/(s^4-\omega^2)$	→	(1/2) { cos(ax)+ ch(ax) }	
$g_3(s) = \exp(s^2t)\, s^3/(s^4-\omega^2)$	→	(1/2) { $\exp(-a^2t)$ cos(ax) + $\exp(a^2t)$ ch(ax) }	// checked 1x

The first two are rock solid, but the third is wrong according to Maple. I think this is the problem: In the third case, the factor $\exp(s^2 t)$ blows up if you close the vertical Mellin-Barnes inversion contour to the left, so adding the four pole residues does not give the answer. For the first two cases, we are OK on this issue. The second case has e^{ts}/s integrand behavior and the e^{ts} allows left closure. But the third case of course has $\exp(s^2 t)\, e^{ts}/s$ behavior and closure is then not allowed.

On the other hand, the vertical integral itself has good convergence at the top and bottom of the imaginary axis direction since $\exp(s^2 t) = \exp(-|s|^2 t)$ in that direction. so probably the right way to do this is to treat the integral as a conventional integral. So let's look at this integral:

$$(1/2\pi i) \int ds \exp(s^2 t)\, e^{xs}\, s^3/(s^4+\omega^2)$$

As shown in the new ILT document this inverse Laplace Transform is given by

$$G(x) = (1/2) \exp(\sqrt{\omega/2}\, x)\ \cos(\sqrt{\omega/2}\, x + \omega t) + (1/2\pi) \int_0^\infty d\rho\ e^{-\rho t} \sin(x\rho^{1/2})\rho\, /(\rho^2+\omega^2)$$

and this is fairly close to p 242 C but x should be -x. But I have not assembled the total answer yet, so let's assemble the pieces:

$$U(s,t) = \alpha \sin(\omega t) + \beta \cos(\omega t) - \beta \exp(s^2 t) \qquad \alpha = A\omega/(\omega^2+s^4) \qquad \beta = -As^3/(\omega^2+s^4)$$
$$= A\omega/(\omega^2+s^4) * \sin(\omega t) - As^3/(\omega^2+s^4) \cos(\omega t) + As^3/(\omega^2+s^4) \exp(s^2 t)$$
$$= A\omega \sin(\omega t) [\, s/(\omega^2+s^4)\,] - A \cos(\omega t) [s^3/(\omega^2+s^4)] + A [\exp(s^2 t)\, s^3/(\omega^2+s^4)\,]$$
$$= A\omega \sin(\omega t) [\, s/(s^4+\omega^2)\,] - A \cos(\omega t) [s^3/(s^4+\omega^2)] + A [\exp(s^2 t)\, s^3/(s^4+\omega^2)\,]$$

To transform the first two terms, we can use our adjusted results from above

$$g_1(s) = s/(s^4+\omega^2) \quad \rightarrow \quad (1/2a^2)\ \{ -\cos(ax) + \cosh(ax)\ \} \qquad a^2 = i\omega$$
$$g_2(s) = s^3/(s^4+\omega^2) \quad \rightarrow \quad (1/2)\ \{\ \cos(ax) + \mathrm{ch}(ax)\ \} \qquad a = i^{1/2}\sqrt{\omega}$$

So these two terms then give $a = (1+i)\sqrt{\omega/2}$

$$= A\omega \sin(\omega t)\, (1/2a^2)\ \{ -\cos(ax) + \cosh(ax)\ \} - A \cos(\omega t)\, (1/2)\ \{\ \cos(ax) + \mathrm{ch}(ax)\ \}$$
$$= A\omega \sin(\omega t)\, (1/2i\omega)\ \{ -\cos(ax) + \cosh(ax)\ \} - A \cos(\omega t)\, (1/2)\ \{\ \cos(ax) + \mathrm{ch}(ax)\ \}$$
$$= A \sin(\omega t)\, (1/2i)\ \{ -\cos(ax) + \cosh(ax)\ \} - A \cos(\omega t)\, (1/2)\ \{\ \cos(ax) + \mathrm{ch}(ax)\ \}$$
$$= A\, (-i)\sin(\omega t)\, (1/2)\ \{ -\cos(ax) + \cosh(ax)\ \} - A \cos(\omega t)\, (1/2)\ \{\ \cos(ax) + \mathrm{ch}(ax)\ \}$$
$$= (A/2)[\ \ (-i)\sin(\omega t)\ \{ -\cos(ax) + \cosh(ax)\ \} - \cos(\omega t)\ \{\ \cos(ax) + \mathrm{ch}(ax)\ \}\ ]$$
$$= (A/2)[\ \ i\sin(\omega t)\ \{ \cos(ax) - \mathrm{ch}(ax)\ \} - \cos(\omega t)\ \{\ \cos(ax) + \mathrm{ch}(ax)\ \}\ ]$$
$$= -(A/2)[\ \ i\sin(\omega t)\ \{ -\cos(ax) + \mathrm{ch}(ax)\ \} + \cos(\omega t)\ \{\ \cos(ax) + \mathrm{ch}(ax)\ \}\ ]$$
$$= -(A/2)[\ \ \cos(ax)\ \{\cos(\omega t) - i \sin(\omega t)\} + \mathrm{ch}(ax)\ \{ \cos(\omega t) + i\sin(\omega t)\ \}\ ]$$
$$= -(A/2)[\ \ \cos(ax)\, e^{-i\omega t} + \mathrm{ch}(ax)\, e^{i\omega t}\ ]$$
$$= -(A/4)\ [\ \ (e^{iax} + e^{-iax})\ e^{-i\omega t} + (e^{ax} + e^{-ax})\, e^{i\omega t}\ ]$$
$$= -(A/4)\ [\ \ \exp(iax - i\omega t) + \exp(-iax - i\omega t) + \exp(ax + i\omega t) + \exp(-ax + i\omega t)\ ]$$

$$= -(A/4)\,[\quad \exp(i(1+i)\sqrt{\omega/2}\,x-i\omega t) + \exp(-i(1+i)\sqrt{\omega/2}\,x-i\omega t)$$
$$+ \exp((1+i)\sqrt{\omega/2}\,x+i\omega t) + \exp(-a(1+i)\sqrt{\omega/2}\,x+i\omega t)\ ]$$
$$= -(A/4)\,[\quad \exp((i-1)\sqrt{\omega/2}\,x-i\omega t) + \exp((1-i)\sqrt{\omega/2}\,x-i\omega t)$$
$$+ \exp((1+i)\sqrt{\omega/2}\,x+i\omega t) + \exp(-(1+i)\sqrt{\omega/2}\,x+i\omega t)\ ]$$
$$= -(A/4)\,[\quad \exp(-\sqrt{\omega/2}\,x)\exp(i[\sqrt{\omega/2}\,x-\omega t]) + \exp(\sqrt{\omega/2}\,x)\exp(-i[\sqrt{\omega/2}\,x+\omega t]\,)$$
$$+ \exp(\sqrt{\omega/2}\,x)\exp(i\,[\sqrt{\omega/2}\,x+\omega t])\ + \exp(-\sqrt{\omega/2}\,x)\exp(-i\,[\sqrt{\omega/2}\,x-\omega t)]\quad]$$
$$= -(A/4)\,[\quad \exp(-\sqrt{\omega/2}\,x)\,\{\exp(i[\sqrt{\omega/2}\,x-\omega t]) + \exp(-i\,[\sqrt{\omega/2}\,x-\omega t)]\,\}$$
$$+ \exp(\sqrt{\omega/2}\,x)\ \{\exp(i\,[\sqrt{\omega/2}\,x+\omega t])\} + \exp(-i[\sqrt{\omega/2}\,x+\omega t]\,)\ \}$$
$$= -(A/4)\,[\quad \exp(-\sqrt{\omega/2}\,x)\,\{2\cos[\sqrt{\omega/2}\,x-\omega t]\ \}$$
$$+ \exp(\sqrt{\omega/2}\,x)\ \{2\cos[\sqrt{\omega/2}\,x+\omega t)]\ \}$$
$$= -(A/2)\ \{\ \exp(-\sqrt{\omega/2}\,x)\cos[\sqrt{\omega/2}\,x-\omega t]\ +\ \exp(\sqrt{\omega/2}\,x)\ \cos[\sqrt{\omega/2}\,x+\omega t)]\ \}$$

Summary to this point: When we do ILT to the first two terms of our U(s,t) solution, we get

$$-(A/2)\ \{\ \exp(-\sqrt{\omega/2}\,x)\cos[\sqrt{\omega/2}\,x-\omega t]\}\quad -(A/2)\ \{\exp(\sqrt{\omega/2}\,x)\ \cos[\sqrt{\omega/2}\,x+\omega t)]\ \}$$

When we do ILT on the third term, that gives

$$-(A/\pi)\,(-1/2)\ \int_0^\infty d\rho\ \exp(-\rho t)\sin(t\rho^{1/2})\rho\,/(\rho^2+\omega^2)\ + (A/2)\exp(\sqrt{\omega/2}\,x)\ \cos(\sqrt{\omega/2}\,x+\omega t)$$

I am absolutely amazed and of course happy to see the left-directed waves cancelling exactly leaving us with this final result

$$u(x,t) = \ A(-1/2)\ \{\ \exp(-\sqrt{\omega/2}\,x)\cos[\sqrt{\omega/2}\,x-\omega t]\}$$
$$-(A/\pi)\,(-1/2)\ \int_0^\infty d\rho\ \exp(-\rho t)\sin(t\rho^{1/2})\rho\,/(\rho^2+\omega^2)$$

I have the same (-1/2) error in all my terms !!!

PAUSE 3A: But I had a wrong sign on A all along, and here I finally fix it to get

$$u(x,t) = \ A(1/2)\ \{\ \exp(-\sqrt{\omega/2}\,x)\cos[\sqrt{\omega/2}\,x-\omega t]\}$$
$$-(A/\pi)\,(1/2)\ \int_0^\infty d\rho\ \exp(-\rho t)\sin(t\rho^{1/2})\rho\,/(\rho^2+\omega^2)\qquad\qquad \text{// final result!}$$

and now I have everything off by a factor of (1/2).

So there it is! Despite the doubly illegal method, I get exactly 1/2 of the exact right answer!

(c3) A more detailed previous solution of the same problem, giving same 1/2 result.

I think somehow this section was written first because you see me doing the pole residues. I later realized I could just look those up, and then I did the third g_3 term from my ILT doc. In any event, this section follows the exact same path as section (c2) above.

(1) Set up the problem in x and s spaces. In regular x space, the problem is this:

$(\partial_t - \partial_x^2)u(x,t) = 0 \qquad 0 \le x < \infty \qquad t \ge 0$

$u(x,0) = 0$ boundary condition BC (a.e. , true that $u(0,0) = A$ by next line)
$u(0,t) = A\cos(\omega t)$ initial condition IC

When the PDE is transformed to s-space by the usual spatial Laplace transform, the PDE becomes

$\partial_t U(s,t) - [\, s^2U(s,t) - s\,u(0,t) - \partial_x u(0,t)] = 0$

Since $u(x,0) = 0$ a.e., we have $\partial_x u(0,t) = 0$ a.e. as well. [My bogus assumption] There is perhaps some issue with the value of this derivative right at t = 0, where one might argue that $\partial_x u(0,0) = \infty$ due to the a.e. situation noted above. But we are going to study our transformed equation for t > 0 so we set $\partial_x u(0,t) = 0$ and the above becomes

$\partial_t U(s,t) - [\, s^2U(s,t) - s\,A\cos(\omega t)\,] = 0$

or

$(\partial_t - s^2)\,U(s,t) = -A\cos(\omega t)$

Since u(x,0) = 0 for x > 0, integrating even with the a.e situation says U(s,0) = 0. We thus arrive at the following s-space first order ODE problem:

$(\partial_t - s^2)\,U(s,t) = -A\cos(\omega t) \qquad t > 0 \qquad U(s,0) = 0$

(2) Find solution to the s space problem. A first order ODE with an initial condition can have only one solution. That solution is this:

$U(s,t) = \alpha \sin(\omega t) + \beta \cos(\omega t) - \beta \exp(s^2 t) \qquad \alpha = -As\omega/(\omega^2+s^4) \qquad \beta = As^3/(\omega^2+s^4)$

Maple confirms directly that this U(s,t) satisfies our system above, and we know that there can only be one solution.

```
restart;
alpha := -A*s*w/(w^2+s^4);
```

$$\alpha := -\frac{A\,s\,w}{w^2+s^4}$$

```
beta := A*s^3/(w^2+s^4);
```

$$\beta := \frac{A\,s^3}{w^2+s^4}$$

```
U := alpha*sin(w*t) + beta*cos(w*t) - beta*exp(s^2*t);
```

$$U := -\frac{A\,s\,w\,\sin(w\,t)}{w^2+s^4}+\frac{A\,s^3\cos(w\,t)}{w^2+s^4}-\frac{A\,s^3\,\mathbf{e}^{(s^2\,t)}}{w^2+s^4}$$

```
t := 0;
```

$$t := 0$$

```
U;
```

$$0$$

```
unassign('t');t;
```

$$t$$

```
diff(U,t) - s^2*U;
```

$$-\frac{A\,s\,w^2\cos(w\,t)}{w^2+s^4}-\frac{A\,s^3\sin(w\,t)\,w}{w^2+s^4}-\frac{A\,s^5\,\mathbf{e}^{(s^2\,t)}}{w^2+s^4}-s^2\left(-\frac{A\,s\,w\,\sin(w\,t)}{w^2+s^4}+\frac{A\,s^3\cos(w\,t)}{w^2+s^4}-\frac{A\,s^3\,\mathbf{e}^{(s^2\,t)}}{w^2+s^4}\right)$$

```
simplify(%);
```

$$-\cos(w\,t)\,A\,s$$

So there is very little doubt but that this U(s,t) is the correct s-space solution. There are no missing powers of 2, for example.

(3) Identify functions needing inverse Laplace transforms. If we install our α and β coefficients, the solution U(s,t) becomes

$$\begin{aligned} U(s,t) &= -As\omega/(\omega^2+s^4)\sin(\omega t) + As^3/(\omega^2+s^4)\cos(\omega t) - As^3/(\omega^2+s^4)\exp(s^2t) \\ &= -A\omega\sin(\omega t)\,[s/(s^4+\omega^2)] + A\cos(\omega t)[s^3/(s^4+\omega^2)] - A\,[\exp(s^2t)\,s^3/(s^4+\omega^2)] \\ &= -A\omega\sin(\omega t)\,[g_1(s)] + A\cos(\omega t)[\,g_2(s)] - A\,[g_3(s)] \end{aligned}$$

where

$$\begin{aligned} g_1(s) &= s/(s^4+\omega^2) \\ g_2(s) &= s^3/(s^4+\omega^2) \\ g_3(s) &= \exp(s^2t)\,s^3/(s^4+\omega^2) \end{aligned}$$

Our task is to find the Inverse Laplace Transform of each of these functions,

$$G_i(x) = (1/2\pi i)\int_C ds\, e^{xs}\, g_i(s)$$

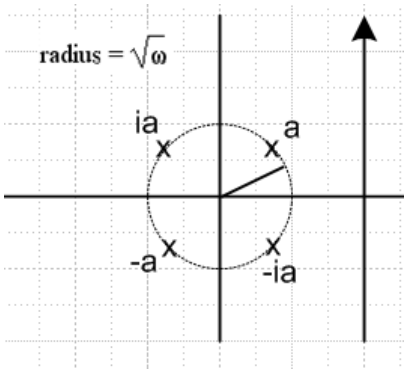
and our final problem solution will then be

$$u(x,t) = -A\omega \sin(\omega t)G_1(x) + A\cos(\omega t)[\,G_2(x)] - A\,[G_3(x)] \qquad (*)$$

Notice that all three $g_i(s)$ have the same denominator which can be factored this way

$$(s^4+\omega^2) = (s+ia)(s-ia)(s+a)(s-a) \qquad \text{where } a \equiv i^{1/2}\sqrt{\omega} \qquad (\text{so } a^2 = i\omega)$$

The point s = a is located at a point with phase π/4 and radius $\sqrt{\omega}$ in the first quadrant of the s plane. We can in fact plot the four pole locations and label them as follows: (i rotates a pole ccw 90 degrees)



(4) Compute the first two terms of result (*). For g_1 and g_2 we can compute the inverse transform by sliding the recovery contour to the left. This picks up four pole residues and when the contour reaches a great half circle on the left, it vanishes due to the factor e^{ts} in the integrand ($t < 0$, $\mathrm{Re}(s) < 0$). I computed $G_{1,2}(x)$ in this manner in full detail. But here, I will instead look up these two results in a table. Schaum page 167 says

$$s/(s^4-a^4) \;\rightarrow\; (1/2)\,[\,\mathrm{ch}(ax)-\cos(ax)\,]/a^2 \qquad (s^4-a^4) = (s^2-a^2)(s^2+a^2)$$
$$s^3/(s^4-a^4) \;\rightarrow\; (1/2)\,[\,\mathrm{ch}(ax)+\cos(ax)\,] \qquad = (s-a)(s+a)(s+ia)(s-ia)$$

These results are valid for a = any complex number and are evaluated by picking of the residues of the four poles, all of which lie to the left of our contour C. As we change the phase of a, the four poles just rotate around on a circle of radius |a| and stay spaced apart by 90 degrees. The poles are located as we see at s = ±a and s=±ia. We want to have $\omega^2 = -a^4$ so $\omega = -ia^2$, say, and then $\sqrt{\omega} = i^{-1/2}a$ and $a = i^{1/2}\sqrt{\omega}$. Thus we get these results

$$s/(s^4+\omega^2) \;\rightarrow\; (1/2)\,[\,\mathrm{ch}(ax)-\cos(ax)\,]/a^2 \qquad \text{where } a = i^{1/2}\sqrt{\omega} \qquad a^2 = i\omega$$
$$s^3/(s^4+\omega^2) \;\rightarrow\; (1/2)\,[\,\mathrm{ch}(ax)+\cos(ax)\,] \qquad a = (1+i)\sqrt{\omega/2}$$

Thus we know the first two terms of our solution. It is quite amazing how many lines of algebra it takes to get things the way we want, and Maple is pretty much useless.

$$u(x,t) = -A\omega \sin(\omega t)G_1(x) + A\cos(\omega t)[\,G_2(x)]$$
$$= -A\omega \sin(\omega t)\,(1/2)\,[\,\mathrm{ch}(ax)-\cos(ax)\,]/a^2 + A\cos(\omega t)\,(1/2)\,[\,\mathrm{ch}(ax)+\cos(ax)\,]$$
$$= -A\omega \sin(\omega t)\,(1/2)\,[\,\mathrm{ch}(ax)-\cos(ax)\,]/(i\omega) + A\cos(\omega t)\,(1/2)\,[\,\mathrm{ch}(ax)+\cos(ax)\,]$$

$= A\, i \sin(\omega t)\, (1/2)\ [\ ch(ax)-\cos(ax)\ ] + A \cos(\omega t)\, (1/2)\ [\ ch(ax)+\cos(ax)\ ]$
$= (A/2)\ ch(ax)\ [\cos(\omega t) + i \sin(\omega t)\ ] + (A/2) \cos(ax)\ [\cos(\omega t) - i \sin(\omega t)\ ]$
$= (A/2)\ [\ ch(ax)\ e^{i\omega t} + \cos(ax) e^{-i\omega t}]$
$= (A/4)\ [\ (e^{ax} + e^{-ax})\ e^{i\omega t} + \ (e^{iax} + e^{-iax}) e^{-i\omega t}]$

Let $\sqrt{\omega/2}$ = c for the moment, so that a = (1+i)c . Then continue along with more algebra:

$= (A/4)\ [\ (e^{(1+i)c\,x} + e^{-(1+i)c\,x})\ e^{i\omega t} + \ (e^{i(1+i)c\,x} + e^{-i(1+i)c\,x}) e^{-i\omega t}]$
$= (A/4)\ [\ e^{(1+i)cx+i\omega t} + e^{-(1+i)cx+i\omega t} + \ e^{i(1+i)cx-i\omega t} + e^{-i(1+i)cx-i\omega t}]$
$= (A/4)\ [\ e^{(1+i)cx+i\omega t} + e^{-(1+i)cx+i\omega t} + \ e^{(i-1)cx-i\omega t} + e^{(1-i)cx-i\omega t}]$
$= (A/4)\ [\ e^{cx+i(cs+\omega t)} + e^{-cx-i(cx-\omega t)} + \ e^{-cx+i(cx-\omega t)} + e^{cx-i(cx+\omega t)}\ ]$
$= (A/4)\ [\ e^{cx} e^{i(cs+\omega t)} + e^{-cx} e^{-i(cx-\omega t)} + \ e^{-cx} e^{i(cx-\omega t)} + e^{cx} e^{-i(cx+\omega t)}\ ]$
$= (A/4)\ [\ e^{cx} e^{i(cs+\omega t)} + e^{cx} e^{-i(cx+\omega t)} \ + \ e^{-cx} e^{-i(cx-\omega t)} + \ e^{-cx} e^{i(cx-\omega t)}\ ]$
$= (A/4)\ [\ e^{cx}\ 2\cos(cx+\omega t) \ + \ e^{-cx}\ 2\cos(cx-\omega t)]$
$= (A/2)\ [\ e^{cx}\ \cos(cx+\omega t) \ + \ e^{-cx}\ \cos(cx-\omega t)]$ $\qquad\qquad c = \sqrt{\omega/2}$

= damped wave going to the left + damped wave going to the right

(5) Compute the third term of the result (*). I did this in full detail. The integral of interest is

$$G_3(x) = (1/2\pi i) \int_C ds\, e^{xs}\, g_3(s) = (1/2\pi i) \int_C ds\, e^{xs} \exp(s^2 t)\, s^3/(s^4+\omega^2)$$

The game plan is to shift the C contour to the left, picking up the two rightmost pole residues, then C is along the imaginary axis, and we can rotate it into a real axis integral. The contribution of the two pole residues is this (see raw notes)

$$G_3(x) \text{ from both poles} = (1/2) \exp(\sqrt{\omega/2}\, x)\ \cos(\sqrt{\omega/2}\, x + \omega t)$$

and the contribution of the residual vertical integral is this

$$G_3(x) \text{ from imaginary axis integral} = (1/2\pi) \int_0^\infty d\rho\ e^{-\rho t} \sin(t\rho^{1/2})\rho\ /(\rho^2+\omega^2)$$

Since the third term in our solution is - A [$G_3(x)$] we have

$$\text{third term} = -(A/2)\, e^{cx}\ \cos(cx+\omega t)\ - A(1/2\pi) \int_0^\infty d\rho\ e^{-\rho t} \sin(t\rho^{1/2})\rho\ /(\rho^2+\omega^2)$$

= - damped wave to the left - transient term

But recall that we have already found that

$$\text{first two terms} = (A/2)\ [\ e^{cx}\ \cos(cx+\omega t) \ + \ e^{-cx}\ \cos(cx-\omega t)]$$

The (unwanted!) damped waves going to the left exactly cancel when we add the three terms and our final result comes out being

$$u(x,t) = (A/2)\, e^{-cx}\, \cos(cx-\omega t) \; - A(1/2\pi) \int_0^\infty d\rho \; e^{-\rho t} \sin(t\rho^{1/2})\rho \,/(\rho^2+\omega^2)$$

or

$$u(x,t) = (1/2) \{ A\, e^{-cx}\, \cos(cx-\omega t) \; - (A/\pi) \int_0^\infty d\rho \; e^{-\rho t} \sin(t\rho^{1/2})\rho \,/(\rho^2+\omega^2)$$

$$= (1/2) \{ \text{the answer appearing in p 242 C and which we verified using time Laplace } \}$$

So we get the right answer apart from a factor of 2 which I cannot understand. It is *as if* the s space equation were really $(\partial_t - s^2)\, U(s,t) = -2A\cos(\omega t)$, but that 2 is not there as you see above. I am at a total loss on this factor of two issue and have had to just give up on finding it. I did recheck the time Laplace calculation and it agrees with p 242 C.

(d) Redo this problem using the *spatial* Fourier Cosine Transform.

I will correctly solve this problem using the Fourier Sine Transform in Section (e). I am keeping this Section (d) for historical interest only. I had *two major misconceptions* here which caused me to flail again and again with various "Plans", and I kept ending up with contradictions.

The **first misconception** was that you can pass $(\partial_t - \partial_x^2)$ through the Fourier Cosine (or Sine) expansion integral and then conclude that the k space ODE is $(\partial_t + k^2)U(k,t) = 0$. This ODE then leads to a contradiction since then no solution U(k,t) is possible since we require U(k,0) = 0. This was the hardest bug to find, and I explain it all in "order interchange for limits.doc" Section 7 and following.

The **second misconception** was that you can only apply a Fourier Sine Transform to an odd function and you can only apply a Fourier Cosine Transform to an even function. The fact is that you can apply either of these transforms to any function you want, as long as the transform integral converges. I updated this in my "transforms.doc" stuff.

In retrospect, correcting these misconceptions was very important for me to do, they surely would have caused more disasters "down the road". So this is the way it works, you sometimes have to flail for a lot of days in order to repair your bad knowledge in some subject. Yes, having human people to talk to would perhaps have cut this short, but I think still that the school or hard knocks has its benefits.

I did derive the four "Dubnath Rules" here, one of which I will in fact use in Section (e). So that part of the work is about all that survives.

Here are some previous frustrated comments I made concerning Section (d):

> Frustrated Comment: Everything below is just scribble which I leave. I could not solve this problem using any kind of spatial Fourier related transform. I ran into endless *logic* problems, not math problems, so I don't even know what to do. The fact that the known solution p 242C is not even in x seems right at the start to rule out the Fourier Cosine transform approach that Stak is suggesting, for example. But deeper down, there is some problem with trying to transform a parabolic PDE using a spatial transform that one does not run into if you transform with a time transform as in an earlier part of this problem. I need more guidance to work in k space with a heat conduction problem, Stak has

done nothing so far. Of course in QED things are always simpler with k space propagators, so there is probably a way to interpret the question and get a reasonable answer, I just don't know what it is.

And my efforts on this problem begin here:

This is what Stakgold asked for, not part c above, but fine. I wonder how the factor of 2 issue will come out here? I am starting this section at 4 PM 4.13.11 after spending a lot of days on (c) above. [And I am now restarting here at 3 PM 4.16.11] Here again is the x-space problem,

$(\partial_t - \partial_x^2)u(x,t) = 0 \qquad 0 \le x < \infty \qquad t \ge 0$

$u(x,0) = 0$ boundary condition BC (a.e. , true that u(0,0) = A by next line)
$u(0,t) = A\cos(\omega t)$ initial condition IC

And here is my standard issue Fourier Cosine transform for x on the interval $(0,\infty)$ (because that is our region of interest in this problem!)

$f(z) = \sqrt{2/\pi} \int_0^\infty dk \cos(kz) f^k$ // expansion

$f^k = \sqrt{2/\pi} \int_0^\infty dz \cos(kz) f(z)$ // projection

$\int_0^\infty dz \cos(kz) \cos(k'z) = (\pi/2)\delta(k-k')$ // orthogonality

$\int_0^\infty dk \cos(kz) \cos(kz') = (\pi/2)\delta(z-z')$ // completeness

Plan A: Try Fourier Cosine. So let's expand

$u(x,t) = \sqrt{2/\pi} \int_0^\infty dk \cos(kx) U(k,t) \qquad x \ge 0$

We insert this into our PDE and get [See "essay on Stak heat k space" for why this next step is wrong!]

$(\partial_t - \partial_x^2)\, u(x,t) = \sqrt{2/\pi} \int_0^\infty dk \cos(kx) (\partial_t + k^2)U(k,t) = 0$

and we conclude from completeness of the EF's that

$(\partial_t + k^2)U(k,t) = 0 \qquad (*)$

Our initial condition is

$U(k,t) = \sqrt{2/\pi} \int_0^\infty dx \cos(kx) u(x,t) \quad \Rightarrow \quad U(k,0) = 0$ since $u(x,0) = 0$

But right here we are Dead in the Water as follows. A solution to (*) is $U(k,t) = B\exp(-k^2 t)/k$ and since this is a first order ODE, this is the only linearly independent solution. If $U(k,0) = 0$ then $B = 0$, except possibly at $k = \infty$. So we then end up with $U(k,t) = 0$ as our only solution and so $u(x,t) = 0$ as well.

Plan B: Try Fourier Sine. Suppose we try a Fourier sine transform? Then we will get

$$U(k,t) = \sqrt{2/\pi} \int_0^\infty dx \sin(kx)\, u(x,t) \qquad \Rightarrow \qquad U(k,0) = 0 \quad \text{since} \quad u(x,0) = 0$$

But this leads to the exact same Dead in the Water problem.

Plan C: Try cosine transform again, two Debnath Rules. Let's try to mimic the Laplace method done in the previous section. There we applied the Laplace transform to each term in the heat equation, so here let's try applying the Fourier Cosine transform to each term:

$$(\partial_t - \partial_x^2) u(x,t) = 0$$

$$\sqrt{2/\pi} \int_0^\infty dx \cos(kx) \left[\partial_x^2 u(x,t) \right] = ?$$

In the Laplace case we just looked this up as a general thing in a Laplace table, which we could derive ourselves from parts integrations. So here let's try parts integrations. Here is the first one:

$$I \equiv \int_0^\infty dx \cos(kx) \left[\partial_x^2 u(x,t) \right] = - \int_0^\infty dx \left[\partial_x \cos(kx) \right] \left[\partial_x u(x,t) \right] + \{ \cos(kx) \left[\partial_x u(x,t) \right] \} |^{x=\infty}_0$$

$$= - J - \left[\partial_x u(x,t) \right] |^{x=0}$$

But $\left[\partial_x u(x,t) \right] |^{x=0}$ is the same thing I did not know in the Laplace problem!!! But let's blindly continue, maybe it will cancel out. We then have a second parts:

$$J \equiv \int_0^\infty dx \left[\partial_x \cos(kx) \right] \left[\partial_x u(x,t) \right] = - \int_0^\infty dx \left[\partial_x^2 \cos(kx) \right] u(x,t) + \{ \left[\partial_x \cos(kx) \right] u(x,t) \} |^{x=\infty}_0$$

$$= -K - \left[\partial_x \cos(kx) \right] |^{x=0}\, u(0,t) = -K \qquad \text{// since } \sin(kx) = 0.$$

So no it does not cancel. [It is true, Fourier cosine is not going to work in this problem.]

[**Let's now just follow through** here. I have just shown above that

$$I = K - \left[\partial_x u(x,t) \right] |^{x=0} = \int_0^\infty dx \left[\partial_x^2 \cos(kx) \right] u(x,t) - \left[\partial_x u(x,t) \right] |^{x=0}$$

$$= - k^2 \int_0^\infty dx \cos(kx)\, u(x,t) - \left[\partial_x u(x,t) \right] |^{x=0}$$

$$= -k^2\, U(k,t) - [\, \partial_x u(x,t)\,]|^{x=0}$$

or

$$\int_0^\infty dx\, \cos(kx)\, [\, \partial_x^2 u(x,t)\,] = -k^2\, U(k,t) - [\, \partial_x u(x,t)\,]|^{x=0}$$

or

$$F_c\,[\partial_x^2 u(x,t)\,] = -k^2\, F_c[u(x,t)\,] - [\, \partial_x u(x,t)\,]|^{x=0}$$

where $F_c = 1$ * integral is my normalization. If you use $F_{c1} = \sqrt{2/\pi}$ *integral normalization, then of course you will have $F_{c1} = \sqrt{2/\pi}\, F_c$. If you then multiply the above line through by $\sqrt{2/\pi}$ you get

$$F_{c1}\,[\partial_x^2 u(x,t)\,] = -k^2\, F_{c1}[u(x,t)\,] - \sqrt{2/\pi}\, [\, \partial_x u(x,t)\,]|^{x=0}$$

and this is in agreement with my **Debnath** quote

$$\mathscr{F}_c\{f''(x)\} = -k^2\ F_c(k) - \sqrt{\frac{2}{\pi}}\, f'(0), \qquad (2.14.4)$$

So at least I have derived this rule properly! While we're at it, my single-parts result above said

$$\int_0^\infty dx\, \cos(kx)\, [\, \partial_x^2 u(x,t)\,] = \int_0^\infty dx\, \cos(kx)\, \partial_x\, [\partial_x u(x,t)\,]$$

$$= -\int_0^\infty dx\, [\partial_x \cos(kx)]\, [\, \partial_x u(x,t)\,] - [\, \partial_x u(x,t)\,]|^{x=0}$$

If I replace $\partial_x u(x,t) \rightarrow u(x,t)$ everywhere, this says

$$\int_0^\infty dx\, \cos(kx)\, [\, \partial_x u(x,t)\,] = -\int_0^\infty dx\, [\partial_x \cos(kx)]\, [u(x,t)\,] - [u(x,t)\,]|^{x=0}$$

$$= +k \int_0^\infty dx\, \sin(kx)\, u(x,t) - [u(x,t)\,]|^{x=0}$$

or

$$F_c\,[\partial_x u(x,t)\,] = +k\ F_s[u(x,t)\,] - [u(x,t)\,]|^{x=0}$$

or

$$F_{c1}\,[\partial_x u(x,t)\,] = +k\ F_{s1}[u(x,t)\,] - \sqrt{2/\pi}\, [u(x,t)\,]|^{x=0}$$

and this last is seen to be in agreement with another Debnath rule

$$\mathscr{F}_c\{f'(x)\} = k\ F_s(k) - \sqrt{\frac{2}{\pi}}\, f(0), \qquad (2.14.3)$$

and be sure to notice that this line involves both the cosine and sine transforms, whereas the ∂_x^2 line only had cosine ones.

Now since the cosine transform of ∂_x^2 brings in $[\, \partial_x u(x,t)\,]|^{x=0}$ which is a piece of information we *don't have* in this problem, let's try the sine transform and see what it does! Paste and edit:]

Plan D. Try sine transform again, two more Debnath Rules. Let's try to mimic the Laplace method done in the previous section. There we applied the Laplace transform to each term in the heat equation, so here let's try applying the **Fourier Sine transform** to each term:

$$(\partial_t - \partial_x^2)u(x,t) = 0$$

$$\sqrt{2/\pi} \int_0^\infty dx \sin(kx) [\, \partial_x^2 u(x,t) \,] = ?$$

In the Laplace case we just looked this up as a general thing in a Laplace table, which we could derive ourselves from parts integrations. So here let's try parts integrations. Here is the first one:

$$I \equiv \int_0^\infty dx \sin(kx) [\, \partial_x^2 u(x,t) \,] = - \int_0^\infty dx [\partial_x \sin(kx)] [\, \partial_x u(x,t) \,] + \{ \sin(kx) [\, \partial_x u(x,t) \,] \} |^{x=\infty}_0$$

$$= - J \qquad \text{// parts = 0 because now } \sin(k0) = 0, \text{ so } [\, \partial_x u(x,t) \,]|^{x=0} \text{ does not appear!}$$

We then have a second parts:

$$J \equiv \int_0^\infty dx [\partial_x \sin(kx)] [\, \partial_x u(x,t) \,] = - \int_0^\infty dx [\partial_x^2 \sin(kx)] u(x,t) + \{ [\partial_x \sin(kx)] u(x,t) \} |^{x=\infty}_0$$

$$= -K + \{ k \cos(kx) u(x,t) \} |^{x=\infty}_0 = -K - k u(0,t)$$

So this looks a little more hopeful. We then have

$$I = - J = - [-K - k u(0,t)] = K + k u(0,t)$$

where

$$K = \int_0^\infty dx [\partial_x^2 \sin(kx)] u(x,t) = - k^2 \int_0^\infty dx \sin(kx) u(x,t)$$

Therefore we have shown that

$$\sqrt{2/\pi} \int_0^\infty dx \sin(kx) [\, \partial_x^2 u(x,t) \,] = \sqrt{2/\pi} \{ - k^2 \int_0^\infty dx \sin(kx) u(x,t) + k u(0,t) \}$$

But of course

$$f(x) = \sqrt{2/\pi} \int_0^\infty dk \sin(kz) F(k) \qquad \text{// expansion}$$

$$F(k) = \sqrt{2/\pi} \int_0^\infty dx \sin(kx) f(x) \qquad \text{// projection}$$

so that

$$\sqrt{2/\pi} \int_0^\infty dx \sin(kx) u(x,t) = U(k,t)$$

so we have then shown that

$$\sqrt{2/\pi} \int_0^\infty dx \sin(kx) [\partial_x^2 u(x,t)] = \{ -k^2 \sqrt{2/\pi} \int_0^\infty dx \sin(kx) u(x,t) + \sqrt{2/\pi}\, k\, u(0,t) \}$$

$$= -k^2 U(k,t) + \sqrt{2/\pi}\, k\, u(0,t)$$

Now it remains to apply our Fourier Sine Transform to the first term in our homo heat equation

$$\sqrt{2/\pi} \int_0^\infty dx \sin(kx) [\partial_t u(x,t)] = \partial_t \sqrt{2/\pi} \int_0^\infty dx \sin(kx) [u(x,t)] = \partial_t U(k,t)$$

Then our homo equation reads, after application of the Fourier Sine Transform:

$$\partial_t U(k,t) - [-k^2 U(k,t) + \sqrt{2/\pi}\, k\, u(0,t)] = 0$$

$$(\partial_t + k^2) U(k,t) = \sqrt{2/\pi}\, k\, u(0,t)$$ // This is the correct result !!!! But I don't use it here.

But we have a paradox as follows. Suppose as in Part A we said

$$u(x,t) = \sqrt{2/\pi} \int_0^\infty dk \sin(kx) U(k,t)$$

$$(\partial_t - \partial_x^2) u(x,t) = \sqrt{2/\pi} \int_0^\infty dk \sin(kx) (\partial_t + k^2) U(k,t) = 0$$

Then I appeal to the completeness of the sin(kx) to get

$$(\partial_t + k^2) U(k,t) = 0$$

Obviously I am missing something here. [I was indeed. the above equation is wrong, see "an essay on Stak heat k space".]

Moreover, looking at p 242 C, if I take x → -x, the integral term is odd in x, but the first term is neither even or odd in x. Thus the solution is neither even nor odd, so neither the Fourier Sine nor Cosine transform is justified! [Another wrong statement !] Only the full Fourier Transform would be justified.

[**Let's now just follow through** here with the sine stuff. I have just shown above that

$$I = \int_0^\infty dx \sin(kx) [\partial_x^2 u(x,t)] = -k^2 \int_0^\infty dx \sin(kx) u(x,t) + k\, u(0,t)$$

or

$F_s [\partial_x^2 u(x,t)] = -k^2 F_s [u(x,t)] + k\, u(0,t)$

or

$F_{s1} [\partial_x^2 u(x,t)] = -k^2 F_{s1} [u(x,t)] + \sqrt{2/\pi}\, k\, u(0,t)$

which is in agreement with Debnath

$$\mathscr{F}_s\{f''(x)\} = -k^2\ F_s(k) + \sqrt{\frac{2}{\pi}}\, k\, f(0). \qquad (2.14.6)$$

The first parts result above was

$$\int_0^\infty dx\, \sin(kx) [\partial_x^2 u(x,t)] = - \int_0^\infty dx\, [\partial_x \sin(kx)] [\partial_x u(x,t)]$$

or taking $\partial_x u(x,t) \rightarrow u(x,t)$ as we did above

$$\int_0^\infty dx\, \sin(kx) [\partial_x u(x,t)] = - \int_0^\infty dx\, [\partial_x \sin(kx)] [u(x,t)] = -k \int_0^\infty dx\, \cos(kx)\, u(x,t)$$

or

$F_s [\partial_x u(x,t)] = -k\, F_c[u(x,t)]$

or

$F_{s1} [\partial_x u(x,t)] = -k\, F_{c1}[u(x,t)]$

which agrees with the fourth Debnath rule

$$\mathscr{F}_s\{f'(x)\} = -k\ F_c(k), \qquad (2.14.5)$$

Therefore I have derived all four Debnath rules, confirming that they do not contain typos:

$$\mathscr{F}_c\{f'(x)\} = k\ F_s(k) - \sqrt{\frac{2}{\pi}}\, f(0), \qquad (2.14.3)$$

$$\mathscr{F}_c\{f''(x)\} = -k^2\ F_c(k) - \sqrt{\frac{2}{\pi}}\, f'(0), \qquad (2.14.4)$$

$$\mathscr{F}_s\{f'(x)\} = -k\ F_c(k), \qquad (2.14.5)$$

$$\mathscr{F}_s\{f''(x)\} = -k^2\ F_s(k) + \sqrt{\frac{2}{\pi}}\, k\, f(0). \qquad (2.14.6)$$

These rules are stated in the convention that the transforms are stated symmetrically with $\sqrt{2/\pi}$ in front of each integral. If you put constant 1 in front of the dx integral (and then $2/\pi$ in front of the dk one), the three factors $\sqrt{2/\pi}$ seen in the above rules are all replaced by 1. For some reason **Schaum** does not state these four rules!!! Nor do they appear in **A&S 2010.** Nor do they appear even in **ET I.**

But **GR7** has lots of data on the transforms of the derivatives on page 1121 in its Integral Transforms section:

$$F_s(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\xi x)\, dx \quad \text{and} \quad F_c(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\xi x)\, dx.$$

17.32 Basic properties of the Fourier sine and cosine transforms

1. For a and b arbitrary constants,

$$\mathcal{F}_s\left[af(x) + bg(x)\right] = aF_s(\xi) + bG_s(\xi)$$

and

$$\mathcal{F}_c\left[af(x) + bg(x)\right] = aF_c(\xi) + bG_c(\xi) \qquad \text{(linearity)}$$

2. If $\lim_{x\to\infty} f^{(r-1)}(x) = 0$ and $\lim_{x\to\infty} \sqrt{\frac{2}{\pi}} f^{(r-1)}(x) = a_{r-1}$, then denoting the Fourier sine and cosine transforms of $f^{(r)}(x)$ by ${F_s}^{(r)}$ and ${F_c}^{(r)}$, respectively,

(i) $${F_c}^{(r)}(\xi) = -a_{r-1} + \xi {F_s}^{(r-1)}.$$

(ii) $${F_s}^{(r)}(\xi) = -\xi {F_c}^{(r-1)}(\xi),$$

(iii) $${F_c}^{(2r)}(\xi) = -\sum_{n=0}^{r-1} (-1)^n a_{2r-2n-1} \xi^{2n} + (-1)^r \xi^{2n} F_c(\xi),$$

(iv) $${F_c}^{(2r+1)}(\xi) = -\sum_{n=0}^{r-1}{}' (-1)^n a_{2r-2n} \xi^{2n} + (-1)^r \xi^{2r+1} F_s(\xi),$$

(v) $${F_s}^{(r)}(\xi) = \xi a_{r-2} - \xi^2 {F_s}^{(r-2)}(\xi),$$

(vi)[6] $${F_s}^{(2r)}(\xi) = -\sum_{n=1}^{r} (-1)^n \xi^{2n-1} a_{2r-2n} + (-1)^r \xi^{2r} F_s(\xi),$$

(vii) $${F_s}^{(2r+1)}(\xi) = -\sum_{n=1}^{r}{}' (-1)^n \xi^{2n-1} a_{2r-2n+1} + (-1)^{r+1}\, \xi^{2r+1} F_c(\xi).$$ SN 28

where the SN reference Sneddon Fourier Transforms. Notice how the $\sqrt{2/\pi}$ is "handled" here by putting it into the a_{r-1} and notice the typo as well. That should be $\lim_{x\to 0}$ in the definition of a_{r-1}. So here you see those "conditions" that Debnath stated, which are basically that things go away at ∞.

I will send Dan an email on this right now. DONE. I also mentioned the discontinuity thing.

Plan E. Try the full Fourier Transform in the same manner. [This section is pretty useless, I leave it though.] But that requires (-∞,∞) for the x integral. So this brings up the question: how do you do Fourier analysis just on (0,∞) where you don't want to rule out either x^2 or x^3 say as a component of the solution? I am puzzled as to why this all seems so difficult and strange? Well just try it anyway and see what happens.

$f(x) = \int_{-\infty}^{\infty} dk\, f^k\, e^{-ikx}$ // expansion

$f^{k} = (1/2\pi) \int_{-\infty}^{\infty} dx\, f(x)\, e^{+ikx}$ // projection

$\int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = \int_{-\infty}^{\infty} dx \cos[(k-k')x] = 2\pi\, \delta(k-k')$ // orthogonality

$\int_{-\infty}^{\infty} dk\, e^{-ik(x'-x)} = \int_{-\infty}^{\infty} dk \cos[k(x-x')] = 2\pi\, \delta(x-x')$ // completeness

So we go with our expansion this way

$u(x,t) = \int_{-\infty}^{\infty} dk\, e^{-ikx}\, U(k,t)$

Apply $(\partial_t - \partial_x^2)$ to get

$0 = \int_{-\infty}^{\infty} dk\, e^{-ikx}\, (\partial_t + k^2) U(k,t)$

Now appeal to the completeness of the e^{-ikx} to get

$(\partial_t + k^2) U(k,t) = 0$

which is what we get in all these transforms. We then have

$U(k,t) = (1/2\pi) \int_{-\infty}^{\infty} dx\, e^{+ikx}\, u(x,t)$

Now the question is: what do we use for u(x,0) ? On the right we want = 0. Perhaps on the left side we need to have the entire negative rod running at A cos(ωt) as if it were a reservoir. Then that at least gives us a model for what to do on the left. We would then get

$U(k,t) = (1/2\pi) \int_{-\infty}^{0} dx\, e^{+ikx}\, A\cos(\omega t) = (1/2\pi)\, A\cos(\omega t) \int_{-\infty}^{0} dx\, e^{+ikx} = (1/2)\, \delta(k)$?

This seems a bit contrived. Suppose instead I go for u(-x,t) = u(x,t) and contrive to make it even in this manner. Then we get U(k,0) = 0 and we are then back to our old problem that this makes U(k,t) = 0. So I need to get over this hump in the road.

It is true that if we consider "distributional solutions" of homo $(\partial_t + k^2) f(k,t) = 0$ we could consider

$f(k,t) = \Sigma_{n=1}^{\infty} k^n\, \delta(k)$ // but this is really the same as saying f(k,t) = 0.

$f(k,t) = C\, \delta(k)$ // this is not the same as zero and solves the homo equation!

$(\partial_t + k^2)\, C\, \delta(k) = 0 + Ck^2\delta(k) = 0 + 0 = 0$

$f(k,t) = C\, \delta(k)/k$ would also work in this manner, but does not Fourier very well! $f(x,t) = 1/0 = \infty$.

So I suppose we could consider this possible solution of our homo equation

$$U(k,t) = B\exp(-k^2 t) + C\delta(k)$$

Then our initial condition let's say is still U(k,0) = 0, we then get $B + C\delta(k) = 0$ which says $B = -C\delta(k)$. Then our solution is this

$$U(k,t) = -C\delta(k)\exp(-k^2 t) + C\,\delta(k) = C\,\delta(k)[\,1 - \exp(-k^2 t)]$$

But again this is just zero !

Plan F: Try cosine transform yet again.

[I keep making the same $(\partial_t + k^2)U(k,t) = 0$ wrong statement!]

Our boundary condition enters the k-space fray this way

$$u(x,t) = \sqrt{2/\pi} \int_0^\infty dk\, \cos(kx)\, U(k,t)$$

$$u(0,t) = \sqrt{2/\pi} \int_0^\infty dk\, \cos(k0)\, U(k,t) = \sqrt{2/\pi} \int_0^\infty dk\, U(k,t) = A\cos(\omega t)$$

which is an "integral condition" on U(k,t). So here then seems to be our k-space problem:
[my usual error below]

$$(\partial_t + k^2)U(k,t) = 0 \qquad U(k,0) = 0 \qquad \int_0^\infty dk\, U(k,t) = A\sqrt{\pi/2}\, \cos(\omega t)$$

I am not used to this kind of BC situation, but let's try to plod ahead. But that journey ends in a hurry. The only solution to this ODE is

$$U(k,t) = B\exp(-k^2 t)/k$$

$$\partial_t U(k,t) = -kB\exp(-k^2 t)$$

and then our condition U(k,0) = 0 requires B/k = 0 and we then have NO solution. Secondly, we seem to have a first order ODE with two BC's which does not sound healthy. Thirdly, how did we know that u(x,t) was even so we could use the cos(kx) expansion? Perhaps it also has sin(kx) contributions.

Consider again our condition above

$$A\cos(\omega t) = \sqrt{2/\pi} \int_0^\infty dk\, U(k,t)$$

Do I know how to solve this integral equation? I could try U(k,t) = g(k) A cos(ωt), then we have

$$A\cos(\omega t) = \sqrt{2/\pi}\ A\cos(\omega t) \int_0^\infty dk\ g(k)$$

$$1 = \sqrt{2/\pi} \int_0^\infty dk\ g(k)$$

There are lots of functions g(k) whose integral would gives this, so there are an infinite number of solutions of this type to my "integral equation". But let's continue and say

$$U(k,t) = g(k)\ A\cos(\omega t) \qquad \text{where} \qquad 1 = \sqrt{2/\pi} \int_0^\infty dk\ g(k)$$

$$U(k,0) = g(k)\ A$$

Then suppose we have our same ODE in k space [My usual repeated error.]

$$(\partial_t + k^2)U(k,t) = 0 \qquad \Rightarrow \qquad U(k,t) = B\exp(-tk^2)$$

Then we can have conflicting facts

$$U(k,t) = g(k)\ A\cos(\omega t) = B\exp(-tk^2)$$

The time dependences don't match. Well suppose we try our second form in the first equation

$$A\cos(\omega t) = \sqrt{2/\pi} \int_0^\infty dk\ U(k,t)$$

$$= \sqrt{2/\pi} \int_0^\infty dk\ B\exp(-tk^2) = \sqrt{2/\pi}\ B\ (1/2)\sqrt{\pi}\ t^{-1/2}$$

and again we get conflicting time dependences. So this Plan gets nowhere.

Plan G : We know our solution is p 242C. Suppose we "evenize" this and say

$$u(x,t) = A\exp(-|x|\sqrt{\omega/2})\cos[\ \omega t - |x|\sqrt{\omega/2}\ ] - (A/\pi) \int_0^\infty d\rho\ e^{-\rho t}\sin(|x|\sqrt{\rho})\ (\rho/(\rho^2+\omega^2))$$

Then we can at least ask this question: What does this solution look like in full Fourier Integral k space? I think it at least converges. The second term would involve the following

$$(1/2\pi) \int_{-\infty}^\infty dx\ \sin(|x|\sqrt{\rho})\ e^{+ikx} = (1/2\pi) \int_{-\infty}^\infty dx\ \sin(|x|\sqrt{\rho})[\ \cos(kx)+i\sin(kx)\ ]$$

$$= (1/2\pi) \int_{-\infty}^{\infty} dx \sin(|x|\sqrt{\rho}\,) \cos(kx) = (1/\pi) \int_{0}^{\infty} dx \sin(x\sqrt{\rho}\,) \cos(kx)$$

$$= (1/\pi)(1/2) \int_{0}^{\infty} dx \, [\, \sin(\sqrt{\rho} - k)x + \sin(\sqrt{\rho} + k)x \,] = 0 + 0 = 0$$

So I have just proven the interesting fact that the Fourier Transform of an "evenized" sine function is 0.

On the other hand, if we let $\rho = k^2$ and $d\rho = 2kdk$ we can write

$$\int_{0}^{\infty} d\rho \, e^{-\rho t} \sin(|x|\sqrt{\rho}\,) \, (\rho/(\rho^2 + \omega^2)) = \int_{0}^{\infty} 2kdk \exp(-k^2 t) \sin(|x|k) \, (k^2/(k^4 + \omega^2)) \equiv u(x,t)$$

which we can interpret as a Fourier Sine Transform and we have

$$u(x,t) = 2 \int_{0}^{\infty} dk \, k \exp(-k^2 t) \sin(|x|k) \, (k^2/(k^4 + \omega^2))$$

Then

$$U(k,t) = 2k \exp(-k^2 t) \, (k^2/(k^4 + \omega^2))$$

So I have shown that the second term in p 242C is the Fourier Sine Transform of the above line, apart from constants. [I think this notion will arise again when I eventually do this thing right below.]

Comment: Stakgold's question is illogical, so I cannot solve his problem. He says to use a Fourier Cosine Transform to obtain the same solution which solution is this,

$$u(x,t) = A\exp(-x\sqrt{\omega/2}\,) \cos[\, \omega t - x\sqrt{\omega/2}\,] - (A/\pi) \int_{0}^{\infty} d\rho \, e^{-\rho t} \sin(x\sqrt{\rho}\,) \, (\rho/(\rho^2 + \omega^2))$$

$$= A\exp(-x\sqrt{\omega/2}\,) \cos[\, \omega t - x\sqrt{\omega/2}\,] - (2A/\pi) \int_{0}^{\infty} dk \, k \exp(-k^2 t) \sin(kx) \, (k^2/(k^4 + \omega^2))$$

The second term is odd in x, while the first term is neither odd nor even. Thus, you CANNOT write this solution u(x,t) as a Fourier Sine or as a Fourier Cosine transform of some U(k,t). [This is wrong and I had a misconception here, which is now fixed in transforms.doc]

Plan H. Try full Fourier Transform again. The only possible thing I can think of that would give some reasonable interpretation of Stakgold's question is to do a full Fourier transform. We might then complexify our solution above and write

$$v(x,t) = A\exp(-x\sqrt{\omega/2}\,) \exp\{-i[\, \omega t - x\sqrt{\omega/2}\,]\} - (2A/\pi)(i) \int_{0}^{\infty} dk \, k \exp(-k^2 t) \, e^{-ikx} \, (k^2/(k^4 + \omega^2))$$

where then $u(x,t) = \mathrm{Re}[\, v(x,t)]$. We might then ask about V(k,t). Data:

$$f(x) = \int_{-\infty}^{\infty} dk\, f^{k}\, e^{-ikx} \qquad \text{// expansion}$$

$$f^{k} = (1/2\pi) \int_{-\infty}^{\infty} dx\, f(x)\, e^{+ikx} \qquad \text{// projection}$$

$$\int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = \int_{-\infty}^{\infty} dx\, \cos[(k-k')x] = 2\pi\, \delta(k-k') \qquad \text{// orthogonality}$$

$$\int_{-\infty}^{\infty} dk\, e^{-ik(x'-x)} = \int_{-\infty}^{\infty} dk\, \cos[k(x-x')] = 2\pi\, \delta(x-x') \qquad \text{// completeness}$$

This would involve for example (let $\alpha = \sqrt{\omega/2}$)

$$(1/2\pi) \int_{-\infty}^{\infty} dx\, e^{+ikx} \{ A\exp(-\alpha x) \exp\{-i [\, \omega t - \alpha x \,] \} \}$$

$$= e^{-i\omega t} (1/2\pi) \int_{-\infty}^{\infty} dx\, e^{+ikx} \{ A\exp(-\alpha x) \exp[i\alpha x \,] \}$$

[STOP. this integral diverges at the $-\infty$ end since $\alpha = \sqrt{\omega/2} > 0$, so go no further! We could try it with an evenized v(x,t), but I will be doing the right solution soon, so don't bother.]

Plan I. Is there some way to do the transform on g rather than on u ?

The general solution for the half-rod problem is given by 7.14 where you would install the half-rod propagator which is given by 7.23. As expressed there, this g is in fact odd in x_0, but that does not fit very well with the Fourier cosine transform which is for even functions.

Plan J: Imagine heating a full infinite rod at the center x = 0, but only LOOK AT the right side. In this case, we would use the full rod propagator of page 200 A. I can see how the heating pattern would be symmetric about x = 0 since the two patterns will just be mirror images. [I think this is in fact an OK way to approach this problem, where you treat the heat source as a q instead of on h!]

Point of Confusion: Consider $y = x^3$ in the region $x > 0$. Imagine reflecting this on the negative side, so we then have the curve $y = |x|^3$ really. So on the right it is $y = x^3 > 0$ and on the left it is $y = -x^3 > 0$. This two region function is certainly even in x, since $f(-x) = |-x|^3 = |x|^3 = f(x)$. Can we expand this function using a Fourier Cosine transform?

$$f(x) = \sqrt{2/\pi} \int_{0}^{\infty} dk\, \cos(kx)\, f^{k} \qquad \text{// expansion}$$

$$f^{k} = \sqrt{2/\pi} \int_{0}^{\infty} dx\, \cos(kx)\, f(x) \qquad \text{// projection}$$

$$\int_{0}^{\infty} dz\, \cos(kx) \cos(k'x) = (\pi/2)\delta(k-k') \qquad \text{// orthogonality}$$

$$\int_{0}^{\infty} dk\, \cos(kx) \cos(kx') = (\pi/2)\delta(x-x') \qquad \text{// completeness}$$

Well, for $f(x) = |x|^3$ we will have a non-convergent projection. But in our actual problem where the temperature propagates "slowly" and has expo decay, maybe it does converge. So in our contrived problem here we have u(x,t) that is even in x, and which we hope decays exponentially so we can actually do a Fourier cosine projection. My "point of confusion" here was really this: the function $f(x) = x^3$ is odd in x, so you wonder how you could use a Fourier Cosine transform. The resolution is that $f(x) = |x|^3$ is even!

So let's continue on our infinite frozen string driven in the middle. In 7.14, the entire solution is given by

$$u(x,t) = \int_0^t dt_0 \int_{-\infty}^{\infty} dx_0 \; g(x|x_0)\, q(t_0,x_0) \quad (*) \qquad q(t_0,x_0) = \delta(x_0)\, A \cos(\omega t_0)$$

$$g(x|x_0) = \text{page 198 7.10}$$

Then it would seem that our answer is

$$u(x,t) = \int_0^t dt_0 \int_{-\infty}^{\infty} dx_0 \; g(x,t|x_0,t_0)\, \delta(x_0)\, A \cos(\omega t_0)$$

$$= A \int_0^t dt_0 \; g(x,t|0,t_0) \cos(\omega t_0)$$

$$= A \int_0^t dt_0 \; H(t-t_0) \exp(-|x|^2/[4(t-t_0)]\,)\, (\,4\pi(t-t_0)\,)^{-1/2} \cos(\omega t_0)$$

Now

$$\int_0^t dt_0 \; H(t-t_0) = \int_0^t dt_0 \; H(t > t_0) = \int_0^t dt_0 \; H(t_0 < t) = \int_0^t dt_0 \;, \text{ no constraint}$$

so it seems that our answer is then

$$u(x,t) = (A/\sqrt{4\pi}\,) \int_0^t dt_0 \exp(-|x|^2/[4(t-t_0)]\,)\, (t-t_0)^{-1/2} \cos(\omega t_0)$$

The claim is that this must be the same as p 242 C, but that would take a long time to prove because it is somewhat of an ugly integral. It is not a Fourier Cosine Transform due to the upper endpoint.

So this does not involve the Fourier cosine transform anywhere! But, we could apply it our starting equation (*) above, which is roughly the same as applying it to the very end here. We would have to look up

$$F(k) = \int_0^{\infty} dx \cos(kx)\, f(x) = \int_0^{\infty} dx \cos(kx) \exp(-|x|^2/[4(t-t_0)]\,)$$

where I take the projection constant as 1 to match Schaum p 176. define

$$b = 1/[4(t-t_0)] \qquad 1/b = 4(t-t_0) \qquad 1/4b = (t-t_0) \qquad b^{-1/2} = 2\sqrt{t-t_0}$$

$$F(k) = \int_0^\infty dx \cos(kx) \exp(-bx^2) = (1/2)\sqrt{\pi/b}\ \exp(-k^2/4b)$$

$$= (1/2)\sqrt{\pi}\ 2\sqrt{t\text{-}t_0}\ \exp(-k^2(t\text{-}t_0)) = \sqrt{\pi}\sqrt{t\text{-}t_0}\ \exp(-k^2(t\text{-}t_0))$$

Then we get above that

$$U(k,t) = (A/\sqrt{4\pi}) \int_0^t dt_0 \{\sqrt{\pi}\sqrt{t\text{-}t_0}\ \exp(-k^2(t\text{-}t_0))\} (t\text{-}t_0)^{-1/2} \cos(\omega t_0)$$

$$= (A/2) \int_0^t dt_0\ \exp(-k^2(t\text{-}t_0))\ \cos(\omega t_0)$$

$$= (A/2)\ \exp(-k^2 t) \int_0^t dt_0 \exp(k^2 t_0)) \cos(\omega t_0)$$

$$= (A/2)\ \exp(-k^2 t) \int_0^t dx \exp(k^2 x) \cos(\omega x)$$

This integral Maple says is this

```
restart;
Int(exp(k^2*x)*cos(w*x),x = 0..t);
```

$$\int_0^t e^{(k^2 x)} \cos(w x)\, dx$$

```
int(exp(k^2*x)*cos(w*x),x = 0..t);
```

$$\frac{e^{(k^2 t)} k^2 \cos(w t) + e^{(k^2 t)} w \sin(w t) - k^2}{k^4 + w^2}$$

So we then get

$$U(k,t) = (A/2)\ [\ k^2(\cos(\omega t) - 1) + \omega\sin(\omega t)\ ]/(k^4+\omega^2)$$

This looks convergent for an inverse transform, and we get

$$u(x,t) = (2/\pi) \int_0^\infty dk \cos(kx)\ \{(A/2)\ [\ k^2(\cos(\omega t) - 1) + \omega\sin(\omega t)\ ]/(k^4+\omega^2)\ \}$$

$$= (A/\pi) \int_0^\infty dk \cos(kx)\ \{\ [\ k^2(\cos(\omega t) - 1) + \omega\sin(\omega t)\ ]/(k^4+\omega^2)\ \}$$

$T_1 = (A/\pi)(\cos(\omega t) - 1) \int_0^\infty dk \cos(kx)\, k^2/(k^4+\omega^2)$

$T_2 = (A/\pi)\,\omega\sin(\omega t) \int_0^\infty dk \cos(kx)\, 1/(k^4+\omega^2)$

But these are not in my Schaum short list. Bateman has this stuff however.

Algebraic functions (cont'd)

	$f(x)$	$g(y) = \int_0^\infty f(x)\cos(xy)dx \qquad y > 0$
(19)	$(x^4+a^4)^{-1} \qquad \lvert\arg a\rvert < \pi/4$	$\frac{1}{2}\pi a^{-3}\exp(-2^{-\frac{1}{2}}ay) \times \sin(\pi/4+2^{-\frac{1}{2}}ay)$

for T_2

Now for T_1 we have to use this Bateman rule with n = 1 to get our extra k^2 factor

(5)	$x^{2n} f(x)$	$(-1)^n \frac{d^{2n} g(y)}{dy^{2n}}$

So we then have where $\alpha^4 = \omega^2$ so that $\alpha^2 = \omega$ and $\alpha = \sqrt{\omega}$

$$\int_0^\infty dk \cos(kx)\, 1/(k^4+\omega^2) = (\pi/2)\,\alpha^{-3}\exp(-\alpha x/\sqrt{2})\sin(\pi/4 + \alpha x/\sqrt{2})$$

$$\int_0^\infty dk \cos(kx)\, k^2/(k^4+\omega^2) = (-1)\,\partial_x^2 \{ (\pi/2)\,\alpha^{-3}\exp(-\alpha x/\sqrt{2})\sin(\pi/4 + \alpha x/\sqrt{2}) \}$$

So here is the very strange answer I have obtained:

$u(x,t) = T_1 + T_2$

$$= (A/\pi)(\cos(\omega t) - 1)(-1)\,\partial_x^2 \{ (\pi/2)\,\alpha^{-3}\exp(-\alpha x/\sqrt{2})\sin(\pi/4 + \alpha x/\sqrt{2}) \}$$
$$+ (A/\pi)\,\omega\sin(\omega t)(\pi/2)\,\alpha^{-3}\exp(-\alpha x/\sqrt{2})\sin(\pi/4 + \alpha x/\sqrt{2}) \qquad \alpha = \sqrt{\omega}$$

$$= (A/2)(\cos(\omega t) - 1)(-1)\,\partial_x^2 \{ \alpha^{-3}\exp(-\alpha x/\sqrt{2})\sin(\pi/4 + \alpha x/\sqrt{2}) \}$$
$$+ (A/2)\sin(\omega t)\,\alpha^{-1}\exp(-\alpha x/\sqrt{2})\sin(\pi/4 + \alpha x/\sqrt{2}) \qquad \alpha = \sqrt{\omega}$$

Note that $\alpha/\sqrt{2} = \sqrt{\omega/2}$ so we seem to have an exponential decay that matches the first term in p 242C. But where is the p 242C integral going to come from? So something is not quite right. [Probably this could be made to fly, but I will not attempt repairs since the correct solution lies ahead.]

Point of Confusion Revisited. Suppose we have some f(x) like this

[Here is where I started to realize that you can Fourier Cosine Transform a non-even function.]

$$f(x) = A\sin(kx)\exp(-x^2) + B\sin^2(kx)\exp(-x^2)$$

which I have tried to make convergent, and which has an odd piece plus an even piece. What is the Fourier Cosine Projection of this odd first piece?

$$\int_0^\infty dx \cos(kx)\sin(kx)\exp(-x^2) = (1/2)\int_0^\infty dx \sin(2kx)\exp(-x^2)$$

Maple tells us this integral

```
int(sin(2*k*x)*exp(-x^2),x=0..infinity);
```

$$-\frac{1}{2}\frac{I\sqrt{\pi}\,\mathrm{erf}(Ik)}{e^{(k^2)}}$$

which is real, despite looking imaginary (there are two I's here, and somewhere I showed that erf(ix) = purely imaginary). In any event, we can see that the integral is real, so we get

$$F(k) = A\,(1/2)\exp(-k^2)\,(-1/2)\sqrt{\pi}\;i\,\mathrm{erf}(ik) \qquad // \; = A\,(1/2)\exp(-k^2)\,(1/2)\sqrt{\pi}\;\mathbf{erfi(k)}\;\;!$$

My point is simply that the Fourier Cosine transform of an odd f(x) term need not be zero! Somewhere I got the idea that you could only apply this transform to f(x) that had only even pieces.

But look at the recovery formula:

$$f(x) = \int_0^\infty dk \cos(kx)\,F(k)$$

You want to say that f(x) is even because cos(kx) is, but we don't have $x < 0$ in this transform!

Another example: The Fourier Cosine transform of e^{-bx} is $b/(k^2+b^2)$. Obviously this transform works both ways. And obviously $e^{-bx} = \Sigma_{n=0}^{\infty} (-bx)^2/n!$ has an odd and an even piece.

(e) Redo this problem using the *spatial* Fourier Sine Transform.

Hopefully all my woes are encapsulated in Section (d) above, and my solution here will "go through" without incident. [Well, there was one "incident": Maple made the first error I have ever seen it make.] We will use the Schaum convention so we can use the Schaum tables:

$$U(k,t) = \int_0^\infty dx\, \sin(kx)\, u(x,t) \qquad \text{// projection}$$

$$u(x,t) = (2/\pi) \int_0^\infty dk\, \sin(kx)\, U(k,t) \qquad \text{// expansion}$$

Statement of Problem: We apply boundary schedule h(t) = Acos(ωt) to the x=0 end of a frozen (u=0) half rod, and we want to know u(x,t). So:

$$(\partial_t - \partial_x^2)u(x,t) = 0 \qquad u(x,0) = 0 \text{ (frozen)} \qquad u(0,t) = h(t) \text{ (schedule)}$$

We know the Debnath rule from Section (d) Plan D above which says (Schaum normalization)

$$F_s [\, \partial_x^2\, u(x,t)\,] = -k^2 F_s [u(x,t)\,] + k\, u(0,t)$$

Since the cosine transform requires knowing u'(0,t) which we don't know, we *must* use the sine transform. I assume this is a typo on Stak's part, he meant the sine transform I think and I will mark that as an error in the book.

So applying the above rule our PDE becomes this ODE

$$(\partial_t + k^2)\, F_s [u(x,t)\,] = k\, h(t)$$

or

$$(\partial_t + k^2)\, U(k,t) = kA\cos(\omega t)$$

Meanwhile we have

$$U(k,0) = \int_0^\infty dx\, \sin(kx)\, u(x,0) = \int_0^\infty dx\, \sin(kx)\, 0 = 0$$

so our ODE system is then this

$$(\partial_t + k^2)\, U(k,t) = kA\cos(\omega t) \qquad t>0,\ x>0 \qquad U(k,0) = 0$$

The most general homogeneous solution to this equation is

$$U_{homo}(k,t) = c \exp(-k^2 t)$$

For a particular solution we try

$$U_{part}(k,t) = \alpha\sin(\omega t) + \beta\cos(\omega t)$$

$$(\partial_t + k^2)\, U_{part}(k,t) = \alpha\omega\cos(\omega t) - \beta\omega\sin(\omega t) + k^2\,\alpha\sin(\omega t) + k^2\,\beta\cos(\omega t)$$

$$= (\alpha\omega + k^2\beta)\cos(\omega t) + (k^2\alpha - \beta\omega)\sin(\omega t)$$

But this has to equal kAcos(ωt) so we have

$$(\alpha\omega + k^2\beta) = kA$$
$$(k^2\alpha - \beta\omega) = 0 \quad \Rightarrow \quad \beta\omega = \alpha k^2 \quad \beta = \alpha k^2/\omega \quad k^2\beta = \alpha k^4/\omega$$

Then first equation gives

$$\alpha\omega + \alpha k^4/\omega = kA$$
$$\alpha\omega^2 + \alpha k^4 = kA\omega$$
$$\alpha(\omega^2 + k^4) = A\omega k \quad \Rightarrow \quad \alpha = A\omega k/(k^4+\omega^2) \quad \beta = Ak^3/(k^4+\omega^2)$$

Our solution is then

$$U(k,t) = c\,\exp(-k^2 t) + \alpha\sin(\omega t) + \beta\cos(\omega t)$$

and then applying our BC we get

$$U(k,0) = c\,1 + 0 + \beta\,1 = c + \beta = 0 \quad \Rightarrow c = -\beta$$

So then our full k-space solution to this problem is:

$$U(k,t) = -\beta\,\exp(-k^2 t) + \alpha\sin(\omega t) + \beta\cos(\omega t) \qquad \alpha = A\omega k/(k^4+\omega^2) \qquad \beta = Ak^3/(k^4+\omega^2)$$

$$= -A\,\exp(-k^2 t)\,k^3/(k^4+\omega^2) + A\omega k\,\sin(\omega t)/(k^4+\omega^2) + Ak^3\cos(\omega t)/(k^4+\omega^2)$$

or

$$U(k,t)/A = -\{\exp(-k^2 t)\,k^3/(k^4+\omega^2)\} + \omega\sin(\omega t)\{k/(k^4+\omega^2)\} + \cos(\omega t)\{k^3/(k^4+\omega^2)\}$$

where we show in {...} the three inverse Fourier Sine transforms we will need. So define

$$g_1(k) = \exp(-k^2 t)\,k^3/(k^4+\omega^2)$$
$$g_2(k) = k/(k^4+\omega^2)$$
$$g_3(k) = k^3/(k^4+\omega^2)$$

and we are then looking for

$$u(x,t)/A = (2/\pi)\int_0^\infty dk\,\sin(kx)\,[-g_1(k) + \omega\sin(\omega t)\,g_2(k) + \cos(\omega t)\,g_3(k)]$$

or

$$(\pi/2A)\,u(x,t) = -\int_0^\infty dk\,\sin(kx)\,g_1(k) + \omega\sin(\omega t)\int_0^\infty dk\,\sin(kx)\,g_2(k) + \cos(\omega t)\int_0^\infty dk\,\sin(kx)\,g_3(k)$$

Schaum does not have these so we hunt in ET I. They are not there either and work is required. I of course dealt with all this in earlier parts of this doc (but in Laplace World). Let's get Maple help to save time. We have

$$\int_0^\infty dk\, \sin(kx)\, g_n(k) = \sqrt{\pi/2}\ \{\sqrt{2/\pi}\ \int_0^\infty dk\, \sin(kx)\, g_n(k)\} = \sqrt{\pi/2}\ \text{fouriersin}(g_n(k),k,x)$$

Maple can do $g_2(k)$ and $g_3(k)$ OK,

```
> assume(w>0,t>0);
> g2 := k/(k^4+w^2);
```

$$g2 := \frac{k}{k^4 + w^2}$$

```
> g3 := k^3/(k^4+w^2);
```

$$g3 := \frac{k^3}{k^4 + w^2}$$

```
> sqrt(Pi/2)*fouriersin(g2,k,x):simplify(%);
```

$$\frac{1}{2}\frac{\pi\, e^{\left(-\frac{1}{2}\sqrt{2}\sqrt{w}\,x\right)} \sin\left(\frac{1}{2}\sqrt{2}\sqrt{w}\,x\right)}{w}$$

```
> sqrt(Pi/2)*fouriersin(g3,k,x):simplify(%);
```

$$\frac{1}{2}\pi\, e^{\left(-\frac{1}{2}\sqrt{2}\sqrt{w}\,x\right)}$$

Digression: Maple has made an error!

But amazingly, Maple is making a mistake here!!!! I don't think I have ever seen that happen. The right answer for g3 is this from Alpha, which has the extra cos factor:

Input:

$$\text{FourierSinTransform}\left[\frac{k^3}{k^4 + w^2}, k, x\right]$$

FourierSinTransform$[f(x), x, \omega]$ is the Fourier sine transform of f(x) with frequency variable ω

Exact result:

$$\sqrt{\frac{\pi}{2}}\ e^{-\frac{x}{\sqrt{2}\sqrt[4]{\frac{1}{w^2}}}} \cos\left(\frac{x}{\sqrt{2}\sqrt[4]{\frac{1}{w^2}}}\right)$$

You also get this answer from the power diff rule (which I did by hand). I tried the integral directly but Maple said it was undefined:

```
Int(sin(k*x)*g3,x=0..infinity);
```

$$\int_0^{\infty} \frac{\sin(k\,x)\,k^3}{k^4+w^2}\,dx$$

```
int(sin(k*x)*g3,x=0..infinity);
```

undefined

So poor Maple is suffering a bit of late. These two integrals are in GR7 p 426, just to confirm:

4. $\int_0^{\infty} \frac{x\sin(ax)}{b^4+x^4}\,dx = \frac{\pi}{2b^2}\exp\left(-\frac{ab}{\sqrt{2}}\right)\sin\frac{ab}{\sqrt{2}}$ $[a>0,\quad b>0]$ BI (160)(23)a

10. $\int_0^{\infty} \frac{x^3\sin(ax)}{b^4+x^4}\,dx = \frac{\pi}{2}\exp\left(-\frac{ab}{\sqrt{2}}\right)\cos\frac{ab}{\sqrt{2}}$ $[a>0,\quad b>0]$ BI (160)(24)

and in this second one you see the cosine factor that Maple omits. The BI refers to Bierens de Haan 1867 which is on the French library system but I could not download it today for some reason.

So let's get these two down before starting into $g_1(k)$: [and I have now included the cos factor]

$$\int_0^{\infty} dk\,\sin(kx)\,g_2(k) = (\pi/2\omega)\exp(-\sqrt{\omega/2}\,x)\sin(\sqrt{\omega/2}\,x)$$

$$\int_0^{\infty} dk\,\sin(kx)\,g_3(k) = (\pi/2)\exp(-\sqrt{\omega/2}\,x)\cos(\sqrt{\omega/2}\,x)$$

Now Maple cannot do g_1 and I think it will just stay in integral form. So we then have

$$(\pi/2A)\,u(x,t) = -\int_0^{\infty} dk\,\sin(kx)\,g_1(k) + \omega\sin(\omega t)\int_0^{\infty} dk\,\sin(kx)\,g_2(k) + \cos(\omega t)\int_0^{\infty} dk\,\sin(kx)\,g_3(k)$$

$$= -\int_0^{\infty} dk\,\sin(kx)\exp(-k^2t)\,k^3/(k^4+\omega^2) + \omega\sin(\omega t)(\pi/2\omega)\exp(-\sqrt{\omega/2}\,x)\sin(\sqrt{\omega/2}\,x)$$

$$+ \cos(\omega t)(\pi/2)\cos(\sqrt{\omega/2}\,x)\exp(-\sqrt{\omega/2}\,x)$$

or

$$u(x,t)/A = -(2/\pi)\int_0^{\infty} dk\,\sin(kx)\exp(-k^2t)\,k^3/(k^4+\omega^2)$$

$$+ \sin(\omega t)\exp(-\sqrt{\omega/2}\,x)\sin(\sqrt{\omega/2}\,x) + \cos(\omega t)\cos(\sqrt{\omega/2}\,x)\exp(-\sqrt{\omega/2}\,x)$$

The last two terms become

$$\exp(-\sqrt{\omega/2}\,x)\,[\sin(\omega t)\sin(\sqrt{\omega/2}\,x) + \cos(\omega t)\cos(\sqrt{\omega/2}\,x)]$$

$$= \exp(-\sqrt{\omega/2}\,x)\,[\cos(\omega t - \sqrt{\omega/2}\,x)]$$

which is our famous damped right-moving wave. Now we can look at the integral term:

$$[u(x,t)/A]_{\text{integral term}} = -(2/\pi) \int_0^\infty dk \sin(kx) \exp(-k^2 t)\, k^3/(k^4+\omega^2)$$

Change to $k^2 = \rho$ and $2kdk = d\rho$ to get

$$= -(2/\pi) \int_0^\infty d\rho (2k)^{-1} \sin(\sqrt{\rho}\, x) \exp(-\rho t)\, k^3/(k^4+\omega^2)$$

$$= -(1/\pi) \int_0^\infty d\rho \sin(\sqrt{\rho}\, x) \exp(-\rho t)\, \rho/(\rho^2+\omega^2)$$

So this thing comes out correct (see p 242C). Our total answer for Section (e) here is then:

$$u(x,t) = A \exp(-\sqrt{\omega/2}\, x) [\cos(\omega t - \sqrt{\omega/2}\, x)] - (A/\pi) \int_0^\infty d\rho \sin(\sqrt{\rho}\, x) \exp(-\rho t)\, \rho/(\rho^2+\omega^2)$$

which agrees with p 242C. [By the way, I started this doc on Exercise 7.16 17 days ago and have now finally finished the last part today 4.23.11.]

Recall that the second term arises because we had u=0 initially, and it takes a while for the wave to travel down the rod and get into a steady state where all of that u=0 initial freeze is removed. At any time then the second term is the difference between the steady state wave and what is actually on the rod. It is a transient in effect and it fades away due to the $e^{-\rho t}$.

Can we do the transient integral?

$$I = \int_0^\infty dk \sin(kx) \exp(-k^2 t)\, k^3/(k^4+\omega^2)$$

The integrand is even in k so we can write

$$= (1/2) \int_{-\infty}^\infty dk \sin(kx) \exp(-k^2 t)\, k^3/(k^4+\omega^2)$$

But the expo blows up or down so we cannot close the contour on the GC. So no go on that. We can simplify into four terms by doing partial fractions this way

```
f := k^3/(k^4+w^2);
```

$$f := \frac{k^3}{k^4 + w^2}$$

```
convert(f,fullparfrac,k);
```

$$\sum_{_\alpha = \%1} \left(\frac{1}{4} \frac{1}{k - _\alpha} \right)$$

$$\%1 := \mathrm{RootOf}(_Z^4 + w^2)$$

.

Then you only need an integral of this type, where a $= \pm e^{\pm i\pi/4}\sqrt{\omega}$ (four points on a circle all at 45 degrees) :

$$\int_{-\infty}^{\infty} dk \sin(kx) \exp(-k^2 t) /(k-a)$$

OK, let's brute force expand sin(ka) and 1/(k-a) in powers of k

$$\sin(kx) = \Sigma_{i=1,3}^{\infty} (-1)^{(i-1)/2} (kx)^i/i!$$

$$(k-a) = -(a-k) = -a(1-k/a)$$

$$(k-a)^{-1} = (-a)^{-1}\, 1/(1-k/a) = (-a)^{-1} \Sigma_{j=0}^{\infty} (k/a)^j$$

So we then have

$$\int_{-\infty}^{\infty} dk \sin(kx) \exp(-k^2 t) /(k-a) =$$

$$= \Sigma_{i=1,3}^{\infty} (-1)^{(i-1)/2} (1/i!)\, x^i\, (-a)^{-1} \Sigma_{j=0}^{\infty} a^{-j} \int_{-\infty}^{\infty} dk \exp(-k^2 t)\, k^{i+j}$$

$$= \Sigma_{i=1,3}^{\infty} (-1)^{(i+1)/2} (1/i!)\, x^i \Sigma_{j=0}^{\infty} a^{-j-1} I(t,i+j)$$

But the integral vanishes if i+j = odd, but i is already only odd, so j must be odd, so

$$= \Sigma_{i=1,3}^{\infty} (-1)^{(i+1)/2} (1/i!)\, x^i \Sigma_{j=1,3..}^{\infty} a^{-j-1}\, 2 \int_{0}^{\infty} dk \exp(-k^2 t)\, k^{i+j}$$

$$= \Sigma_{i=1,3}^{\infty} (-1)^{(i+1)/2} (1/i!)\, x^i \Sigma_{j=1,3..}^{\infty} a^{-j-1}\, \Gamma(i/2+j/2+1/2)\, t^{i/2+j/2+1/2}$$

and then finally we have to sum this on the four on-circle values of constant a $= \pm e^{\pm i\pi/4}\sqrt{\omega}$. This one of the ugliest real integrals I have ever seen!

Comment: Remember that for ω = 0 we know the solution to this problem is

$$u(x,t) = A[1 - \mathrm{erf}(x/(2\sqrt{t}))]$$

This reminds us that the "influence" propagates instantly onto the rod so that at all $x > 0$ we have an influence (might be very small) for any t. Thus, this "heat wave" certainly goes faster than the speed of light. You feel a little temperature rise a light year away after a mere 1 sec of heat application time. Obviously the heat conduction equation does not really model the real world in this respect.