Exercise Chap 7 PL1 PhL 4.28.11

The Problem. 1
Comments: 1
Replacing the Problem with a Simpler Problem in 2 instead of 1+3 variables. 1
Show that (3) is true 2
Can Maple do this stuff? 5

The Problem. On page 62 Stak shows that $C_2 = \theta(t)\delta(t-R)/4\pi R$ using a time Laplace transform. How would you show *directly* that this is a solution to the wave equation? That is to say, we want to show that

$$(\partial_t^2-\nabla_x^2)\,[\theta(t)\delta(t-R)/4\pi R] = \delta(t)\delta^3(x-x_0) \qquad (1)$$

where $R = |x-x_0|$ and we are in 3D and our region is all of R_3.

Comments: I successfully show this below, but it is amazingly complicated with lots of "delta function algebra" involving many of the various Delta Function Rules (see doc of that name in distributions folder). Just threading through this algebra is non-trivial. Maple at least knows about this stuff, but it could not finish the simplification of the problem (as shown below). Notice that ∇^2 acts on the *product of* $\delta(t-R)$ and $(1/4\pi R)$, so the known fact that $-\nabla^2(1/4\pi R) = \delta^3(\mathbf{x}-\mathbf{x_0})$ was not as useful as I thought it was going to be, and in fact I never use this fact below. With $\mathbf{x_0} = 0$ this says $-\nabla^2(1/4\pi r) = \delta(r)/4\pi r^2$ but even that was not directly used below. I just did it all brute force.

I thought I was going to find a very simple way of extending $-\nabla^2(1/4\pi R) = \delta^3(\mathbf{x}-\mathbf{x_0})$ to result (1) above, but I did not discover such a simple way. The Green's function is a "delta shell" that radiates out from a spacetime disturbance at $\delta(t)\delta^3(x-x_0)$ but of course only for $t > 0$. There is no "wake" behind this shell, and I remember this is true in general for an odd number of dimensions for space. The "influence" of this disturbance exists at some point in space exists only when $t = R$. There is no influence whatsoever when $t < R$ and $t > R$. In contrast, the 1D Green's function is $C_1 = \theta(t)\, H(t-R)/2$ and the "wake" in this case is permanent and is the same size as the wavefront. In 2D we get halfway between these two solutions. There is a wake, but it is maximal just behind the wavefront which is infinite but integrable.We have $C_2 = (1/2\pi)\, H(t-R)/\sqrt{t^2-R^2}$, and this looks like the charge density on a disk!

Replacing the Problem with a Simpler Problem in 2 instead of 1+3 variables.

Suppose we knew the above were true for $\mathbf{x_0} = 0$, so we would then know

$$(\partial_t^2-\nabla_x^2)\,[\theta(t)\delta(t-|x|)/4\pi|x|] = \delta(t)\delta^3(x) \qquad (2)$$

If we knew this last was true for all $\mathbf{x}$, we could just replace $\mathbf{x} \rightarrow \mathbf{x}-\mathbf{x_0}$ to get our desired result (1). The key fact is that $\nabla_{x-x0}^2 = \nabla_x^2$ and this makes it work. So we only have to show it for $\mathbf{x_0} = 0$.We need to show this, where r is the usual spherical coordinate,

$$(\partial_t^2-\nabla_x^2)\,[\theta(t)\delta(t-r)/4\pi r] = \delta(t)\delta^3(x) \;= \delta(t)\delta(r)/4\pi r^2$$

where we have replaced $\delta^3(x) = \delta(r)/4\pi r^2$ which can be verified by integrating f(r) over a small spherical volume. We also know that

$$\nabla_x^2 f(r) = r^{-2}\,\partial_r(r^2\partial_r\, f(r))$$

so we have reduced our problem to one of just two variables, r and t , which we proceed to rewrite several times:

$$(\partial_t^2-\nabla_x^2)\,[\theta(t)\delta(t-r)/4\pi r] = \delta(t)\delta(r)/4\pi r^2 \qquad ?$$

$$\partial_t^2\,[\theta(t)\delta(t-r)/4\pi r] \;- r^{-2}\,\partial_r(r^2\partial_r\,[\theta(t)\delta(t-r)/4\pi r]) \;= \delta(t)\delta(r)/4\pi r^2 \qquad ?$$

$$(1/4\pi r)\,\partial_t^2\,[\theta(t)\delta(t-r)] - (1/4\pi)\,\theta(t)\,r^{-2}\,\partial_r(r^2\partial_r[\delta(t-r)/r]) = \delta(t)\delta(r)/4\pi r^2 \qquad ?$$

$$(1/r)\,\partial_t^2\,[\theta(t)\delta(t-r)] - \theta(t)\,r^{-2}\,\partial_r(r^2\partial_r[\delta(t-r)/r]) = \delta(t)\delta(r)/\,r^2 \qquad ?$$

$$Q \equiv\; r\,\partial_t^2\,[\theta(t)\delta(t-r)] - \theta(t)\,\partial_r(r^2\partial_r[\delta(t-r)/r]) = \delta(t)\delta(r) \qquad ? \qquad (3)$$

We shall now attempt to show that the last line (3) is in fact true, and if we can show that, we have finished our direct verification!

Show that (3) is true

Now start in the second term of (3) with

$$\partial_r[\delta(t-r)/r] \;= \partial_r[\delta(t-r)/t] \;= (1/t)\,\partial_r\delta(t-r) = -\,(1/t)\,\delta'(t-r)$$

$$r^2\partial_r[\delta(t-r)/r] \;=\; -\,(1/t)\,r^2\,\delta'(t-r)$$

$$\partial_r(r^2\partial_r[\delta(t-r)/r]) = -\,(1/t)\{\,r^2\,\partial_r\,\delta'(t-r) + 2r\,\delta'(t-r)\,\} \;=\; -\,(1/t)\{-\;r^2\,\delta''(t-r) + 2r\,\delta'(t-r)\,\}$$

Now move to the first term in (3) (omit the r for the moment)

$$\partial_t\,[\theta(t)\delta(t-r)] \;= \delta(t)\,\delta(t-r) + \theta(t)\,\delta'(t-r)$$

$$\partial_t^2\,[\theta(t)\delta(t-r)] \;= \delta'(t)\,\delta(t-r) + \delta(t)\,\delta'(t-r) + \delta(t)\,\delta'(t-r) + \theta(t)\,\delta''(t-r)$$

$$= \delta'(t)\,\delta(t-r) + 2\delta(t)\,\delta'(t-r) + \theta(t)\,\delta''(t-r)$$

Now let's add the first and second terms

$$r\,\partial_t^2\,[\theta(t)\delta(t-r)] - \theta(t)\,\partial_r(r^2\partial_r[\delta(t-r)/r])$$

$= r\{ \delta'(t)\,\delta(t-r) + 2\delta(t)\,\delta'(t-r) + \theta(t)\,\delta''(t-r)\} + \theta(t)\,(1/t)\{- r^2\delta''(t-r) + 2r\,\delta'(t-r)\}$

$= \delta'(t)\,r\,\delta(t-r) + 2\delta(t)\,r\,\delta'(t-r) + \theta(t)\,r\,\delta''(t-r) - \theta(t)(1/t)\,r^2\delta''(t-r) + 2\,\theta(t)(1/t)\,r\,\delta'(t-r)$

$\equiv Q$

How to proceed? It is really quite non-obvious what to do next! Let's try to remove factors of r using our various Delta Rules :

(a) For example, consider this quantity which appears above

$r\,\delta''(t-r) = r\,\delta''(r-t) = r\,\partial_r^2\,\delta(r-t)$

We can apply our "rule"

$f(x)\,\delta''(x-b) = f''(b)\delta(x-b) - 2f'(b)\delta'(x-b) + f(b)\,\delta''(x-b)$ Set x = r and b = t :

$f(r)\,\delta''(r-t) = f''(t)\delta(r-t) - 2f'(t)\delta'(r-t) + f(t)\,\delta''(r-t)$

$r\,\delta''(r-t) = 0\,\delta(r-t) - 2\,1\,\delta'(r-t) + t\,\delta''(r-t) = 2\,\delta'(r-t) + t\,\delta''(r-t)$

So we have shown that

$r\,\delta''(t-r) = 2\,\delta'(r-t) + t\,\delta''(r-t) = -2\,\delta'(t-r) + t\,\delta''(t-r)$

(b) Next, consider

$r^2\delta''(t-r) = r^2\,\partial_r^2\,\delta(r-t)$

We can apply our "rule"

$f(x)\,\delta''(x-b) = f''(b)\delta(x-b) - 2f'(b)\delta'(x-b) + f(b)\,\delta''(x-b)$ Set x = r and b = t :

$f(r)\,\delta''(r-t) = f''(t)\delta(r-t) - 2f'(t)\delta'(r-t) + f(t)\,\delta''(r-t)$

$r^2\,\delta''(r-t) = 2\,\delta(r-t) - 2\,2t\,\delta'(r-t) + t^2\,\delta''(r-t) = 2\,\delta(r-t) - 4\,t\,\delta'(r-t) + t^2\,\delta''(r-t)$

So we have shown that

$r^2\,\delta''(t-r) = 2\,\delta(t-r) + 4\,t\,\delta'(t-r) + t^2\,\delta''(t-r)$

(c) Next consider

$r\,\delta'(t-r)$

We can apply our "rule"

$f(x)\,\delta'(x-b) = f(b)\,\delta'(x-b) - f'(b)\,\delta(x-b)$

$f(r)\,\delta'(r-t) = f(t)\,\delta'(r-t) - f'(t)\,\delta(r-t)$

$r\,\delta'(r-t) = t\,\delta'(r-t) - 1\,\delta(r-t)$

So we have shown that

$$-r\,\delta'(t-r) = -t\,\delta'(t-r) - 1\,\delta(t-r)$$

$$r\,\delta'(t-r) = t\,\delta'(t-r) + \delta(t-r)$$

So gather up our three results

$$r\,\delta''(t-r) = 2\,\delta'(r-t) + t\,\delta''(r-t) = -2\,\delta'(t-r) + t\,\delta''(t-r) \quad (4a)$$

$$r^2\,\delta''(t-r) = 2\,\delta(t-r) + 4t\,\delta'(t-r) + t^2\,\delta''(t-r) \quad (4b)$$

$$r\,\delta'(t-r) = t\,\delta'(t-r) + \delta(t-r) \quad (4c)$$

and jam these into Q and see what happens:

$$Q = \delta'(t)\,r\,\delta(t-r) + 2\delta(t)\,r\,\delta'(t-r) + \theta(t)\,r\,\delta''(t-r) - \theta(t)(1/t)\,r^2\delta''(t-r) + 2\,\theta(t)(1/t)\,r\,\delta'(t-r)$$

$$= \delta'(t)\,t\,\delta(t-r) + 2[\delta(t)+\theta(t)(1/t)]\,[t\,\delta'(t-r) + \delta(t-r)] + \theta(t)\,[-2\,\delta'(t-r) + t\,\delta''(t-r)]$$

$$- \theta(t)(1/t)\,[2\,\delta(t-r) + 4t\,\delta'(t-r) + t^2\,\delta''(t-r)]$$

and at least we have gotten rid of all factors of r, the goal we set. Now group by δ type:

$$Q = \delta(t-r)\,\{\,\delta'(t)\,t + 2[\delta(t)+\theta(t)(1/t)] - 2\theta(t)(1/t)\,\}$$

$$+ \delta'(t-r)\,\{\,2t\,[\delta(t)+\theta(t)(1/t)] - 2\,\theta(t)\,\} + \delta''(t-r)\,\{\,t\,\theta(t) - \theta(t)(1/t)t^2\,\}$$

and at least the last term has 0 coefficient. We are left with

$$Q = \delta(t-r)\,\{\,\delta'(t)\,t + 2[\delta(t)+\theta(t)(1/t)] - 2\theta(t)(1/t)\,\} + \delta'(t-r)\,\{\,2t\,[\delta(t)+\theta(t)(1/t)] - 2\,\theta(t)\,\}$$

$$= \delta(t-r)\,\{\,\delta'(t)\,t + 2[\delta(t)]\,\} + \delta'(t-r)\,\{\,2t\,[\delta(t)]\,\}$$

$$= \delta(t-r)\,\{\,\delta'(t)\,t + 2\delta(t)\,\} + \delta'(t-r)\,2t\,\delta(t)$$

$$= t\,\delta(t-r)\,\delta'(t) + 2\delta(t)\,\delta(t-r) + 2t\,\delta'(t-r)\,\delta(t)$$

$$= r\,\delta(t-r)\,\delta'(t) + 2\delta(t)\,\delta(t-r) == r\,\delta(t-r)\,\delta'(t) + 2\delta(t)\delta(r) \quad (5)$$

where we use $t\,\delta(t)=0$ (one of our Delta rules) to remove the last term

(d) Now let's try a Delta Rule on the first term

$$f(x)\,\delta'(x-b) = f(b)\,\delta'(x-b) - f'(b)\,\delta(x-b)$$

$$f(t)\,\delta'(t) = f(0)\,\delta'(t) - f'(0)\,\delta(t) \qquad f(t) = r\,\delta(t-r) \quad f(0) = r\delta(r) \quad f'(0) = -r\delta'(r)$$

$$r\,\delta(t-r)\,\delta'(t) = r\delta(r)\,\delta'(t) + r\delta'(r)\,\delta(t) = -\delta(r)\delta(t) \qquad \text{// since } r\delta(r) = 0 \text{ and } r\delta'(r) = -\delta(r)$$

Then we have

$$Q = \delta(t-r)\,\delta'(t) + 2\delta(t)\,\delta(r)$$

$$= -\delta(r)\delta(t) + 2\delta(t)\,\delta(r) = \delta(t)\delta(r)$$

and this is the desired result, absolutely amazing!! I have never done stuff like this before.

Can Maple do this stuff?

I enter Q :

```
> restart;
> Q := r*Diff((Heaviside(t)*Dirac(t-r)),t,t)-Heaviside(t)*Diff(r^2*Diff(Dirac(t-r)/r,r),r);
```

$$Q := r\left(\frac{\partial^2}{\partial t^2}\,\text{Heaviside}(t)\,\text{Dirac}(t-r)\right) - \text{Heaviside}(t)\left(\frac{\partial}{\partial r}\,r^2\left(\frac{\partial}{\partial r}\,\frac{\text{Dirac}(t-r)}{r}\right)\right)$$

```
> Q := r*diff(Heaviside(t)*Dirac(t-r),t,t)-Heaviside(t)*diff(r^2*diff(Dirac(t-r)/r,r),r);
```

$$Q := r\,(\text{Dirac}(1,t)\,\text{Dirac}(t-r) + 2\,\text{Dirac}(t)\,\text{Dirac}(1,t-r) + \text{Heaviside}(t)\,\text{Dirac}(2,t-r))$$
$$- \text{Heaviside}(t)\left(2r\left(-\frac{\text{Dirac}(1,t-r)}{r} - \frac{\text{Dirac}(t-r)}{r^2}\right) + r^2\left(\frac{\text{Dirac}(2,t-r)}{r} + 2\,\frac{\text{Dirac}(1,t-r)}{r^2} + 2\,\frac{\text{Dirac}(t-r)}{r^3}\right)\right)$$

```
> expand(%);
```

$$r\,\text{Dirac}(1,t)\,\text{Dirac}(t-r) + 2\,r\,\text{Dirac}(t)\,\text{Dirac}(1,t-r)$$

```
> simplify(%);
```

$$0$$

Let's ignore the last simplification (which is true except when r=0 and t=0) and consider the previous line:

$$Q = r\delta'(t)\delta(t-r) + 2r\delta(t)\delta'(t-r) \qquad (8)$$

This has the same first term as (5) above, but the second term is different. Back in section (d) we learned that the first term is just -δ(r)δ(t), so the second term must be +2 δ(r)δ(t). So all we need to show now is that

$$r\delta(t)\delta'(t-r) = \delta(r)\delta(t)$$

So apply our rule to the left hand side

$$f(x)\,\delta'(x-b) = f(b)\,\delta'(x-b) - f'(b)\,\delta(x-b)$$
$$f(t)\,\delta'(t-r) = f(r)\,\delta'(t-r) - f'(r)\,\delta(t-r)$$
$$\delta(t)\,\delta'(t-r) = \delta(r)\,\delta'(t-r) - \delta'(r)\,\delta(t-r)$$

$$r\delta(t)\delta'(t-r) = r\,\delta(r)\,\delta'(t-r) - r\,\delta'(r)\,\delta(t-r) = -r\,\delta'(r)\,\delta(t-r) = -t\,\delta'(t)\,\delta(t-r)$$

$$= -\text{ first term in (8) or (5)} = \delta(r)\delta(t) \qquad \text{QED}$$

So Maple was some help at least, but could not take it all the way.