Separation and Transforms PhL 8.9.11

I have been working in Stak Chapter 7 where Stak is fiddling with polar coordinates in different ways, and I feel the need to generalize the discussion there somewhat.

0. Overview (written 12.9.11, 2.5 pages) **1**

1. The PDE equation and BC's. **3**

2. Doing the separation of the *homo* PDE and identifying the EV equations. **4**

3. Expansions, projections and transforms associated with L_x and L_y **6**

4. Double Expansion Solution for the *inhomo* equation Lu = f. **6**

5. Going back to the homo equation Lu=0 ..... **8**

6. What about the Green's Function equation Lu=δ? **9**

7. Big Example: Let (x,y) = (r,θ), polar coordinates, with r in (0,a) and θ in (-π,π), and L = -∇². 10

(a) The two Sturm-Liouville problems and their solutions 10

(b) Solve the Lu = f. 12

(c) Solve Lu = 0 13

(d) The Green's Function for a circle with point charge at (r_0,θ_0) 15

8. The Partial Eigenfunction Method of Analysis. **15**

(a) Use the x eigenfunctions 15

(b) Compute the Green's Function for the y-equation 17

(c) Use the y eigenfunctions 19

Comments. 20

9. Curvilinear Coordinates **20**

(a) 2D systems 20

(b) 3D systems with solutions not dependent one of the variables 22

10. A review of Stak examples in light of this doc **22**

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0. Overview (written 12.9.11, 2.5 pages)

Section 1 considers a 2D PDE having a certain simple form in hopes that it will be separable,

$$L = g_x(y)h_x(x)L_x + g_y(x)\,h_y(y)L_y \qquad Lu = 0 \quad \text{homo} \qquad Lu = f \quad \text{inhomo}$$

If the operators L_x and L_y are self-adjoint, then each offers the possibility of a 1D Sturm-Liouville problem. This 2D PDE need not be the 2D Laplace or Helmholtz equation. For that reason, and also since the coordinates are not necessarily "curvilinear coordinates", the Stackel theory (as I wrote about it) does not in general apply, though of course it does apply in the special case (see Appendix A).

Section 2 then shows that the homo equation Lu=0 can indeed be separated with u = X(x)Y(y). The separated equations are these, written in standard Stakgold Sturm-Liouville form,

$L_x X_{\lambda x}(x) = \lambda_x s_x(x) X_{\lambda x}(x)$ $\quad \lambda_x = \nu$ $\quad s_x(x) \equiv g_y(x)/ h_x(x)$

$L_y Y_{\lambda y}(y) = \lambda_y s_y(y) Y_{\lambda y}(y)$ $\quad \lambda_y = -\nu$ $\quad s_y(y) \equiv g_x(y)/ h_y(y)$

where $X_{\lambda x}(x)$ are EF's of L_x and $Y_{\lambda y}(y)$ are EF's of L_y, and then it is verified that $L\, X_{\lambda x}(x)\, Y_{\lambda y}(y) = 0$.

Section 3 writes down some presumed S-L transforms for these X and Y systems.

Section 4 then does a "**full eigenfunction method**" (double expansion) solution of inhomo Lu=f. In the "double diagonalized space" the solution is given by $u_{\lambda x,\lambda y} = F_{\lambda x,\lambda y}/ (\lambda_x + \lambda_y)$ where the F projections are obtained in a certain way from f as shown below, so the final solutions is

$$u(x,y) = \Sigma_{\lambda x} \Sigma_{\lambda y} X_{\lambda x}(x)\, Y_{\lambda y}(y)\, u_{\lambda x,\lambda y} = \Sigma_{\lambda x} \Sigma_{\lambda y} X_{\lambda x}(x)\, Y_{\lambda y}(y)\, F_{\lambda x,\lambda y}/ (\lambda_x+\lambda_y)$$

where

$$F_{\lambda x,\lambda y} = \int_{ax}^{bx} dx\, X_{\lambda x}^*(x) \int_{ay}^{by} dy\, Y_{\lambda y}^*(y)\; f(x,y)/ [\, h_x(x)\, h_y(y)\,]$$

where it is understood that each spectrum could be discrete, continuous or mixed.

Section 5 sets f=0 in Lu=f to reconsider the homo equation Lu=0 and one finds these equivalent atomic forms

$$u(x,y) = \Sigma_{\lambda x}\; u_{\lambda x,-\lambda x}\; X_{\lambda x}(x)\, Y_{-\lambda x}(y)$$
$$u(x,y) = \Sigma_{\lambda y}\; u_{-\lambda y,\lambda y}\; X_{-\lambda y}(x)\, Y_{\lambda y}(y)$$

Section 6 then solves the Lu=δ problem where $\delta = \delta(x-x_0)\, \delta(y-y_0)\, p(x_0,y_0)$ as a special case of Section 4 above and finds then that

$$g(x,y|\, x_0,y_0) = \{\, p(x_0,y_0) / [\, h_x(x_0)\, h_y(y_0)\}$$
$$* \;\; \Sigma_{\lambda x} \Sigma_{\lambda y} [\, X_{\lambda x}(x)\, Y_{\lambda y}(y)]\; [X_{\lambda}(x_0)\, Y_{\lambda y}(y_0)]^* / (\lambda_x +\lambda_y)$$

$$= \text{ " } \Sigma_{\lambda}\, \varphi_{\lambda}(\mathbf{r})\varphi_{\lambda}(\mathbf{r}_0)^* /(\lambda) \text{ "}$$

which is a generic form for a green's function where here λ is the sum of λ's probably due to the simple PDE form we started with.

Section 7 considers the example of **polar coordinates** and L = Laplace,. There are 4 parts:

In part (a), everything is transcribed from x,y to r,θ and the two Sturm-Liouville problems are discussed in a typical BC situation which is periodicity in θ and u vanishing at r=a and interval (0,a). The associated transforms are **Expo Fourier Series** for θ and **Mellin Sine** for r.

Then in part (b), the solution to Lu=f is transcribed from above and the appropriate helper g and h functions are inserted.

In part (c), Lu=0 is solved and we have transcribed the atomic forms obtained earlier. These forms are then written just as I wrote them for 3D in a separate doc, and the atoms are shown to match Stak's 2D potential theory.

Finally, part (d) sets f = δ for a point charge Green's problem with u vanishing at r=a and we get a well-defined solution. So all this work is using the full or double eigenfunction expansion idea.

Section 8 considers the **partial-eigenfunction method**, and there are 3parts:

In part (a) the expansion onto x eigenfunctions is $u(x,y) = \Sigma_{\lambda x}\, u_{\lambda x}(y)\, X_{\lambda x}(x)$ and similarly for f/F, where here $u_{\lambda x}(y)$ are the usual "coefficient functions" of the other variable. It is found that $u_{\lambda x}(y)$ solves a certain ODE $L_y\, u_{\lambda x}(y) + \lambda_x\; s_y(y) u_{\lambda x}(y) = s_y(y)\, F_{\lambda x}(y)$ which one then has to solve.

In part (b) the above equation is solved with RHS = $\delta(y-y_0)$ as a first solution step. This is a standard 1D Green's function problem, and we consider the "left and right sides" of the Green's function in various situations such as regular or limit circle at both ends, then limit point at one end, then at both ends. This just controls whether each "side" has one or two terms. In the last case, $g(y|y_0) = \alpha\ [Y_{\lambda y}(y_<)]\ [\mathcal{Y}_{\lambda y}(y_>)]$ for example, where the two Y functions shown are those of finite s-norm at their respective endpoints. I did not finish the task, but in theory then you can use this Green's Function to solve the above inhomo equation for $u_{\lambda x}(y)$ and then, finally, you have your solution u(x,y).

In part (c) we just restate the equations in the case the one uses the Y eigenfunctions. It is noted that the spectrum in this case of ν might be completely different from the X eigenfunction case.

Section 9 then considers the Laplace and Helmholtz equations in 2D curvilinear coordinates, 2 parts:

In part (a) these 2D systems are considered. Basically for all such systems one finds that $Q_1 = Q_2$ and then $\nabla^2 f = (1/g_{11}(12))\ (\partial^2_1 + \partial^2_2)f$. For Laplace Lu=0, those looks just like the 2D Cartesian case and one gets the expected separated solutions with trivial functions. For Helmholtz, separation is only possible if $g_{11}(x,y) = a(x) + b(y)$. This only occurs in alternate polar, parabolic, and elliptic 2D coordinates. Most of M&S 2D systems are therefore not separable for Helmholtz. The conclusion reached here agrees with the conclusion of my N=2 Stackel doc section.

In part (b) it is noted that these 2D systems effectively occur in 3D curvilinear coordinate work when a certain 3rd coordinate is symmetric and so does not enter the solution.

Section 10 then reviews a lot of the Stak 2D Helmholtz applications in his Chapter 7, and I show how all these fit right into the mold of this document and this explains why "things always work". Whether you apply a partial EF "projector" to the PDE, or whether you expand the solution in those EF's, you always end up with an ODE for your coefficient functions of the other variable, and you never have any surviving integrals on the LHS of your ODE. This is what got me writing this doc! Here I show this for a very general class of PDE's, and all his examples are either polars or ρ-z in cylindricals.

Appendix A considers the special "2D Helmholtz equation + curvilinear coordinates" case of our general 2D PDE form, and then makes the connection between our helper functions and the Stackel matrix functions.

Comment: What led to this doc was Stak's "bandying around" with Helmholtz in polar coordinates

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1. The PDE equation and BC's.

To that end, assume we have some PDE of the following form

$$L\,u(x,y) = f(x,y) \qquad\qquad L = g_x(y)h_x(x)L_x + g_y(x)\,h_y(y)L_y$$

$$[g_x(y)h_x(x)L_x + g_y(x)\,h_y(y)L_y]\,u(x,y) = f(x,y)$$

where x and y are arbitrary coordinates (perhaps x=r and y=θ), and where f(x,y) is some driving function, perhaps a distribution such as "a ring of source". One hidden assumption here is that when we say L_x, we mean that L_x is a differential operator which is self-adjoint and does not involve the other variable y, and similarly for L_y. In our full L, we allow that L_x might be multiplied by some functions as shown, so this is a fairly restrictive form for L, but I think it covers our important cases. More generally we *could* assume

$L = q_x(x,y)L_x + q_y(x,y)L_y$ where the q functions don't factor, and this would no doubt be a much harder problem to deal with. In practice, we will find that these helper g and h functions are just scale factors encountered when changing to non-Cartesian coordinate systems, with possible constants to cause the resulting L operators to be standard ones you find in a book.

2. Doing the separation of the *homo* PDE and identifying the EV equations.

Our usual starting point is to study the homo version of this equation

$$[g_x(y)h_x(x)L_x + g_y(x)\, h_y(y)L_y]\, u(x,y) = 0$$

We shall now assume that there are some *boundary conditions* for our x and y systems which serve to determine the eigenfunctions for those systems. We assume that these BC's are not "interlinked" with each other, which means that for each coordinate, the boundaries lie along a surface of constant coordinate, which we sometimes call a "level surface". So maybe the interval for x is (a_x, b_x) and similarly for y. Sometimes a BC says u=0 at both ends of the interval, but sometimes (as for θ) the BC has the form that u and u' are continuous at a certain point ($\theta=\pm\pi$).

Comment added 12.9.11. The Stackel theory, as I present it in my paper, starts specifically with the Helmholtz equation (a homo equation) and specifically with some generic orthogonal coordinates ξ_n with scale factors h_n. Here, I am *not* assuming Helmholtz, and I am assuming nothing about coordinates x and y, and below I will be seeking only "simple separation" meaning R = 1. I think one could apply the Stackel concept to the generic PDE above, and maybe that is what I am doing below.

We are now ready to do our separation attempt. We are just searching for solutions of the separated form, and we are not claiming that this exhausts all possible solutions (though it probably does in the atomic sense). So

$$u(x,y) = X(x)Y(y)$$

Installing this into our homo equation gives

$$[g_x(y)h_x(x)L_x + g_y(x)\, h_y(y)L_y]X(x)Y(y) = 0$$

$$[g_x(y)\, h_x(x)L_x\, X(x)]\, Y(y) + g_y(x)\, h_y(y)X(x)\, [L_y\, Y(y)] = 0$$

Divide both sides by $X(x)Y(y)\, g_x(y)\, g_y(x)$ to get

$$[\, h_x(x)L_x\, X(x)] \,/\, [g_y(x)\, X(x)] = -\, h_y(y)L_y\, Y(y)/[g_y(y)\, Y(y)]$$

The LHS is independent of y so it could be some t(x). The RHS is independent of x, so it could be some w(y). We then have t(x) = w(y). The only functions that satisfy this equation are that both are constants, and they are of course the same constant, and we shall call that ν, the separation constant. So we have

$$[h_x(x)L_x\, X(x)] \,/[\, g_y(x)X(x] = -\, [h_y(y)L_y\, Y(y)]/\, [\, g_x(y)Y(y)\,] = \nu$$

$$h_x(x)L_x\,X(x) = +\nu\, g_y(x)X(x)$$
$$h_y(y)L_y\,Y(y) = -\nu\, g_x(y)Y(y)$$

and these then our the "separated equations". Rewrite them now as

$$L_x\,X(x) = \nu\,[g_y(x)/\,h_x(x)]\,X(x)$$
$$L_y\,Y(x) = -\nu\,[g_x(y)/\,h_y(y)]\,Y(y)$$

Anticipating that ν will the ±λ, we put this into "standard EV form" by defining

$$s_x(x) \equiv g_y(x)/\,h_x(x) \quad \Rightarrow \quad g_y(x) = h_x(x)\,s_x(x)$$
$$s_y(y) \equiv g_x(y)/\,h_y(y) \quad \Rightarrow \quad g_x(y) = h_y(y)\,s_y(y)$$

Then our separated equations are

$$L_x\,X(x) = \nu\,s_x(x)\,X(x)$$
$$L_y\,Y(x) = -\nu\,s_y(y)\,Y(y)$$

If we now set $\nu = \lambda_x = -\lambda_y$, these equations become

$$L_x\,X_{\lambda x}(x) = \lambda_x\,s_x(x)X_{\lambda x}(x) \qquad \lambda_x = \nu$$
$$L_y\,Y_{\lambda y}(y) = \lambda_y\,s_y(y)Y_{\lambda y}(y) \qquad \lambda_y = -\nu$$

And these are our x and y EV equations now in "standard form". Meanwhile, we can rewrite L in a slightly more useful form:

$$L = [\,g_x(y)h_x(x)L_x + s_x(x)\,h_y(y)L_y]$$

$$= [h_y(y)\,s_y(y)h_x(x)L_x + h_x(x)\,s_x(x)\,h_y(y)L_y]$$

so

$$L = h_x(x)\,h_y(y)\,[s_y(y)\,L_x + s_x(x)\,L_y]$$

Notice what happens when we apply L to a 2D basis function:

$$L\,X_{\lambda x}(x)\,Y_{\lambda y}(y) = h_x(x)\,h_y(y)\,[s_y(y)\,\lambda_x s_x(x) + s_x(x)\,\lambda_y s_y(y)]\,X_{\lambda x}(x)\,Y_{\lambda y}(y)$$

$$= h_x(x)\,h_y(y)\,s_x(x)\,s_y(y)\,[\,\lambda_x+\lambda_y]\,X_{\lambda x}(x)\,Y_{\lambda y}(y)$$

$$= (\lambda_x+\lambda_y)\,\{\,h_x(x)\,h_y(y)\,s_x(x)\,s_y(y)\}\,X_{\lambda x}(x)\,Y_{\lambda y}(y)$$

$$= 0 \qquad \text{// since by their definition, } \lambda_x+\lambda_y = 0$$

[stuff was here which is now Appendix A.]

3. Expansions, projections and transforms associated with L_x and L_y

Since we can find complete eigenfunctions for each variable, assuming the BC's provide a clean SL problem in each, we know we can expand either separated function in them as shown here on the left:

$$X(x) = \Sigma_{\lambda x}\, a_{\lambda x} X_{\lambda x}(x) \qquad a_{\lambda x} = \int_{ax}^{bx} dx\, s_x(x)\, X_{\lambda x}^*(x)\, X(x)$$

$$Y(y) = \Sigma_{\lambda y}\, a_{\lambda y} Y_{\lambda y}(y) \qquad a_{\lambda y} = \int_{ay}^{by} dy\, s_y(y)\, Y_{\lambda x}^*(y)\, Y(y)$$

where the $a_{\lambda x}$ are "coefficients" which are constants, and where the sums could be integrals or even combinations of sums and integrals. Since the eigenfunctions form a complete set, we know we can invert our expansions to get the projections as shown. The combination of expansion and projection is called a transform, so there are then two transforms associated with our problem, one in x and one in y.

4. Double Expansion Solution for the *inhomo* equation Lu = f.

This discussion aligns with what Stak would call the "full eigenfunction method" of doing something, as opposed to the "partial EF method" which we shall do later. Since we are going to expand u in basis functions $X_{\lambda x}(x)Y_{\lambda y}(y)$, our solutions are appropriate for u(x,y) meeting the boundary conditions which are met by $X_{\lambda x}(x)Y_{\lambda y}(y)$.

We start with Lu=f which reads

$$h_x(x)\, h_y(y)\, [s_y(y)\, L_x + s_x(x)\, L_y]\, u(x,y) = f(x,y)$$

Now since we are here doing a *double* expansion, we might as well write

$$u(x,y) = \Sigma_{\lambda x}\, \Sigma_{\lambda y}\, X_{\lambda x}(x)\, Y_{\lambda y}(y)\, u_{\lambda x,\lambda y}$$

Comparing to the above, we can identify $u_{\lambda x,\lambda y} = a_{\lambda x}\ a_{\lambda y}$ but here only the product will occur, so we will just use $u_{\lambda x,\lambda y}$. Inserting our expansion we get

$$h_x(x)\, h_y(y)\, [s_y(y)\, L_x + s_x(x)\, L_y]\, \Sigma_{\lambda x}\, \Sigma_{\lambda y}\, X_{\lambda x}(x)\, Y_{\lambda y}(y)\, u_{\lambda x,\lambda y} = f(x,y)$$

Move in the operators and move out the sums and coefficients

$$h_x(x)\, h_y(y)\, \Sigma_{\lambda x}\, \Sigma_{\lambda y}\, u_{\lambda x,\lambda y}\, [s_y(y)\, L_x + s_x(x)\, L_y]\, X_{\lambda x}(x)\, Y_{\lambda y}(y) = f(x,y)$$

$$h_x(x)\, h_y(y)\, \Sigma_{\lambda x}\, \Sigma_{\lambda y}\, u_{\lambda x,\lambda y}\, [s_y(y)\, L_x\, X_{\lambda x}(x)\, Y_{\lambda y}(y) + s_x(x)\, L_y\, X_{\lambda x}(x)\, Y_{\lambda y}(y)] = f(x,y)$$

$$h_x(x)\, h_y(y)\, \Sigma_{\lambda x}\, \Sigma_{\lambda y}\, u_{\lambda x,\lambda y}\, [s_y(y)\, \lambda_x\, s_x(x)\, X_{\lambda x}(x)\, Y_{\lambda y}(y) + s_x(x)\, \lambda_y\, s_y(y)\, X_{\lambda x}(x)\, Y_{\lambda y}(y)] = f(x,y)$$

$$h_{\mathbf{x}}(x)\, h_{\mathbf{y}}(y)\, s_{\mathbf{x}}(x)\, s_{\mathbf{y}}(y) \sum_{\boldsymbol{\lambda x}} \sum_{\boldsymbol{\lambda y}} u_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}} \left[\lambda_{\mathbf{x}}\, X_{\boldsymbol{\lambda x}}(x)\, Y_{\boldsymbol{\lambda y}}(y) + \lambda_{\mathbf{y}}\, X_{\boldsymbol{\lambda x}}(x)\, Y_{\boldsymbol{\lambda y}}(y)\right] = f(x,y)$$

$$\left[\, h_{\mathbf{x}}(x)\, h_{\mathbf{y}}(y)\, s_{\mathbf{x}}(x)\, s_{\mathbf{y}}(y)\,\right] \sum_{\boldsymbol{\lambda x}} \sum_{\boldsymbol{\lambda y}} X_{\boldsymbol{\lambda x}}(x)\, Y_{\boldsymbol{\lambda y}}(y)\, u_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}} \left[\lambda_{\mathbf{x}} + \lambda_{\mathbf{y}}\right] = f(x,y)$$

$$\sum_{\boldsymbol{\lambda x}} \sum_{\boldsymbol{\lambda y}} X_{\boldsymbol{\lambda x}}(x)\, Y_{\boldsymbol{\lambda y}}(y)\, u_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}} \left(\lambda_{\mathbf{x}} + \lambda_{\mathbf{y}}\right) = f(x,y) / \left[\, h_{\mathbf{x}}(x)\, h_{\mathbf{y}}(y)\, s_{\mathbf{x}}(x)\, s_{\mathbf{y}}(y)\,\right]$$

We now see that we want to define a new RHS function F(x,y) according to

$$F(x,y) \equiv f(x,y) / \left[\, h_{\mathbf{x}}(x)\, h_{\mathbf{y}}(y)\, s_{\mathbf{x}}(x)\, s_{\mathbf{y}}(y)\,\right]$$

and then we want to do a double expansion of F

$$F(x,y) = \sum_{\boldsymbol{\lambda x}} \sum_{\boldsymbol{\lambda y}} X_{\boldsymbol{\lambda x}}(x)\, Y_{\boldsymbol{\lambda y}}(y)\, F_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}}$$

When our equation Lu=f is fully diagonalized and we obtain

$$u_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}} \left(\lambda_{\mathbf{x}} + \lambda_{\mathbf{y}}\right) = F_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}}$$

Thus we have replaced a complicated PDE in two variables with this one line of algebra which contains no integrals, derivatives, etc. I would call this a double diagonalization. This equation must be true for all pairs $\{\lambda_{\mathbf{x}}, \lambda_{\mathbf{y}}\}$ where each $\lambda_{\mathbf{x}}$ runs over its complete spectrum, whatever that might be (determined by its SL problem). For any pair $\{\lambda_{\mathbf{x}}, \lambda_{\mathbf{y}}\}$ which has the property $(\lambda_{\mathbf{x}} + \lambda_{\mathbf{y}}) = 0$, we shall need $F_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}} = 0$, and we would call this a "consistency condition" on the driving function, in line with Stak's vocabulary. If this property is not true, then our solution is

$$u_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}} = F_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}} / \left(\lambda_{\mathbf{x}} + \lambda_{\mathbf{y}}\right)$$

So all we have to do is compute the $F_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}}$ using our transform projections stated above,

$$F_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}} = \int_{\mathbf{ax}}^{\mathbf{bx}} dx\, s_{\mathbf{x}}(x)\, X_{\boldsymbol{\lambda x}}{}^{*}(x) \int_{\mathbf{ay}}^{\mathbf{by}} dy\, s_{\mathbf{y}}(y)\, Y_{\boldsymbol{\lambda y}}{}^{*}(y)\, F(x,y)$$

$$= \int_{\mathbf{ax}}^{\mathbf{bx}} dx\, s_{\mathbf{x}}(x)\, X_{\boldsymbol{\lambda x}}{}^{*}(x) \int_{\mathbf{ay}}^{\mathbf{by}} dy\, s_{\mathbf{y}}(y)\, Y_{\boldsymbol{\lambda y}}{}^{*}(y)\, f(x,y) / \left[\, h_{\mathbf{x}}(x)\, h_{\mathbf{y}}(y)\, s_{\mathbf{x}}(x)\, s_{\mathbf{y}}(y)\,\right]$$

$$= \int_{\mathbf{ax}}^{\mathbf{bx}} dx\, X_{\boldsymbol{\lambda x}}{}^{*}(x) \int_{\mathbf{ay}}^{\mathbf{by}} dy\, Y_{\boldsymbol{\lambda y}}{}^{*}(y)\, f(x,y) / \left[\, h_{\mathbf{x}}(x)\, h_{\mathbf{y}}(y)\,\right]$$

Our solution is then

$$u_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}} = F_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}} / \left(\lambda_{\mathbf{x}} + \lambda_{\mathbf{y}}\right)$$

and in coordinate space we have

$$u(x,y) = \sum_{\boldsymbol{\lambda x}} \sum_{\boldsymbol{\lambda y}} X_{\boldsymbol{\lambda x}}(x)\, Y_{\boldsymbol{\lambda y}}(y)\, u_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}} = \sum_{\boldsymbol{\lambda x}} \sum_{\boldsymbol{\lambda y}} X_{\boldsymbol{\lambda x}}(x)\, Y_{\boldsymbol{\lambda y}}(y)\, F_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}} / \left(\lambda_{\mathbf{x}} + \lambda_{\mathbf{y}}\right)$$

and the problem is completely solved, although the solution appears as a double sum/integral which is not always what we would like. As a final check of this result, consider:

$$L\,u(x,y) = \Sigma_{\lambda x}\,\Sigma_{\lambda y}\,L[X_{\lambda x}(x)\,Y_{\lambda y}(y)]\,F_{\lambda x,\lambda y}/(\lambda_x+\lambda_y)$$

$$= \Sigma_{\lambda x}\,\Sigma_{\lambda y}\,L[X_{\lambda x}(x)\,Y_{\lambda y}(y)]\,F_{\lambda x,\lambda y}/(\lambda_x+\lambda_y)$$

$$= \Sigma_{\lambda x}\,\Sigma_{\lambda y}\,(\lambda_x+\lambda_y)\,s(x,y)\,[X_{\lambda x}(x)\,Y_{\lambda y}(y)]\,F_{\lambda x,\lambda y}/(\lambda_x+\lambda_y)$$

$$= s(x,y)\Sigma_{\lambda x}\,\Sigma_{\lambda y}\;X_{\lambda x}(x)\,Y_{\lambda y}(y)\,F_{\lambda x,\lambda y}$$

$$= s(x,y)F(x,y) = h_x(x)\,h_y(y)\,s_x(x)\,s_y(y)\,F(x,y)$$

$$= f(x,y)$$

5. Going back to the homo equation Lu=0

We obtain

$$u_{\lambda x,\lambda y}\,(\lambda_x+\lambda_y) = 0$$

Now the only contributions to the solution u(x,y) come from pairs $\{\lambda_x, \lambda_y\}$ such that $(\lambda_x+\lambda_y) = 0$. And for such pairs we can have $u_{\lambda x,\lambda y}$ be an arbitrary value. Thus, we can write our homo solution as

$$u(x,y) = \Sigma_{\lambda x+\lambda y=0}\;u_{\lambda x,\lambda y}\;X_{\lambda x}(x)\,Y_{\lambda y}(y)$$

When we separated our PDE earlier, we had separation constant ν and we identified

$$\nu = \lambda_x = -\lambda_y \qquad \Rightarrow \qquad \lambda_x+\lambda_y = 0 \textit{ all the time}$$

We could then write the above sum as

$$u(x,y) = \Sigma_{\lambda x}\;u_{\lambda x,-\lambda x}\;X_{\lambda x}(x)\,Y_{-\lambda x}(y)$$

or

$$u(x,y) = \Sigma_{\lambda y}\;u_{-\lambda y,\lambda y}\;X_{-\lambda y}(x)\,Y_{\lambda y}(y)$$

where again the $u_{a,b}$ are arbitrary numbers. It is obvious that these are homo equation solutions since, for example using the first line above,

$$Lu = h_x(x)\,h_y(y)\,[s_y(y)\,L_x + s_x(x)\,L_y]\;\Sigma_{\lambda x}\;u_{\lambda x,-\lambda x}\;X_{\lambda x}(x)\,Y_{-\lambda x}(y)$$

$$= h_x(x)\,h_y(y)\,\Sigma_{\lambda x}\;u_{\lambda x,-\lambda x}\;[s_y(y)\,L_x + s_x(x)\,L_y]\,X_{\lambda x}(x)\,Y_{-\lambda x}(y)$$

$$= h_x(x)\,h_y(y)\,\Sigma_{\lambda x}\;u_{\lambda x,-\lambda x}\;[s_y(y)\,s_x(x)\lambda_x + s_x(x)(-\lambda_x)s(y)]\,X_{\lambda x}(x)\,Y_{-\lambda x}(y)$$

$$= h_{\mathbf{x}}(x)\, h_{\mathbf{y}}(y)\, s_{\mathbf{x}}(x)\, s_{\mathbf{y}}(y)\, \Sigma_{\boldsymbol{\lambda x}}\; u_{\boldsymbol{\lambda x},-\boldsymbol{\lambda x}}\; [\lambda_{\mathbf{x}} + (-\lambda_{\mathbf{x}})]\, X_{\boldsymbol{\lambda x}}(x)\, Y_{-\boldsymbol{\lambda x}}(y)$$

$$= 0$$

I refer to the product $X_{\boldsymbol{\lambda x}}(x)\, Y_{-\boldsymbol{\lambda x}}(y)$ as an "atom", and point out that the factors of this atom are "cross linked" in that the label $\lambda_{\mathbf{x}}$ appears in both factors. The solution of the homo equation is a linear combination of atoms! I have a document that lists the triple atomic forms for various curvilinear coordinate systems in 3D, but here we happen to be working in 2D.

6. What about the Green's Function equation Lu=δ?

We take our inhomo case now with f = δ. The solution will be

$$g(x,y) = \Sigma_{\boldsymbol{\lambda x}}\, \Sigma_{\boldsymbol{\lambda y}}\, X_{\boldsymbol{\lambda x}}(x)\, Y_{\boldsymbol{\lambda y}}(y)\, u_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}}$$

$$u_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}} = F_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}} / (\lambda_{\mathbf{x}} + \lambda_{\mathbf{y}})$$

$$F_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}} = \int_{\mathbf{ax}}^{\mathbf{bx}} dx\, X_{\boldsymbol{\lambda x}}^{*}(x) \int_{\mathbf{ay}}^{\mathbf{by}} dy\, Y_{\boldsymbol{\lambda y}}^{*}(y)\; f(x,y) / [\, h_{\mathbf{x}}(x)\, h_{\mathbf{y}}(y)\,]$$

The question now arises of what kind of δ you have in mind! To be flexible, let's allow some arbitrary function here and write (this would account for any curvilinear coordinates for example)

$$\delta = \delta(x-x_0)\, \delta(y-y_0)\, p(x_0,y_0)$$

Then we get

$$F_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}} = \int_{\mathbf{ax}}^{\mathbf{bx}} dx\, X_{\boldsymbol{\lambda x}}^{*}(x) \int_{\mathbf{ay}}^{\mathbf{by}} dy\, Y_{\boldsymbol{\lambda y}}^{*}(y)\; f(x,y) / [\, h_{\mathbf{x}}(x)\, h_{\mathbf{y}}(y)\,]$$

$$= \int_{\mathbf{ax}}^{\mathbf{bx}} dx\, X_{\boldsymbol{\lambda x}}^{*}(x) \int_{\mathbf{ay}}^{\mathbf{by}} dy\, Y_{\boldsymbol{\lambda y}}^{*}(y)\, \delta(x-x_0)\, \delta(y-y_0)\, p(x_0,y_0) / [\, h_{\mathbf{x}}(x)\, h_{\mathbf{y}}(y)\,]$$

$$= X_{\boldsymbol{\lambda x}}^{*}(x_0)\, Y_{\boldsymbol{\lambda y}}^{*}(y_0)\, p(x_0,y_0) / [\, h_{\mathbf{x}}(x_0)\, h_{\mathbf{y}}(y_0)\,]$$

We then obtain

$$g(x,y) = \Sigma_{\boldsymbol{\lambda x}}\, \Sigma_{\boldsymbol{\lambda y}}\, X_{\boldsymbol{\lambda x}}(x)\, Y_{\boldsymbol{\lambda y}}(y)\, u_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}}$$

$$= \Sigma_{\boldsymbol{\lambda x}}\, \Sigma_{\boldsymbol{\lambda y}}\, X_{\boldsymbol{\lambda x}}(x)\, Y_{\boldsymbol{\lambda y}}(y)\, F_{\boldsymbol{\lambda x},\boldsymbol{\lambda y}} / (\lambda_{\mathbf{x}} + \lambda_{\mathbf{y}})$$

$$= \Sigma_{\boldsymbol{\lambda x}}\, \Sigma_{\boldsymbol{\lambda y}}\, X_{\boldsymbol{\lambda x}}(x)\, Y_{\boldsymbol{\lambda y}}(y)\, X_{\boldsymbol{\lambda x}}^{*}(x_0)\, Y_{\boldsymbol{\lambda y}}^{*}(y_0)\, p(x_0,y_0) / [\, h_{\mathbf{x}}(x_0)\, h_{\mathbf{y}}(y_0)\,] / (\lambda_{\mathbf{x}} + \lambda_{\mathbf{y}})$$

$$= \{\, p(x_0,y_0) / [\, h_{\mathbf{x}}(x_0)\, h_{\mathbf{y}}(y_0)\}\;\; \Sigma_{\boldsymbol{\lambda x}}\, \Sigma_{\boldsymbol{\lambda y}}\, X_{\boldsymbol{\lambda x}}(x)\, Y_{\boldsymbol{\lambda y}}(y)\, X_{\boldsymbol{\lambda x}}^{*}(x_0)\, Y_{\boldsymbol{\lambda y}}^{*}(y_0) / (\lambda_{\mathbf{x}} + \lambda_{\mathbf{y}})$$

$$= \{ p(x_0,y_0) / [h_x(x_0) h_y(y_0)\}$$
$$* \quad \Sigma_{\lambda x} \Sigma_{\lambda y} [X_{\lambda x}(x) Y_{\lambda y}(y)] \ [X_\lambda(x_0) Y_{\lambda y}(y_0)]^* / (\lambda_x + \lambda_y)$$

Recall from "review of the generic regular BV problem" that the 1D situation was this

$$g(x|\xi;0) = \Sigma_n \varphi_n(x) F_n/(\lambda_n) = \Sigma_n \varphi_n(\xi)^* \varphi_n(x) /(\lambda_n)$$

Apart from our overall constant, which only controls scaling in a Green's function problem, we see how our 1D formula is generalized. We would regard this as the properly normalized EF

$$\varphi_{\lambda x,\lambda y}(x,y) = X_{\lambda x}(x) Y_{\lambda y}(y)$$

and then we have

$$g(x,y|x_0,y_0) = \{ p(x_0,y_0) / [h_x(x_0) h_y(y_0)\} \quad \Sigma_{\lambda x,\lambda y} \varphi_{\lambda x,\lambda y}(x,y) \varphi_{\lambda x,\lambda y}^*(x_0,y_0) / (\lambda_x + \lambda_y)$$

$$g(\mathbf{x}|\mathbf{x}_0) = \Sigma_\lambda \varphi_\lambda(\mathbf{x})^* \varphi_\lambda(\mathbf{x}_0) /(\lambda)$$

Notice that $g(x,y|x_0,y_0)$ will have our usual symmetry only if $p(x_0,y_0) = K\, h_x(x_0)\, h_y(y_0)$. Surely this is what happens when you work with Laplace or Helmholtz converted to some curvilinear coordinates where only two of the coordinates are active because it is a 2D problem or a 3D one with a symmetry.

7. Big Example: Let (x,y) = (r,θ), polar coordinates, with r in (0,a) and θ in (-π,π), and L = -∇².

(a) The two Sturm-Liouville problems and their solutions

Consider $-\nabla^2 u = f$ in polar coordinates so $L = -\nabla^2$. This is our example Lu=f.

$$L\, u(r,\theta) = f(r,\theta) \qquad L = g_r(\theta)h_r(r)L_r + g_\theta(r)h_\theta(\theta)L_\theta = -\nabla^2$$

$$[g_r(\theta)h_r(r)L_r + g_\theta(r)h_\theta(\theta)L_\theta\,]u(r,\theta) = f(r,\theta)$$

$$[\, -(1/r)\partial_r(r\partial_r) - (1/r)^2\partial_\theta^2)\, u(r,\theta) = f(r,\theta)$$

$$g_r(\theta)\, h_r(r)L_r = -(1/r)\partial_r(r\partial_r)$$
$$g_\theta(r)\, h_\theta(\theta)L_\theta = -(1/r)^2\partial_\theta^2$$

And this point we make the choice that $L_r = -\partial_r(r\partial_r)$ and $L_\theta = -\partial_\theta^2$, and this then forces values onto the four g and h functions. We then have (we select things to get "standard L's")

$L_r = -\partial_r(r\partial_r)$	$g_r(\theta)= 1$	$h_r(r) = 1/r$	$s_r(r) = 1/r$
$L_\theta = -\partial_\theta^2$	$g_\theta(r) = 1/r^2$	$h_\theta(\theta) = 1$	$s_\theta(\theta) = 1$

Notice that both L's are formally self-adjoint. We then find that

$s_r(r) = g_\theta(r)/ h_r(r) = (1/r^2)/(1/r) = 1/r$
$s_\theta(\theta) = g_r(\theta)/ h_\theta(\theta) = 1/1$

Our two eigenfunction equations are then

$L_r R_{\lambda r}(r) = \lambda_r (1/r) R_{\lambda r}(r)$ $\qquad\qquad$ $\lambda_r = -\lambda_\theta$
$L_\theta \Theta_{\lambda\theta}(\theta) = \lambda_\theta \Theta_{\lambda\theta}(\theta)$

Let's write them both out:

$-\partial_r(r\partial_r) R_{\lambda r}(r) = \lambda_r (1/r) R_{\lambda r}(r)$
$-\partial_\theta^2 \Theta_{\lambda\theta}(\theta) = \lambda_\theta \Theta_{\lambda\theta}(\theta)$

The actual eigenvalues λ_r and λ_θ are determined by the details of the boundary conditions for our problem.

If we have full θ azimuth (-π.π), for example, then we know $\lambda_\theta = n^2$ where n is an integer and $\Theta_{\lambda\theta}(\theta) \sim e^{in\theta}$. The corresponding transform is the **Exponential Fourier Series Transform** as in transforms.doc. The correctly normalized eigenfunctions are $\Theta_{\lambda\theta}(\theta) = (\sqrt{2\pi})^{-1} e^{-in\theta}$ where $\lambda_\theta = n^2$. We often use fuzzy notation and say that $\Theta_n(\theta) = (\sqrt{2\pi})^{-1} e^{-in\theta}$ even through $\lambda_\theta = n^2$. Proper normalization here means that $\int_{-\pi}^{\pi} d\theta\, \Theta_n(\theta)\, \Theta_m(\theta) = \delta_{n,m}$. The "measure" used in the eigenvalue summation is this:

$\Sigma_{\lambda\theta} = \Sigma_{\lambda\theta = 0,1,4,9..} = \Sigma_{n=0,1,2..} = \Sigma_{n=0}^{\infty}$

Now consider instead the r equation say on an interval (0,a). It is easy to show that $R_{\lambda r} = r^{\pm i\alpha}$ with $\alpha = \sqrt{\lambda_r}$. If for example we want R to vanish at r = a, we get the situation I wrote up in "the Mellin Transforms" . You get the **Mellin Sine Transform** which says this: (I refer to λ_r just as λ here)

$-\partial_r(r\partial_r) R_{\lambda r}(r) = \lambda (1/r) R_{\lambda r}(r)$
$-(rR_{\lambda r}')' - \lambda (1/r) R_{\lambda r} = 0$

$R_{\lambda r}(r) = (\pi\sqrt{\lambda})^{-1/2} \sin[\sqrt{\lambda}\, \ln(r/a)]$ // normalized eigenfunctions

$\delta(r-r')r = (2/\pi) \int_0^{\infty} dk \sin[k \ln(r/a)] \sin[k \ln(r'/a)]$ // completeness $\lambda = k^2$

$F(k) = \int_0^a dr \sin[k \ln(r/a)] f(r)$ // the transform $\lambda = k^2$

$f(r) = (2/\pi) \int_0^{\infty} dk \sin[k \ln(r/a)] F(k)$

The $R_\lambda(r)$ shown form a complete set on the interval (0,a), the spectrum is the entire positive real λ axis, and we have $\lambda = k^2$ as the relation between λ and the transform conjugate parameter k. The measure in our summation is now

$$\Sigma_{\lambda r} = \int_0^\infty d\lambda_r = \int_0^\infty [2k_r dk_r]$$

If we choose to express things in terms of $\lambda_r = k_r^2$, we note that $d\lambda_r = 2k_r dk_r$ which is the second form shown above. This is analogous to using n instead of λ_θ to enumerate the summation.

So in this case we find that λ_r = all positive real numbers and it is not quantized the way $\lambda_\theta = n^2$ was.

(b) Solve the Lu = f.

Now let's assume these two sets of boundary conditions and solve Lu=f using our double eigenfunction expansion. We have, as just noted,

$$R_{\lambda r}(r) = (\pi\sqrt{\lambda_r})^{-1/2} \sin[\sqrt{\lambda_r}\ \ln(r/a)] \qquad \lambda_r > 0$$
$$g_r(\theta) = 1 \qquad h_r(r) = 1/r \qquad s_r(r) = 1/r$$

$$\Theta_{\lambda\theta}(\theta) = (\sqrt{2\pi})^{-1}\ e^{-in\theta} \qquad \lambda_\theta = n^2,\ n=0,1,2...$$
$$g_\theta(r) = 1/r^2 \qquad h_\theta(\theta) = 1 \qquad s_\theta(\theta) = 1$$

We then do the following expansion for $u(r,\theta)$

$$u(r,\theta) = \Sigma_{n=0}^\infty \int_0^\infty d\lambda_r\, R_{\lambda r}(r)\, \Theta_{\lambda\theta}(\theta)\, u_{\lambda r,\lambda\theta}$$

We then define $F(r,\theta)$ this way

$$F(r,\theta) \equiv f(r,\theta)/[\, h_r(r)\, h_\theta(\theta)\, s_r(r)\, s_\theta(\theta)\,] = f(r,\theta)/[(1/r)\ 1\ (1/r)\ ] = r^2 f(r,\theta)$$

The double-diagonalized solution is

$$u_{\lambda r,\lambda\theta} = F_{\lambda r,\lambda\theta}/(\lambda_r + \lambda_\theta)$$

and in coordinate space we have

$$u(r,\theta) = \Sigma_{\lambda r}\, \Sigma_{\lambda\theta}\, R_{\lambda r}(r)\, \Theta_{\lambda\theta}(\theta)\, F_{\lambda r,\lambda\theta}/(\lambda_r + \lambda_\theta)$$

$$= \int_0^\infty [2k_r dk_r]\ \ \Sigma_{n=0}^\infty R_{\lambda r}(r)\, \Theta_{\lambda\theta}(\theta)\, F_{\lambda r,\lambda\theta}/(k_r^2 + n^2) \qquad \lambda_r = k_r^2 \quad \lambda_\theta = n^2$$

where

$$\Theta_{\lambda\theta}(\theta) = \{\, (\sqrt{2\pi})^{-1}\ e^{-in\theta}\,\} \qquad n = \sqrt{\lambda_\theta}$$
$$R_{\lambda r}(r) = \{\, (\pi k_r)^{-1/2} \sin[\, k_r \ln(r/a)\,]\,\} \qquad k_r = \sqrt{\lambda_r}$$

We do have to compute our $F_{\lambda r,\lambda\theta}$ coefficients this way

$$F_{\lambda x,\lambda y} = \int_{ax}^{bx} dx\, X_{\lambda x}^*(x) \int_{ay}^{by} dy\, Y_{\lambda y}^*(y)\ f(x,y)/ [\, h_x(x)\, h_y(y)\,]$$

$$F_{\lambda r,\lambda\theta} = \int_0^a dr\, R_{\lambda r}^*(r) \int_{-\pi}^{\pi} d\theta\, \Theta_{\lambda\theta}^*(\theta)\ f(r,\theta)/ [\, h_r(r)\, h_\theta(\theta)\,]$$

$$= \int_0^a dr\, R_{\lambda r}^*(r) \int_{-\pi}^{\pi} d\theta\, \Theta_{\lambda\theta}^*(\theta)\ f(r,\theta)/ [\, (1/r)\,]$$

$$= \int_0^a dr\, r\, R_{\lambda r}^*(r) \int_{-\pi}^{\pi} d\theta\, \Theta_{\lambda\theta}^*(\theta)\ f(r,\theta)$$

So one could try out some sample f(r,θ) and carry this all through.

(c) Solve Lu = 0

As for the homo equation Lu=f , we have two options for writing the solution

$$u(x,y) = \Sigma_{\lambda x}\ u_{\lambda x,-\lambda x}\ X_{\lambda x}(x)\, Y_{-\lambda x}(y)$$
$$u(x,y) = \Sigma_{\lambda y}\ u_{-\lambda y,\lambda y}\ X_{-\lambda y}(x)\, Y_{\lambda y}(y)$$

which now become

$$u(r,\theta) = \Sigma_{\lambda r}\ u_{\lambda r,-\lambda r}\ R_{\lambda r}(r)\, \Theta_{-\lambda r}(\theta)$$
$$u(r,\theta) = \Sigma_{\lambda\theta}\ u_{-\lambda\theta,\lambda\theta}\ R_{-\lambda\theta}(r)\, \Theta_{\lambda\theta}(\theta)$$

Here we have to pay attention more then normal:

$$\Theta_{\lambda\theta}(\theta) = \{\, (\sqrt{2\pi}\,)^{-1}\ \exp(-in\theta)\,\} = \{\, (\sqrt{2\pi}\,)^{-1}\ \exp(-i\sqrt{\lambda_\theta}\,\theta)\,\}$$

$$\Theta_{-\lambda r}(\theta) = \{\, (\sqrt{2\pi}\,)^{-1}\ \exp(-i\sqrt{-\lambda_r}\,\theta)\,\} = \{\, (\sqrt{2\pi}\,)^{-1}\ \exp(-k_r\theta)\,\}$$

We think of our λ_r integration variable as running along the "top of the cut", so $-\lambda_r$ is below the real axis on the left side, and therefore $\sqrt{-\lambda_r} = -i\sqrt{\lambda_r}$ and therefore $\exp(-i\sqrt{-\lambda_r}\,\theta)\,\} = \exp(-\sqrt{\lambda_r}\,\theta) = \exp(-k_r\theta)$. Now consider

$$R_{\lambda r}(r) = \{\, (\pi k_r)^{-1/2} \sin[\, k_r \ln(r/a)\,]\,\} \qquad k_r = \sqrt{\lambda_r}$$

If we want $R_{-\lambda\theta}(r)$, we need $\lambda_r = -\lambda_\theta = -n^2$. If we think of λ_θ as just above the cut (though no cut, just in the upper half plane), then $-\lambda_\theta$ is lower left at phase $-\pi$ so $\sqrt{-\lambda_\theta} = -i\,n$ so then $\sqrt{\lambda_r} = k_r = \sqrt{-\lambda_\theta} = -in$. Then we have

$$R_{-\lambda\theta}(r) = \{ (-in\pi)^{-1/2} \sin[-in \ln(r/a)] \} = \{ - (-in\pi k_r)^{-1/2} \sinh[n \ln(r/a)] \}$$

So our two forms for the homo solution were

$$u(r,\theta) = \Sigma_{\lambda r} \; u_{\lambda r,-\lambda r} \; R_{\lambda r}(r) \; \Theta_{-\lambda r}(\theta)$$
$$u(r,\theta) = \Sigma_{\lambda\theta} \; u_{-\lambda\theta,\lambda\theta} \; R_{-\lambda\theta}(r) \; \Theta_{\lambda\theta}(\theta)$$

and we insert to get

$$u(r,\theta) = \int_0^\infty [2k_r dk_r] \; u_{\lambda r,-\lambda r} \; \{ \quad (\pi k_r)^{-1/2} \quad \sin[k_r \ln(r/a)] \} \; \{ (\sqrt{2\pi})^{-1} \; \exp(-k_r\theta) \}$$

$$u(r,\theta) = \Sigma_{n=0}^\infty \; u_{-\lambda\theta,\lambda\theta} \; \{ - (-in\pi)^{-1/2} \quad \sinh[n \ln(r/a)] \} \; \{ (\sqrt{2\pi})^{-1} \; \exp(-in\theta) \}$$

where the coefficients can be arbitrary numbers. Basically L acting on either {..} gives 0, so the coefficients don't matter. Now if we think of the potential inside a disk with u=0 on the perimeter r=a, then yes, all the coefficients will be 0. But these "atomic expansions" would be appropriate for a situation where we have charges inside, but we are away from those charges, such as in a Green's Function situation. Then the coefficients are not 0 and have to be adjusted to match the charges somehow.

We can identify the basic "atoms" here as

osc expo			
$(1/\sqrt{k_r}) \sin[k_r \ln(r/a)] \; e^{-k_r\theta}$	$\lambda_r = k_r^2$ = positive real		r in (0,a)
expo osc			
$(1/\sqrt{n}) \sinh[n \ln(r/a)] \; e^{-in\theta}$	$n^2 = \lambda_\theta$ = 1,4,9...	n = 1,2..	θ in (-π,π)
[ln(r/a), 1] 1	n=0		

where in the second case we have to treat n=0 as a special case with different atoms.

Notice that a set of atoms like this is specific to the boundary conditions. In this example, our r BC was really that $R_{\lambda r}(r=a) = 0$, and for θ was the usual continuity stuff. Had we chosen $R_{\lambda r}'(r=a) = 0$, both atomic forms would be different.

So any linear combination of the above atoms solves the Laplace equation for the inside of a grounded circle of radius a.

In the second case above, notice that

$$\sinh[n \ln(r/a)] = (1/2)[e^{n\ln(r/a)} - e^{-n\ln(r/a)}] = (1/2)[(r/a)^n - (r/a)^{-n}]$$

and one could more generally think of the atoms as

$$(r/a)^{\pm n} \; e^{-in\theta}$$

which is how things appear in Stak p 128.

(d) The Green's Function for a circle with point charge at (r_0,θ_0)

We now have u = g and $f(r,\theta) = \delta(\mathbf{r}-\mathbf{r_0}) = \delta(\theta-\theta_0)\,\delta(r-r_0)/r_0$. According to work above, we know that

$$\delta = \delta(x-x_0)\,\delta(y-y_0)\,p(x_0,y_0)$$

$$g(x,y) = \{\, p(x_0,y_0) / [\, h_{\mathbf{x}}(x_0)\, h_{\mathbf{y}}(y_0)\}$$
$$* \quad \Sigma_{\lambda\mathbf{x}}\,\Sigma_{\lambda\mathbf{y}}\,[\, X_{\lambda\mathbf{x}}(x)\, Y_{\lambda\mathbf{y}}(y)]\;\; [X_{\lambda\mathbf{x}}(x_0)\, Y_{\lambda\mathbf{y}}(y_0)]^* / (\lambda_{\mathbf{x}} + \lambda_{\mathbf{y}})$$

$$g(r,\theta) = \{\, p(r_0,\theta_0) / [\, h_{\mathbf{r}}(r_0)\, h_{\theta}(\theta_0)\}$$
$$* \quad \Sigma_{\lambda\mathbf{r}}\,\Sigma_{\lambda\theta}\,[\, R_{\lambda\mathbf{r}}(r)\, \Theta_{\lambda\theta}(\theta)]\;\; [R_{\lambda\mathbf{r}}(r_0)\, \Theta_{\lambda\theta}(\theta_0)]^* / (\lambda_{\mathbf{r}} + \lambda_{\theta})$$

$$= \{[1/r_0] / [1/r_0]\,\}$$
$$* \quad \Sigma_{\lambda\mathbf{r}}\,\Sigma_{\lambda\theta}\,[\, R_{\lambda\mathbf{r}}(r)\, \Theta_{\lambda\theta}(\theta)]\;\; [R_{\lambda\mathbf{r}}(r_0)\, \Theta_{\lambda\theta}(\theta_0)]^* / (\lambda_{\mathbf{r}} + \lambda_{\theta})$$

$$= \Sigma_{\lambda\mathbf{r}}\,\Sigma_{\lambda\theta}\,[\, R_{\lambda\mathbf{r}}(r)\, \Theta_{\lambda\theta}(\theta)]\;\; [R_{\lambda\mathbf{r}}(r_0)\, \Theta_{\lambda\theta}(\theta_0)]^* / (\lambda_{\mathbf{r}} + \lambda_{\theta})$$

$$g(r,\theta|r_0,\theta_0) = \int_0^{\infty} [2k_{\mathbf{r}}dk_{\mathbf{r}}]\, \Sigma_{n=0}^{\infty}\; [\, R_{\lambda\mathbf{r}}(r)\, \Theta_{\lambda\theta}(\theta)]\;\; [R_{\lambda\mathbf{r}}(r_0)\, \Theta_{\lambda\theta}(\theta_0)]^* / (k_{\mathbf{r}}^2 + n^2)$$

$$g(\mathbf{x}|\mathbf{x_0}) = \Sigma_{\lambda}\, \varphi_{\lambda}(\mathbf{x})^* \varphi_{\lambda}(\mathbf{x_0}) / (\lambda)$$

where
$$\Theta_{\lambda\theta}(\theta) = \{\, (\sqrt{2\pi}\,)^{-1}\; e^{-in\theta}\,\}$$
$$R_{\lambda\mathbf{r}}(r) = \{\, (\pi k_{\mathbf{r}})^{-1/2} \sin[\, k_{\mathbf{r}} \ln(r/a)\,]\,\}$$

and there is our final answer. The double sum answers are never very convenient of course, and we know other more compact expressions of g(r,θ) such as from the image method. Remember this is g inside a grounded disk, it is not the free space g.

Since $g(\mathbf{x}|\mathbf{x_0}) = -\ln(1/R)$ in 2D, we can regard the above as just one of our many "1/R expansions" but in 2D we get the extra log

8. The Partial Eigenfunction Method of Analysis.

We first did the double expansion stuff shown above mainly just to get it out of the way. In practice, we are going to "work a problem" by using the complete set of eigenfunctions in *either* x or y.

(a) Use the x eigenfunctions

When we "separated variables" writing u(x,y) = X(x)Y(y), we got

$$L_{\mathbf{x}}\, X(x) = \nu\, s_{\mathbf{x}}(x)\, X(x)$$
$$L_{\mathbf{y}}\, Y(x) = -\nu\, s_{\mathbf{y}}(y)\, Y(y)$$

If we now set $\nu = \lambda_{\mathbf{x}} = -\lambda_{\mathbf{y}}$, these equations become

$L_x\, X_{\lambda x}(x) = \lambda_x\, s_x(x) X_{\lambda x}(x)$ $\qquad\qquad$ $\lambda_x = \nu$

$L_y\, Y_{\lambda y}(y) = \lambda_y\, s_y(y) Y_{\lambda y}(y)$ $\qquad\qquad$ $\lambda_y = -\nu$

Suppose we ignore the second equation for the moment, and solve the first equation for its EF's and its EV's, We then obtain some set of legal values $\nu = \{\lambda_x\}$ and we know the $X_{\lambda x}(x)$ functions. So consider our inhomo equation

$$Lu=f$$

$$h_x(x)\, h_y(y)\, [s_y(y)\, L_x + s_x(x)\, L_y]\, u(x,y) = f(x,y)$$

We as before write

$$f(x,y) \equiv F(x,y)\, [\, h_x(x)\, h_y(y)\, s_x(x)\, s_y(y)\,]$$

$$u(x,y) = \Sigma_{\lambda x}\; u_{\lambda x}(y)\, X_{\lambda x}(x)$$
$$F(x,y) = \Sigma_{\lambda x}\; F_{\lambda x}(y)\, X_{\lambda x}(x)$$

and we obtain for the LHS of Lu=f

$$h_x(x)\, h_y(y)\, [s_y(y)\, L_x + s_x(x)\, L_y]\, \Sigma_{\lambda x}\; u_{\lambda x}(y)\, X_{\lambda x}(x)$$

$$h_x(x)\, h_y(y)\, \Sigma_{\lambda x}\; [s_y(y)\, L_x\, u_{\lambda x}(y)\, X_{\lambda x}(x) + s_x(x)\, L_y\, u_{\lambda x}(y)\, X_{\lambda x}(x)]$$

$$h_x(x)\, h_y(y)\, \Sigma_{\lambda x}\; [s_y(y)\, \lambda_x\, s_x(x)\, u_{\lambda x}(y)\, X_{\lambda x}(x) + s_x(x)\, (L_y\, u_{\lambda x}(y))\, X_{\lambda x}(x)]$$

$$h_x(x)\, h_y(y)\, s_x(x)\, s_y(y)\, \Sigma_{\lambda x}\; X_{\lambda x}(x)\, [\lambda_x\; u_{\lambda x}(y) + \{(L_y\, u_{\lambda x}(y))/\, s_y(y)\}]$$

We now set this to the RHS which is

$$f(x,y) \equiv [\, h_x(x)\, h_y(y)\, s_x(x)\, s_y(y)\,]\; F(x,y) = h_x(x)\, h_y(y)\, s_x(x)\, s_y(y)\, \Sigma_{\lambda x}\; F_{\lambda x}(y)\, X_{\lambda x}(x)$$

We then cancel the four helper functions and use completeness of the $X_{\lambda x}(x)$ to get

$$\lambda_x\; u_{\lambda x}(y) + \{(L_y\, u_{\lambda x}(y))/\, s_y(y)\} = F_{\lambda x}(y)$$

$$L_y\, u_{\lambda x}(y) + \lambda_x\; s_y(y) u_{\lambda x}(y) = s_y(y)\, F_{\lambda x}(y) \qquad \text{// KEY RESULT!}$$

This is then a 2nd order 1D ODE in the variable y. If we can somehow solve this equation, somehow paying attention to the boundary conditions, then we would have the solution to our entire problem as :

$$u(x,y) = \Sigma_{\lambda x}\; u_{\lambda x}(y)\, X_{\lambda x}(x)$$

and we would then have completely avoided finding the $Y_{\lambda_y}(y)$eigenfunctions and eigenvalues $\{\lambda_y\}$. Notice that it is an eigenvalue λ_x of the X problem which is sitting quietly in our 1D $u_{\lambda_x}(y)$ y equation as a parameter, and of course $u_{\lambda_x}(y)$ will thus depend on λ_x, as our notation indicates. Notice also that our ODE in y for the function $u_{\lambda_x}(y)$ is NOT the ODE $L_y\, Y_{\lambda_y}(y) = \lambda_y\, s_y(y) Y_{\lambda_y}(y)$. Let's compare

$$L_y\, u_{\lambda_x}(y) + \lambda_x\, s_y(y) u_{\lambda_x}(y) = s_y(y)\, F_{\lambda_x}(y) \qquad (*)$$

$$L_y\, Y_{\lambda_y}(y) - \lambda_y\, s_y(y) Y_{\lambda_y}(y) = 0$$

If f(x,y) = 0, these two equations are the same apart from the eigenvalue sign and name. But if we have the inhomo case with some f, then the equations are not the same.

(b) Compute the Green's Function for the y-equation.

One *modus operandi* is to compute the Green's function for (*) and then use that to find the solution to the equation with the full RHS. So we now shall compute that Green's function for $g(a_y) = g(b_y) = 0$ (assume Dirichlet). We start with

$$L_y\, u_{\lambda_x}(y) + \lambda_x\, s_y(y) u_{\lambda_x}(y) = \delta(y - y_0)$$

This is a Standard Issue 1D Green's function problem which we know how to solve.

In the rest of this section, we shall *not* assume that $Y_{\lambda_y}(y)$ are the Y eigenvalue functions as described above, but instead they are just "some" generic solution of the EV equation without concern for its BC's.

We will continue to use the λ_y label to avoid a bunch of minus signs. If the endpoints are not limit-point singular, we know that at each endpoint we are free to add in a mixture of a second independent solution which we might call $\mathcal{Y}_{\lambda_y}(y)$. If we are limit-point at only one endpoint, there is only one possible s-norm solution at that end point, which we can again call $\mathcal{Y}_{\lambda_y}(y)$. So our general plan is to write (assuming **not limit point**)

$$g(y|y_0) = [A\, Y_{\lambda_y}(y_<) + B\, \mathcal{Y}_{\lambda_y}(y_<)]\, [A'\, Y_{\lambda_y}(y_>) + B'\, \mathcal{Y}_{\lambda_y}(y_>)]$$

Note that a form like this is automatically continuous at $y=y_0$. At that point we have

$$g(y_0|y_0) = [A\, Y_{\lambda_y}(y_0) + B\, \mathcal{Y}_{\lambda_y}(y_0)]\, [A'\, Y_{\lambda_y}(y_0) + B'\, \mathcal{Y}_{\lambda_y}(y_0)]$$

whether we let $y \to y_0$ from the left or right. Our (assumed here as Dirichlet type) end conditions are

$$A\, Y_{\lambda_y}(a_y) + B\, \mathcal{Y}_{\lambda_y}(a_y) = 0$$
$$A'\, Y_{\lambda_y}(b_y) + B'\, \mathcal{Y}_{\lambda_y}(b_y) = 0$$

Finally we require that the jump be equal to $1/a_0$, whatever that is for L_y

$$L_y = a_0(y)\, \partial_y^2 + \text{etc}$$

The jump is given by

$$\partial_y g^+ = [A\, Y_{\lambda_y}(y_0) + B\, \mathcal{Y}_{\lambda_y}(y_0)]\,[A'\, Y_{\lambda_y}'(y_0) + B'\, \mathcal{Y}_{\lambda_y}'(y_0)]$$
$$\partial_y g^- = [A\, Y_{\lambda_y}'(y) + B\, \mathcal{Y}_{\lambda_y}'(y)]\,[A'\, Y_{\lambda_y}(y_0) + B'\, \mathcal{Y}_{\lambda_y}(y_0)]$$

$$\begin{aligned}\text{jump} = {} & [A\, Y_{\lambda_y}(y_0) + B\, \mathcal{Y}_{\lambda_y}(y_0)]\,[A'\, Y_{\lambda_y}'(y_0) + B'\, \mathcal{Y}_{\lambda_y}'(y_0)] \\ & - [A\, Y_{\lambda_y}'(y_0) + B\, \mathcal{Y}_{\lambda_y}'(y_0)]\,[A'\, Y_{\lambda_y}(y_0) + B'\, \mathcal{Y}_{\lambda_y}(y_0)]\end{aligned}$$

$$\begin{aligned}= {} & AA'\,[\, Y_{\lambda_y}(y_0)\, Y_{\lambda_y}'(y_0) - Y_{\lambda_y}'(y_0)\, Y_{\lambda_y}(y_0)\,] \\ & + AB'\,[Y_{\lambda_y}(y_0)\, \mathcal{Y}_{\lambda_y}'(y_0) - Y_{\lambda_y}'(y_0)\, \mathcal{Y}_{\lambda_y}(y_0)\,] \\ & + A'B\,[\mathcal{Y}_{\lambda_y}(y_0)\, Y_{\lambda_y}'(y_0) - \mathcal{Y}_{\lambda_y}'(y_0)\, Y_{\lambda_y}(y_0)\,] \\ & + A'B'[\mathcal{Y}_{\lambda_y}(y_0)\, \mathcal{Y}_{\lambda_y}'(y_0) - \mathcal{Y}_{\lambda_y}'(y_0)\, \mathcal{Y}_{\lambda_y}(y_0)]\end{aligned}$$

$$= AB'\, W[Y_{\lambda_y}(y_0), \mathcal{Y}_{\lambda_y}(y_0)] + A'B\, W[\mathcal{Y}_{\lambda_y}(y_0), Y_{\lambda_y}(y_0)]$$

$$= (AB' - A'B)\, W[Y_{\lambda_y}(y_0), \mathcal{Y}_{\lambda_y}(y_0)]$$

The Wronskian is always some simple expression, and we then have our third condition which we then follow by our first two conditions

$$(AB' - A'B)\, W[Y_{\lambda_y}(y_0), \mathcal{Y}_{\lambda_y}(y_0)] = 1/a_0(y_0)$$
$$A\, Y_{\lambda_y}(a_y) + B\, \mathcal{Y}_{\lambda_y}(a_y) = 0$$
$$A'\, Y_{\lambda_y}(b_y) + B'\, \mathcal{Y}_{\lambda_y}(b_y) = 0$$

How can we have four coefficients and only 3 conditions? Well think of the 3 conditions this way

$$(AB' - A'B) = \alpha$$
$$A = \beta B$$
$$A' = \beta' B'$$

Insert into the first to get

$$\beta BB' - \beta' B'B = \alpha \quad \Rightarrow \quad BB'(\beta - \beta') = \alpha \quad BB' = \alpha/(\beta - \beta')$$

So we know BB'. Now multiply 2nd equation by B'

$$AB' = \beta BB' = \beta\, \alpha/'(\beta - \beta')$$

So we know AB'. Now multiply 3rd equation by A:

$$AA' = \beta' AB' = \beta^2\, \alpha/'(\beta - \beta')$$

So we know AA'. Now multiply 3rd equation by B

$$BA' = \beta' BB' = = \beta'\, \alpha/(\beta - \beta')$$

So we know BA'. Thus we know all four coefficients appearing in g

$$g(y|y_0) = [A\, Y_{\lambda_y}(y_<) + B\, \mathcal{Y}_{\lambda_y}(y_<)]\,[A'\, Y_{\lambda_y}(y_>) + B'\, \mathcal{Y}_{\lambda_y}(y_>)]$$

$$= AA'\, Y_{\lambda_y}(y_<)\, Y_{\lambda_y}(y_>) + BA'\, \mathcal{Y}_{\lambda_y}(y_<)\, Y_{\lambda_y}(y_>) + \text{etc}$$

It is true that we don't know the individual coefficients so this is the degree of freedom that is missing by having only 3 equations on 4 unknowns. Fine, we have what we need.

If we are **limit point at the right end**, say, then we will have

$$g(y|y_0) = [A\, Y_{\lambda_y}(y_<) + B\, \mathcal{Y}_{\lambda_y}(y_<)]\,[\, \mathcal{Y}_{\lambda_y}(y_>)]$$

which is as if A'=0 and B'=1. Note that this $\mathcal{Y}_{\lambda_y}(y)$ is now a specific function, whereas before it could be any function independent of $Y_{\lambda_y}(y)$. We no longer have a right-end g=0 condition so here are our equations this time

left end:	$A\, Y_{\lambda_y}(a_y) + B\, \mathcal{Y}_{\lambda_y}(a_y) = 0$
jump:	$(A)\, W[Y_{\lambda_y}(y_0), \mathcal{Y}_{\lambda_y}(y_0)] = 1/a_0(y_0)$

and this lets us determine A and B.

If we are **limit point at both ends**, then we know that

$$g(y|y_0) = \alpha\, [Y_{\lambda_y}(y_<)]\,[\, \mathcal{Y}_{\lambda_y}(y_>)]$$

where Y is the only solution that works at the left end and $\mathcal{Y}$ is the only one that works at the right end. Now we have no endpoint conditions and only the jump condition and that determines α. If we think of this case as A=α, B=0, A' = 0, B'=1 then the jump condition can be read from above,

$$(AB' - A'B)\, W[Y_{\lambda_y}(y_0), \mathcal{Y}_{\lambda_y}(y_0)] = 1/a_0(y_0)$$
$$\alpha\, W[Y_{\lambda_y}(y_0), \mathcal{Y}_{\lambda_y}(y_0)] = 1/a_0(y_0)$$

So notice that we have successfully found the Green's Function without computing the actual Y eigenfunctions. We only needed to have two independent solutions of the EV equation ignoring BC's. So a big point is that *we avoided* that detailed Y eigenfunction work, and we were probably able to select our generic Y and $\mathcal{Y}$ functions in some simple way to naturally meet our BC's at the individual end.

(c) Use the y eigenfunctions

We just repeat everything stated above with x↔y. This time we have to compute the Y eigenfunctions, but we don't have to do the X ones.

Here are some of the equations:

$$f(x,y) \equiv F(x,y)\,[\,h_{\mathbf{x}}(x)\,h_{\mathbf{y}}(y)\,s_{\mathbf{x}}(x)\,s_{\mathbf{y}}(y)\,]$$

$$u(x,y) = \Sigma_{\boldsymbol{\lambda y}}\; u_{\boldsymbol{\lambda y}}(x)\,Y_{\boldsymbol{\lambda y}}(y)$$
$$F(x,y) = \Sigma_{\boldsymbol{\lambda y}}\; F_{\boldsymbol{\lambda y}}(x)\,Y_{\boldsymbol{\lambda y}}(y)$$

We end up with a 1D equation in x, where now $\lambda_{\mathbf{y}} = -\lambda_{\mathbf{x}}$ but we write it as $\lambda_{\mathbf{y}}$, and we compare

$$L_{\mathbf{x}}\,u_{\boldsymbol{\lambda y}}(x) + \lambda_{\mathbf{y}}\; s_{\mathbf{x}}(x)u_{\boldsymbol{\lambda y}}(x) = s_{\mathbf{x}}(x)\,F_{\boldsymbol{\lambda y}}(x)$$

$$L_{\mathbf{x}}\,X_{\boldsymbol{\lambda y}}(x) - \lambda_{\mathbf{x}}\; s_{\mathbf{x}}(x)X_{\boldsymbol{\lambda y}}(x) = 0 \qquad \text{// the EV equation for X}$$

We can compute the Green's Function as before, but now in the x world.

$$g(x|x_0) = [A\,X_{\boldsymbol{\lambda x}}(x_<) + B\,\mathcal{X}_{\boldsymbol{\lambda x}}(x_<)]\,[A'\,X_{\boldsymbol{\lambda x}}(x_>) + B'\,\mathcal{X}_{\boldsymbol{\lambda x}}(x_>)] \qquad \text{// regular or limit circle}$$

$$g(x|x_0) = [A\,X_{\boldsymbol{\lambda x}}(x_<) + B\,\mathcal{X}_{\boldsymbol{\lambda x}}(x_<)]\,[\mathcal{X}_{\boldsymbol{\lambda x}}(x_>)] \qquad \text{// right end limit point}$$

$$g(x|x_0) = \alpha\,[X_{\boldsymbol{\lambda x}}(x_<)]\,[\mathcal{X}_{\boldsymbol{\lambda x}}(x_>)] \qquad \text{// both ends limit point}$$

Comments.

(a) Notice that in our first method, we used $\nu = \lambda_{\mathbf{x}}$ as our spectrum (since we used the X eigenfunctions). In the second method we used $\nu = -\lambda_{\mathbf{y}}$ and in this is *a completely different spectrum*!

(b) If we did the Helmholtz equation, we would have an extra parameter "λ" floating around everywhere which is separate from our $\lambda_{\mathbf{x}}$ and $\lambda_{\mathbf{y}}$ here.

9. Curvilinear Coordinates

(a) 2D systems

The general Laplacian formula is this for orthogonal 2D systems

$$\nabla^2 f = (1/\sqrt{g'})\,\{\,\partial_1\,[\sqrt{g'}\,g'^{11}\,(\partial_1 f)\,] + \partial_2\,[\sqrt{g'}\,g'^{22}\,(\partial_2 f)\,]\,\}$$

$$\sqrt{g'} = \sqrt{\det(g')} = Q_1 Q_2 \qquad g'^{ii} = Q_i^{-2}$$

So we can write this out a bit as

$$\nabla^2 f = (1/Q_1Q_2)\,\{\,\partial_1\,[(Q_2/Q_1)\,(\partial_1 f)\,] + \partial_2\,[(Q_1/Q_2)\,(\partial_2 f)\,]\,\}$$

In general,

$$\nabla^2 f = (1/Q_1Q_2) \{ \partial_1 [(Q_2/Q_1) (\partial_1 f)] + \partial_2 [(Q_1/Q_2) (\partial_2 f)] \}$$

Now all the usual 2D systems listed in M&S Chapter 2 (including polars treated in the special way), have $Q_1 = Q_2$ due to their conformal mapping origin, so the above simplifies quite a bit

$$\nabla^2 f = (1/g_{11}(12)) (\partial^2_1 + \partial^2_2) f$$

where the g_{11} functions are listed off on page 52-55, and in general one sees that $g_{11}(12)$ does not factor in to a form a(1)b(2).

For the *Laplace* equation, we would then have

$$(1/g_{11}(x,y)) [\partial_x^2+\partial_y^2] u(x,y) = 0 \qquad \Rightarrow \; [\partial_x^2+\partial_y^2] u(x,y) = 0$$

and then we are back into the mold of this document (starting with Section 1 above). So we know we can still find separation in the sense u(x,y) = X(x)Y(y).

Helmholtz is a different story:

$$(1/g_{11}(x,y)) [\partial_x^2+\partial_y^2] + \kappa^2) X(x)Y(y) = 0$$

$$(1/g_{11}(x,y)) [\partial_x^2 X(x)Y(y)+\partial_y^2 X(x)Y(y)] + \kappa^2 X(x)Y(y)) = 0$$

$$(1/g_{11}(x,y)) [(\partial_x^2 X(x))Y(y)+ X(x) (\partial_y^2 Y(y))] + \kappa^2 X(x)Y(y)) = 0$$

$$[(\partial_x^2 X(x))/X(x)+ (\partial_y^2 Y(y)/Y(y)] + \kappa^2 g_{11}(x,y)) = 0$$

How can this possibly separate? Well the trick is that $g_{11}(x,y)$ must have the form

$$g_{11}(x,y) = a(x) + b(y)$$

Looking through the tables p 52-55 we see this is true only for: (1) E1 special **polar** coordinates [a(x) only]; (2) P2 which is 2D **parabolic** coordinates; (3) H1 (Fig 214 p 69) **elliptic** coordinates. This agrees with the conclusions of my N=2 Stackel paper section.

Looking at one of these three cases, for elliptic coordinates, if one writes (as on page 53)

$x = a \cosh u \cos v$
$y = a \sinh u \sin v$

$g_{11} = (a\cosh u)^2 - (a \cos v)^2 = a(x) + b(y) = Q_1^2 = Q_2^2$

So if the separation shown above into a(x) + b(y) then we DO get EV problems.

$$[(\partial_x^2 X(x))/X(x)+ (\partial_y^2 Y(y)/Y(y)] + \kappa^2 a(x) + \kappa^2 b(y)) = 0$$

$$(\partial_x^2 X(x))/X(x) + \kappa^2 a(x) = - (\partial_y^2 Y(y)/Y(y)] - \kappa^2 b(y) = \nu$$

$$\partial_x^2 X(x) + \kappa^2 a(x) X(x) - \nu X(x) = 0$$
$$\partial_y^2 Y(y) + \kappa^2 b(y) Y(y) + \nu Y(y) = 0$$

$L_x X(x) = \nu X(x)$ $\quad L_x = \partial_x^2 + \kappa^2 a(x)$

$L_y Y(y) = -\nu Y(a)$ $\quad L_y = \partial_y^2 + \kappa^2 b(y)$

(b) 3D systems with solutions not dependent one of the variables

If the 3D system was formed by extruding or rotating a 2D system, and if it is that special 3rd variable that the solution does not depend on, then I think there is nothing new to talk about.

For example, M&S talk about elliptic-cylinder coordinates on page 20, and z is the ignored variable, then this is the same as elliptical coordinates under section (a) above. It happens that for these a(x) and b(y) functions we then have

$L_x X(x) = \nu X(x)$ $\quad L_x = \partial_x^2 + \kappa^2 a^2 \cosh^2 x$

$L_y Y(y) = -\nu Y(a)$ $\quad L_y = \partial_y^2 - \kappa^2 a^2 \cos^2 y$

and you see these equations on page 20. The eigenfunctions happen to be some horrible ce_m class functions which are Mathieu functions about which I known nothing whatsoever.

Comment: When you include the z coordinate dependence in this last case, you get the tangle of two separation constants for the η and ψ equations as shown page 19. If you fix one separation constant α_2, you might think of it as an EV problem for the other separation constant α_1. But then your spectrum will be some $\alpha_1 = f(\alpha_2)$. For each α_2, you might get a spectrum $\{\alpha_1\}$, but all the points will "move" as you change α_2. This means the spectrum is really a set of trajectories in α_1 space, $\alpha_{1i} = f_i(\alpha_2)$. So this is they very far from my doc here. But still you can consider one of the separated tangled equations with its BC on level surface, and the BC will then give you some relation between α_1 and α_2. I think the Lamé functions case is typical of this separation constant tangle. I cannot remember enough to make sense of it right now. There are two quantum numbers, but what are the BC's that make them quantize? I quote my Hobson notes:

"We are made aware that the $E_{n,p}(\mu)E_{n,p}(\nu)$ situation is a 2D Sturm Liouville problem which quantizes the n and p parameters all at once, and cannot be decomposed into a product of 1D systems. "

I think the {n,p} eigenvalues are both determined by finiteness requirements, the way the n in $P_n(\cos\theta)$ is quantized to integer.

Comment: None of the 2D systems considered in this paper is separable with R ≠ constant, that is to say, they are all "simple separable".

10. A review of Stak examples in light of this doc

Problems "in more than one dimension" is Section 7.9 starting page 253, but it is not really until Section 7.12 on Helmholtz that things start moving

Page 268 (7.160) Lu=0. Here we have our classic polar separated equations for the first time I think at least in this chapter. On page 269 he considers $Lu=\delta(r-r_0)\delta(\varphi)$ for the point charge, an example of our

Section 6 above. Stak uses φ expo eigenfunctions in his "partial EF analysis" and obtains the 1D Green's in variable r which he then solves. This is done with no finite BC's,. so you end up with a fancy way to write the E_2 free-space Helmholtz solution, and this then leads to Bessel function sum rules.

Now in his analysis he applies the φ projector to both sides of the PDE and I kept wondering: how do you know this is going to separate? How do you know you won't end up with some integral on the LHS that you cannot get rid of? Well, I show in Section 8 (a) above that, since this PDE has the simple form considered in this doc, you really *do* get an ODE for the coefficient function $u_{\lambda x}(y)$, so this shows that things *must* separate in this sense, and there are no "integrals" in this ODE. Then the Green's function is just a special case of equation (*) at the end of Section 8 (a). Alternatively, we know it separates because polar coordinates separate according to Stackel theory or just according to the fact that the classical cylindrical system separates for Helmholtz.

Page 272, 2D Green's function inside a wedge. We already did this earlier with Green's 7.173 as the partial EF solution where sines are the eigenfunctions leaving a 1D Green's in r. On page 273, he instead does the partial EF solution using r as the EF's and this is Kantorovich with the K_{iY} functions. So this is like switching from x EF's to y EF's as I did above in Section 8 (c). Again, we know ahead of time that everything is going to separate "with no integrals on the LHS side".

Page 276: 3D Helmholtz with ring source in cylindrical coords, no φ dependence, so only ρ and z present. We know that even when all three dependencies are present, things here will be simple-separable, so clearly that will still be true when you are dealing only with ρ and z. Here he first does the problem using the r EF's $J_0(\gamma r)$ (Hankel transform) with coefficient functions $U_H(z)$, and he obtains a 1D Greens ODE for $U_H(z)$. Again, no integrals on the LHS. We know it has to work because we know things separate. Just as I did this in Section 8 (a) in "the general case". Then on page 278 he does it all over again using z EF's (Fourier Transform)

Page 281: We are back for more! This is the point source near grounded half-wire in 2D. He expands on the sin φ type EF's and gets a 1D Green's problem in r and solves it and replicates an earlier result.

Page 290 Exercises: 3D point charge on z axis on φ dependence, spherical coordinates. So here our two coordinates of interest are r and θ. But again, we know Helmholtz separates in r,θ,φ so it surely separates in this r,θ only problem and we get another sum rule this time involving Legendre functions. This same stuff appears in more of these exercises, they are all special cases of my assumed form above, and of course they are all Helmholtz in known-separable systems!

Appendix A: stuff moved out of the main text flow above, where marked.

Comment added 12.9.11. If our PDE really was the Helmholtz equation, and if the variables really were some orthogonal curvilinears, then I would just USE the Stackel theory, find the matrix Φ, and know the ODE's for the two coordinates. I would not use this document at all. So I guess therefore I don't now see the point of the following digression. I am therefore right now demoting this section to an Appendix A.

New Section Added 8.30.11. How does the above relate to 2D Stackel separation? We know in that case that if we are starting with the Helmholtz equation, we get this, where $\kappa_1^2 = K_1^2$,

$$L_n X_n = (1/f_n)\partial_n[f_n(\partial_n X_n)] + [\, K_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n)]X_n = 0$$
$$L_n X_n = (1/f_n)\partial_n[f_n(\partial_n X_n)] + [\, K_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n)]X_n = 0$$

or

$$L_x X = (1/f_x)\partial_x[f_x(\partial_x X)] + [\, K_1^2\Phi_{x1}(x) + k_2^2\Phi_{x2}(x)]X = 0$$
$$L_y Y = (1/f_y)\partial_y[f_y(\partial_y X)] + [\, K_1^2\Phi_{y1}(y) + k_2^2\Phi_{y2}(y)]Y = 0$$

which we can compare to our forms above, for which I will now use script L

$$\mathcal{L}_x\, X(x) = \nu\, s_x(x)\, X(x)$$
$$\mathcal{L}_y\, Y(x) = -\nu\, s_y(y)\, Y(y)$$

so rewrite the forms above as

$$(1/f_x)\partial_x[f_x(\partial_x X)] + [\, K_1^2\Phi_{x1}(x)\,]X = -\, k_2^2\Phi_{x2}(x)]X$$
$$(1/f_y)\partial_y[f_y(\partial_y X)] + [\, K_1^2\Phi_{y1}(y)\,]Y = -\, k_2^2\Phi_{y2}(y)\, Y$$

$$\{(1/f_x)\partial_x[f_x(\partial_x)] + K_1^2\Phi_{x1}(x)\ \}\, X = -\, k_2^2\Phi_{x2}(x)]X$$
$$\{(1/f_y)\partial_y[f_y(\partial_y)] + K_1^2\Phi_{y1}(y)\ \}Y = -\, k_2^2\Phi_{y2}(y)\, Y$$

We can then make the connection by saying

$$\mathcal{L}_x = \{(1/f_x)\partial_x[f_x(\partial_x)] + K_1^2\Phi_{x1}(x)\ \} \qquad \nu = -k_2^2 \qquad \Phi_{x2}(x) = +s(x)$$
$$\mathcal{L}_y = \{(1/f_y)\partial_y[f_y(\partial_y)] + K_1^2\Phi_{y1}(y)\ \} \qquad \nu = -k_2^2 \qquad \Phi_{y2}(y) = -s(y)$$

Now we would like to somehow "add these" to get some total L object. Stackel (3.10) says

$$(1/\psi)[L\psi] = \Sigma_n(1/[h_n^2 X_n])\,(1/f_n)\partial_n[f_n(\partial_n X_n)] + k_1^2/Q + K_1^2 = 0 \qquad (3.10)$$

and for simple-separation we set $k_1^2 = 0$, so we have

$$(1/\psi)[L\psi] = \Sigma_n(1/[h_n^2 X_n])\,(1/f_n)\partial_n[f_n(\partial_n X_n)] + K_1^2$$

$$= (1/[h_x^2 X])(1/f_x)\,\partial_x[f_x(\partial_x X)] + (1/[h_y^2 Y])(1/f_y)\,\partial_y[f_y(\partial_y Y)] + K_1^2$$

$$= (1/[h_x^2 X])\{\mathcal{L}_x - K_1^2\Phi_{x1}(x)\}X + (1/[h_y^2 Y])\,\{\mathcal{L}_y - K_1^2\Phi_{y1}(y)\}Y + K_1^2$$

Now multiply each term by $\psi = XY$ to get

$$[L\psi] = (1/[h_x^2])\{\mathcal{L}_x - K_1^2\Phi_{x1}(x)\}XY + (1/[h_y^2])\,\{\mathcal{L}_y - K_1^2\Phi_{y1}(y)\}XY + K_1^2\, XY$$

$$[L\psi] = (1/[h_x^2])\{\mathcal{L}_x - K_1^2\Phi_{x1}(x)\}\psi + (1/[h_y^2])\,\{\mathcal{L}_y - K_1^2\Phi_{y1}(y)\}\psi + K_1^2\,\psi$$

Finally we can identify

$$L = (1/[h_x^2])\{ \mathcal{L}_x - K_1^2\Phi_{x1}(x)\} + (1/[h_y^2]) \{ \mathcal{L}_y - K_1^2\Phi_{y1}(y)\} + K_1^2$$

$$L = (1/h_x^2)\, \mathcal{L}_x + (1/h_y^2)\, \mathcal{L}_y - \{ (1/h_x^2)\, K_1^2\Phi_{x1}(x) + (1/h_y^2)\, K_1^2\Phi_{y1}(y)\} + K_1^2$$

$$L = (1/h_x^2)\, \mathcal{L}_x + (1/h_y^2)\, \mathcal{L}_y - \{ (1/h_x^2)\, K_1^2\Phi_{x1}(x) + (1/h_y^2)\, K_1^2\Phi_{y1}(y) - K_1^2 \}$$

$$L = (1/h_x^2)\, \mathcal{L}_x + (1/h_y^2)\, \mathcal{L}_y - K_1^2 \{ (1/h_x^2)\Phi_{x1}(x) + (1/h_y^2)\Phi_{y1}(y) - 1 \}$$

This is the first time I have ever written L as a sum of $\mathcal{L}_x$ and $\mathcal{L}_y$. The surprise is that there are "extra terms" which don't fit my starting mold! I should write this down at once for polar coordinates.

Meanwhile, we know that the full Helmholtz equation has this form

$$H^{-1}\{ \partial_1[(H/h_1^2)(\partial_1\psi)] + \partial_2[(H/h_2^2)(\partial_2\psi)]\}\psi + K_1^2\psi = 0 \qquad H \equiv h_1h_2$$

$$H^{-1}\{ \partial_x[(H/h_x^2)(\partial_x\psi)] + \partial_y[(H/h_y^2)(\partial_y\psi)]\}\psi + K_1^2\psi = 0 \qquad H \equiv h_1h_2$$

Now we have been assuming simple separation in our $\psi = XY$ above, so set $R = Q = 1$, and now also assume (3.5) which says

$$(H/[h_x^2]) = f_x(x)g_x(y)$$
$$(H/[h_y^2]) = f_y(y)g_y(x)$$

Then our Helmholtz above reads

$$H^{-1}\{ \partial_x[f_x(x)g_x(y)\, (\partial_x\psi)] + \partial_y[f_y(y)g_y(x)\, (\partial_y\psi)]\}\psi + K_1^2\psi = 0$$

or

$$H^{-1}\{ g_x(y)\partial_x[f_x(x)\, (\partial_x\psi)] + g_y(x)\, \partial_y[f_y(y)\, (\partial_y\psi)]\}\psi + K_1^2\psi = 0$$
