**Exercises 7.31 Page 290, Second Chapter 7 Midterm Exam** 8.2.11

It took 15 pages to do the first problem, so it went into a separate document rather than try to pack this into the raw3 notes.


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**Overview from Meta Notes:**

**Exercise 7.31 ( p 290).** This problem (and the next) are solved in a separate doc in the Ch 7 support section. We consider here the 3D Helmholtz Green's PDE with a point source on the + z axis, so there is φ azisym. Our task is to solve the PDE for u(r,θ), so we have a 2-variable problem. We select θ for eigenfunctions which are $P_n(\cos\theta)$, and in the usual way this leads to a 1D Green's in variable r which we solve in terms of certain J and H Bessel functions, and the resulting u(r,θ) is the RHS of 7.224. But since there were no local BC's in this problem, that RHS must also be $E_3$ which is then the LHS of 7.224. In the usual way we rewrite this equation for λ = $-k^2$ and then the Bessel I and K functions appear in 7.225.

**Exercise 7.32 ( p 290).** Same problem as above, but replace the z-axis point source with a ring source. It turns out that, when you write the δδ for this ring source, it is exactly the same as the δδ source for the previous problem except now we have δ(θ-$\theta_0$) instead of δ(θ). We are still azisym so still an r,θ problem. The exact same solution method then applies and you get the exact same result except where we used to have a 1, we now have $P_n(\cos\theta_0)$.

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**Exercise 7.31: An expansion for exp(i$\sqrt{\lambda}$ R)/4πR for $r_0$ on the z axis (290)**

We are given the Helmholtz equation in 3D driven by a point charge on the + z axis. I confirmed the distribution description of this point charge by integrating over a small curved frustum containing such a point. The meaning of δ(θ) is $\lim_{\varepsilon\to 0}\delta(\theta+\varepsilon)$, by which I mean it is a "full" delta, not a half delta. This is always an issue for a δ at a finite range endpoint. Note that this is an azisym situation.

We know the fundy solution of this problem from page 55 to be

$u = E(\mathbf{r}|\mathbf{r}_0) = \exp(i\sqrt{\lambda}\, R)/4\pi R$ where $R = |\mathbf{r}-\mathbf{r}_0|$

This azisym problem involves coordinates r and θ only. A complete set of functions for the θ variable is the Legendre polys $P_n(\cos\theta)$. So we construct

$$u(r,z) = \Sigma_{n=0}^{\infty} a_n(r)\, P_n(z). \qquad a_n(r) = (n+1/2) \int_{-1}^{1} dz\, P_n(z)\, u(r,z)$$

where I use my result in transforms.doc with m=0 (any m is valid). Notice that this applies to θ in the full sphere range (0,π) which is appropriate for a fundy situation. Here is the full transforms.doc quote:

$$g(z) = \Sigma_{n=|m|}^{\infty} g_{nm}\, P_n^m(z) \qquad \text{// expansion}$$

$$g_{nm} = (1/K_n^m) \int dz\, P_n^m(z)\, g(z) \qquad K_n^m = (n+1/2)^{-1}\, f(n,m) \qquad \text{// projection}$$

**Obtain the 1D Green's equation in r.**

Now the usual step is to apply $(n+1/2) \int_{-1}^{1} dz\, P_n(z)$ to each term in our PDE, so we better state that PDE:

$$(-\nabla^2-\lambda)u(r,\theta) = \delta(r-r_0)\delta(\theta)/[2\pi r^2\sin\theta] \qquad \text{// RHS is unit charge on +z axis !}$$

$$-r^{-2}\partial_r(r^2\partial_r u) - r^{-2}/\sin\theta * \partial_\theta(\sin\theta\partial_\theta u) - \lambda u = \delta(r-r_0)\delta(\theta)/[2\pi r^2\sin\theta]$$

$$-\partial_r(r^2\partial_r u) - 1/\sin\theta * \partial_\theta(\sin\theta\partial_\theta u) - \lambda r^2 u = \delta(r-r_0)\delta(\theta)/[2\pi\sin\theta]$$

So let's stick to θ coordinates and apply $(n+1/2) \int_0^{\pi} d\theta\, \sin\theta\, P_n(\cos\theta)$. There are three terms on the LHS and we will do them one at a time:

$$(n+1/2) \int_0^{\pi} d\theta\, \sin\theta\, P_n(\cos\theta)\, \{ -\partial_r(r^2\partial_r u(r,\theta)\} =$$

$$= -\partial_r(r^2\partial_r \{ (n+1/2) \int_0^{\pi} d\theta\, \sin\theta\, P_n(\cos\theta)\, u(r,\theta) \} = -\partial_r(r^2\partial_r \{ a_n(r) \} \quad \text{// first term}$$

$$(n+1/2) \int_0^{\pi} d\theta\, \sin\theta\, P_n(\cos\theta)\, \{ -\lambda r^2 u \} = -\lambda\, r^2 a_n(r) \qquad \text{// third term}$$

We saved the harder middle term for last:

$$(n+1/2) \int_0^{\pi} d\theta\, \sin\theta\, P_n(\cos\theta)\, \{ -1/\sin\theta * \partial_\theta(\sin\theta\partial_\theta u)\} \qquad \text{// middle term}$$

$$= -(n+1/2)\int_0^{\pi} d\theta\, P_n(\cos\theta)\ \partial_\theta(\sin\theta\partial_\theta u(r,\theta)) \equiv -(n+1/2)\, q_n(r)$$

We are going to do parts integration twice. We need to know certain facts:

$$P_n(\cos\theta)|^{\theta=\pi} = 1 \qquad P_\nu{}^\mu(1) = \delta_0{}^\mu \qquad \mu=\text{integer}$$
$$P_n(\cos\theta)|^{\theta=-\pi} = (-1)^n$$

Let's just work now with

$$q_n(r) \equiv \int_0^{\pi} d\theta\, P_n(\cos\theta)\ \partial_\theta(\sin\theta\partial_\theta u(r,\theta))$$

Here is our **first parts**:

$$= -\int_0^{\pi} d\theta\, \partial_\theta P_n(\cos\theta)\ (\sin\theta\partial_\theta u(r,\theta)) + [\, P_n(\cos\theta)\ (\sin\theta\partial_\theta u(r,\theta))]\,|^{\pi}_{-\pi}$$

The parts part involves $\sin(\pm\pi)$ at each endpoint times finite functions, so vanishes. We know ahead of time in any SL problem like this that the parts will always vanish since we have a self-adjoint L, but we will carry these parts and check them anyway, it does not hurt. So we have

$$= -\int_0^{\pi} d\theta\, \partial_\theta P_n(\cos\theta)\ \ (\sin\theta\partial_\theta u(r,\theta))$$

$$= -\int_0^{\pi} d\theta\, [\, \sin\theta\, \partial_\theta P_n(\cos\theta)]\ \partial_\theta u(r,\theta)$$

Now here comes our **second parts**:

$$= +\int_0^{\pi} d\theta\, \partial_\theta[\, \sin\theta\, \partial_\theta P_n(\cos\theta)]\, u(r,\theta) - [\, \sin\theta\, \partial_\theta P_n(\cos\theta)\, u(r,\theta)]\,|^{\pi}_{-\pi}$$

and the parts contributions vanish as before, leaving us with

$$= +\int_0^{\pi} d\theta\, \partial_\theta\, [\, \sin\theta\, \partial_\theta P_n(\cos\theta)]\, u(r,\theta)$$

At this point I am sure we need **Legendre's equation**

$$(1-z^2)P_n''(z) - 2zP_n'(z) + n(n+1)P_n(z) = 0 \qquad z = \cos\theta \qquad dz = -\sin\theta d\theta$$

$$\partial_z = \partial/\partial z = \partial\theta/\partial z\ \partial/\partial\theta = (-\sin\theta)^{-1}\partial_\theta$$

$$P_n'(z) = \partial_z\, P_n(z) = (-\sin\theta)^{-1}\partial_\theta\, P_n(z)$$

$$P_n''(z) = \partial_z P_n'(z) = (-\sin\theta)^{-1}\partial_\theta [(-\sin\theta)^{-1}\partial_\theta P_n(z)]$$

So in terms of ugly θ, we have

$$\sin^2\theta \{ (\sin\theta)^{-1}\partial_\theta [(\sin\theta)^{-1}\partial_\theta P_n(z)] \} + 2\cos\theta \{ (\sin\theta)^{-1}\partial_\theta P_n(z)\} + n(n+1)P_n(z) = 0$$

$$\{ (\sin\theta) \partial_\theta [(\sin\theta)^{-1}\partial_\theta P_n(z)] \} + 2\cos\theta \{ (\sin\theta)^{-1}\partial_\theta P_n(z)\} + n(n+1)P_n(z) = 0$$

This does not seem to be helping me much! But persist a little longer. The above is T1+T2+T3=0. We have

$$T1 = (\sin\theta) \partial_\theta [(\sin\theta)^{-1}\partial_\theta P_n(z)]$$

$$= (\sin\theta) \{ (\sin\theta)^{-1}\partial_\theta^2 P_n(z) - (\sin\theta)^{-2}\cos\theta \partial_\theta P_n(z) \}$$

$$= \partial_\theta^2 P_n(z) - (\sin\theta)^{-1}\cos\theta \partial_\theta P_n(z)$$

Then we have

$$T1+T2 = \partial_\theta^2 P_n(z) + (\sin\theta)^{-1}\cos\theta \partial_\theta P_n(z)$$

Now consider

$$T4 \equiv \partial_\theta [ \sin\theta \partial_\theta P_n(\cos\theta)]$$

$$= \sin\theta \partial_\theta^2 P_n(z) + \cos\theta \partial_\theta P_n(z)$$

$$(\sin\theta)^{-1}T4 = \partial_\theta^2 P_n(z) + (\sin\theta)^{-1}\cos\theta \partial_\theta P_n(z)$$

Thus I have proven that

$$(\sin\theta)^{-1}T4 = T1+T2 = -T3 = - n(n+1)P_n(z)$$

which is to say

$$(\sin\theta)^{-1}\partial_\theta [ \sin\theta \partial_\theta P_n(\cos\theta)] = - n(n+1)P_n(z)$$

In retrospect, this just says $-LY_{n0} = -n(n+1)Y_{n0}$ as I review below.

Now **recap** what happened earlier. We showed that, after two parts integrations,

$$q_n(r) = + \int_0^\pi d\theta \, \partial_\theta [ \sin\theta \partial_\theta P_n(\cos\theta)] \, u(r,\theta)$$

And now we see that we must then have

$$q_n(r) = \int_0^{\pi} d\theta \sin\theta \{ (\sin\theta)^{-1}\partial_\theta [ \sin\theta \partial_\theta P_n(\cos\theta)]\} u(r,\theta)$$

$$= \int_0^{\pi} d\theta \sin\theta \{ - n(n+1)P_n(\cos\theta)\} u(r,\theta)$$

$$= - n(n+1) \int_0^{\pi} d\theta \sin\theta P_n(\cos\theta)\} u(r,\theta)$$

$$= - n(n+1) \{(n+1/2)^{-1}\} a_n(r)$$

Now our "middle term" above was

$$\text{middle term} = - (n+1/2)\, q_n(r) = +n(n+1)\, a_n(r)$$

and THAT certainly took a long time to obtain. So we add the three terms and we have for our transformed LHS of our original equation

$$- \partial_r(r^2\partial_r \{ a_n(r) \} + n(n+1)\, a_n(r) - \lambda r^2\, a_n(r)$$

The transformed RHS is this

$$(n+1/2) \int_{-1}^{1} dz\, P_n(z) \{ \delta(r-r_0)\delta(\theta)/[2\pi\sin\theta] \}$$

$$= (n+1/2) \int_0^{\pi} d\theta \sin\theta\, P_n(\cos\theta) \{ \delta(r-r_0)\delta(\theta)/[2\pi\sin\theta] \}$$

$$= (n+1/2) \int_0^{\pi} d\theta\, P_n(\cos\theta) \{ \delta(r-r_0)\delta(\theta)/[2\pi] \}$$

$$= (n+1/2)\, P_n(1) \{ \delta(r-r_0) /[2\pi] \}$$

$$= \{ (n+1/2)\, \delta(r-r_0) /[2\pi] \} = \delta(r-r_0)\, (n+1/2) /[2\pi] = \delta(r-r_0)\, (2n+1)/4\pi$$

and then here is our transformed equation

$$- \partial_r(r^2\partial_r \{ a_n(r) \} + n(n+1)\, a_n(r) - \lambda r^2 a_n(r) = \delta(r-r_0)\, (2n+1)/4\pi$$

$$- (r^2 a_n'(r))' + n(n+1)\, a_n(r) - \lambda r^2 a_n(r) = \delta(r-r_0)\, (2n+1)/4\pi$$

$$- (r^2 a_n'(r))' - [\lambda r^2 - n(n+1)]\, a_n(r) = \delta(r-r_0)\, (2n+1)/4\pi$$

**The Legendre Equation in θ.**

This really was the missing piece of information above. Nobody ever writes the Legendre equation in θ, it is always in z. For example Schaum p 146 which I quoted above,

$$(1-z^2)P_n''(z) - 2zP_n'(z) + n(n+1)P_n(z) = 0$$

Rather than convert this from z =cosθ to θ, here is a faster way to get the same answer**.**

From my phys/qm/angmom we know that

$$L^2 = -(1/S_\theta)\,\partial_\theta\,[\,S_\theta\,\partial_\theta\,] - (1/\sin\theta)^2\,\partial_\phi^2$$

and that

$$\nabla^2 = \nabla_r^2 - r^{-2}L^2$$

We know that

$$L^2\,Y_{\ell m}(\theta,\varphi) = \ell(\ell+1)\,Y_{\ell m}(\theta,\varphi)$$

and in particular we know that

$$L^2\,Y_{\ell 0}(\theta,\varphi) = \ell(\ell+1)\,Y_{\ell 0}(\theta,\varphi)$$

Regardless of normalization of the Y, this does tell us that

$$L^2\,P_\ell(\cos\theta) = \ell(\ell+1)\,P_\ell(\cos\theta)$$

and thus we know that

$$-(1/S_\theta)\,\partial_\theta\,([\,S_\theta\,\partial_\theta\,]\,P_\ell(\cos\theta)) = \ell(\ell+1)\,P_\ell(\cos\theta)$$

and so this is the Legendre equation in θ coordinates. Rewrite as

$$-(1/\sin\theta)\,\partial_\theta[\,\sin\theta\,\partial_\theta P_n(\cos\theta)] = n(n+1)\,P_n(\cos\theta)$$

or

$$(1/\sin\theta)\,\partial_\theta[\,\sin\theta\,\partial_\theta P_n(\cos\theta)] + n(n+1)\,P_n(\cos\theta) = 0$$

**Obtain the 1D Green's equation in r by a different method.**

Go back to the full PDE,

$$-\partial_r(r^2\partial_r u) - 1/\sin\theta * \partial_\theta(\sin\theta\partial_\theta u) - \lambda r^2 u = \delta(r-r_0)\delta(\theta)/[2\pi\sin\theta]$$

Insert the expansion for u

$$u(r,\theta) = \Sigma_{n=0}^{\infty} a_n(r)\, P_n(\cos\theta)$$

First term:

$$-\partial_r(r^2\partial_r u) = \Sigma_{n=0}^{\infty} P_n(\cos\theta)\ \{-\partial_r(r^2\partial_r\, a_n(r))\}$$

Third term:

$$-\lambda r^2 u = \Sigma_{n=0}^{\infty} P_n(\cos\theta)\ \{-\lambda r^2 a_n(r)\}$$

Second term:

$$-1/\sin\theta * \partial_\theta(\sin\theta\partial_\theta\{\ \Sigma_{n=0}^{\infty} a_n(r)\, P_n(\cos\theta)\})$$

$$= \Sigma_{n=0}^{\infty} a_n(r)\{-1/\sin\theta * \partial_\theta(\sin\theta\partial_\theta\{\ P_n(\cos\theta)\})\}$$

Apply our Legendre equation in θ as just derived above to get

$$= \Sigma_{n=0}^{\infty} a_n(r)\{\ n(n+1)\, P_n(\cos\theta)\}$$

$$= \Sigma_{n=0}^{\infty} P_n(\cos\theta)\, n(n+1)\, a_n(r)$$

Thus we can write our LHS in this way, including all three terms

$$LHS = \Sigma_{n=0}^{\infty} P_n(\cos\theta)\ \{\ -\partial_r(r^2\partial_r\, a_n(r)) + n(n+1)\, a_n(r) - \lambda r^2 a_n(r)\ \}$$

Now what about the RHS, which was this

$$RHS = \delta(r-r_0)\delta(\theta)/[2\pi\sin\theta]$$

From transforms.doc I quote

$$\Sigma_{n=|m|}^{\infty}(1/K_n^m)\, P_n^m(z')\, P_n^m(z) = \delta(z'-z) \qquad K_n^m = (n+1/2)^{-1}\, f(n,m) \qquad \text{// completeness}$$

or

$$\Sigma_{n=0}^{\infty}(1/K_n^0)\, P_n(z')\, P_n(z) = \delta(z'-z)$$

or

$$\Sigma_{n=0}^{\infty}(n+1/2)\, P_n(z')\, P_n(z) = \delta(z'-z)$$

Set z=1 and z'=cosθ to get

$$\Sigma_{n=0}^{\infty}(n+1/2)\, P_n(\cos\theta)\, P_n(1) = \delta(\cos\theta-1)$$

or

$$\Sigma_{n=0}^{\infty}(n+1/2)\, P_n(\cos\theta) \;=\; \delta(\theta)/\,|\sin\theta|$$

Since we are concerned only with $(0,\pi)$, we can remove the $\|$. Then we have shown that

$$\text{RHS} = \delta(r-r_0)\delta(\theta)/[2\pi\sin\theta] \;=\; \delta(r-r_0)(1/2\pi)\,\Sigma_{n=0}^{\infty}(n+1/2)\, P_n(\cos\theta)$$

$$= \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, \{\, \delta(r-r_0)(1/2\pi)\,(n+1/2)\}$$

$$= \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, \{\, \delta(r-r_0)(1/4\pi)\,(2n+1)\}$$

Now using the fact that the $P_n$ are a complete set, we set LHS = RHS to get

$$-\,\partial_r(r^2\partial_r\, a_n(r)) + n(n+1)\, a_n(r) - \lambda r^2 a_n(r) \;=\; \delta(r-r_0)\,(2n+1)/(4\pi)$$

$$-\,(r^2 a_n'(r))' - [\lambda r^2 - n(n+1)]\, a_n(r) \;=\; \delta(r-r_0)\,(2n+1)/4\pi$$

and this agrees with our result obtained by the previous method.

**Compute the 1D Green's function for the ODE in variable r.**

Here is one of our Bessel equation forms which we then process (peeking at Stak's solution)

| | | |
|---|---|---|
| $w(x) = J_\nu(\rho x)$ | solves | $-(xw')' - \rho^2 xw + \nu^2\, w/x = 0.$ |
| $w(x) = J_{n+1/2}(\rho x)$ | solves | $-(xw')' - \rho^2 xw + (n+1/2)^2\, w/x = 0.$ |
| $w(r) = J_{n+1/2}(\sqrt{\lambda}\, r)$ | solves | $-(rw')' - \lambda rw + (n+1/2)^2\, w/r = 0.$ |

*Our* equation however has this form

$$-\,(r^2u')' - [\lambda r^2 - n(n+1)]\, u \;=\; \text{RHS}$$

so I will conjecture this connection:

$$u = r^{-1/2}w$$

$$u' = r^{-1/2}w' - (1/2)r^{-3/2}\, w$$

$$r^2u' = r^{3/2}w' - (1/2)r^{1/2}\, w$$

$$(r^2u')' = r^{3/2}w'' + (3/2)r^{1/2}\, w' - (1/2)\, r^{1/2}\, w' - (1/2)^2 r^{-1/2}w$$

$$= r^{3/2}w'' + r^{1/2}\, w' \; - (1/4)r^{-1/2}w$$

But we know that

$$(rw')' = rw'' + w'$$

so we have shown that

$$(r^2u')' = r^{1/2}(rw'' + w') - (1/4)r^{-1/2}w$$

or

$$(r^2u')' = r^{1/2}(rw')' - (1/4)r^{-1/2}w$$

Therefore, our equation in u becomes

$$-(r^2u')' - [\lambda r^2 - n(n+1)]\, u = RHS$$

$$-r^{1/2}(rw')' + (1/4)r^{-1/2}w - [\lambda r^2 - n(n+1)]\, r^{-1/2}w = RHS \qquad \text{ok}$$

$$-(rw')' + (1/4)r^{-1}w - [\lambda r^2 - n(n+1)]r^{-1}w = r^{-1/2}\, RHS$$

$$-(rw')' + (1/4)r^{-1}w - [\lambda r - n(n+1)r^{-1}]\, w = r^{-1/2}\, RHS$$

$$-(rw')' - [\lambda r - \{n(n+1)-1/4\}r^{-1}]w = r^{-1/2}\, RHS$$

$$-(rw')' - [\lambda r - (n+1/2)^2\, r^{-1}]w = r^{-1/2}\, RHS$$

$$-(rw')' - \lambda rw + (n+1/2)^2\, w/r = r^{-1/2}\, RHS$$

and, when RHS = 0, this matches our Bessel w equation above.

Conclusion:

We have this 1D Green's Function equation:

$$-(r^2 a_n'(r))' - [\lambda r^2 - n(n+1)]\, a_n(r) = \delta(r-r_0)\,(2n+1)/4\pi$$

where we think of $a_n(r)$ as our u(r) above. We then define

$$u = r^{-1/2}w$$

$$a_n(r) = r^{-1/2}\, A_n(r)$$

and our equation for $A_n(r)$ is this

$$-(r\, A_n')' + (1/4)\, A_n - [\lambda r^2 - (n+1/2)^2]\, A_n = r^{1/2}\, \delta(r-r_0)\,(2n+1)/4\pi$$

and our homo solutions are then of this form

$$A_n(r) = J_{n+1/2}(\sqrt{\lambda}\, r)$$

$$a_n(r) = r^{-1/2}\, J_{n+1/2}(\sqrt{\lambda}\, r)$$

We know from the asymptotic behaviors that

$$a_n(r) = C\,[r_<^{-1/2}\, J_{n+1/2}(\sqrt{\lambda}\, r_<)][r_>^{-1/2} H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_>)]$$

and our only task is to find C.

Question: Why am I putting $H^{(1)}$ on the RHS here? What asymptotic behavior am I thinking of?
Answer: Although I write $\sqrt{\lambda}$ all the time, I really have in mind that $\lambda = -k^2$ where $\lambda$ is "clean", and the we have in mind $\sqrt{\lambda} = +ik$ with $k>0$, so it is then $H^{(1)}_\nu(ikr)$ that is of interest, and this goes as $e^{i(ikr)} = e^{-kr}$ and that is the asymptotic behavior I had in mind, so this is really the $\sim K_\nu(kr)$ decaying.

**Compute the jump constant C**

Let $a_n(r) = u$ again and we have

$$-(r^2u')' - [\lambda r^2 - n(n+1)]\, u = \delta(r-r_0)\,(2n+1)/4\pi$$

$$-r^2u'' - 2ru' - [\lambda r^2 - n(n+1)]\, u(r) = \delta(r-r_0)\,(2n+1)/4\pi$$

We integrate this as usual from $r_0-\varepsilon$ to $r_0+\varepsilon_0$.

$$-r^2u'' - 2ru' - [\lambda r^2 - n(n+1)]\, u = \delta(r-r_0)\,(2n+1)/4\pi$$

$$\int_{r_0-\varepsilon}^{r_0+\varepsilon} dr\, \{ -r^2 du'/dr - 2ru' - [\lambda r^2 - n(n+1)]\, u \} = (2n+1)/4\pi$$

$$-r_0^2 \int_{r_0-\varepsilon}^{r_0+\varepsilon} dr\, du'/dr - 2r_0 \int_{r_0-\varepsilon}^{r_0+\varepsilon} dr\, du/dr - [\lambda r_0^2 - n(n+1)] \int_{r_0-\varepsilon}^{r_0+\varepsilon} dr\, u = (2n+1)/4\pi$$

$$-r_0^2 \int_{r_0-\varepsilon}^{r_0+\varepsilon} du' - 2r_0 \int_{r_0-\varepsilon}^{r_0+\varepsilon} du - [\lambda r_0^2 - n(n+1)] \int_{r_0-\varepsilon}^{r_0+\varepsilon} dr\, u = (2n+1)/4\pi$$

$$-r_0^2 \int_{r_0-\varepsilon}^{r_0+\varepsilon} du' - 2r_0 \int_{r_0-\varepsilon}^{r_0+\varepsilon} du - [\lambda r_0^2 - n(n+1)]\, u(r_0) \int_{r_0-\varepsilon}^{r_0+\varepsilon} dr = (2n+1)/4\pi$$

Since u is continuous, the third integral vanishes as shown, the second integral trivially vanishes, so

$$-r_0^2 \int_{r_0-\varepsilon}^{r_0+\varepsilon} du' = (2n+1)/4\pi$$

$$-r_0^2[\, u'(r)]|_-^+ = (2n+1)/4\pi$$

$$\text{jump in } u' = [\,(2n+1)/4\pi\,]/(-r_0^2)$$

where we recognize our familiar result that jump = $1/a_0$ but I always do it longhand every time. So

$$u(r) = C\,[r_<^{-1/2} J_{n+1/2}(\sqrt{\lambda}\, r_<)][r_>^{-1/2} H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_>)]$$

$$u(r_0+\varepsilon) = C\,[r_0^{-1/2} J_{n+1/2}(\sqrt{\lambda}\, r_0)][r^{-1/2} H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r)]$$
$$u(r_0-\varepsilon) = C\,[r^{-1/2} J_{n+1/2}(\sqrt{\lambda}\, r)][r_0^{-1/2} H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_0)]$$

$$u'(r_0+\varepsilon) = C\,[r_0^{-1/2} J_{n+1/2}(\sqrt{\lambda}\, r_0)]\,[r^{-1/2} H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r)]'$$
$$u'(r_0-\varepsilon) = C\,[r^{-1/2} J_{n+1/2}(\sqrt{\lambda}\, r)]'\,[r_0^{-1/2} H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_0)]$$

$$u'(r_0+\varepsilon) = C\,[r_0^{-1/2} J_{n+1/2}(\sqrt{\lambda}\, r_0)]\,[r^{-1/2}\partial_r H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r) - (1/2) r^{-3/2} H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r)\,]$$
$$u'(r_0-\varepsilon) = C\,[r^{-1/2}\partial_r J_{n+1/2}(\sqrt{\lambda}\, r) - (1/2) r^{-3/2} J(\sqrt{\lambda}\, r)\,]\,[r_0^{-1/2} H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_0)]$$

Now use

$$\partial_r J_{n+1/2}(\sqrt{\lambda}\, r) = J'_{n+1/2}(\sqrt{\lambda}\, r)\ \sqrt{\lambda}$$

and then we have

$$u'(r_0+\varepsilon) = C\,[r_0^{-1/2} J_{n+1/2}(\sqrt{\lambda}\, r_0)]\,[r^{-1/2}\sqrt{\lambda}\, H^{(1)\prime}_{n+1/2}(\sqrt{\lambda}\, r) - (1/2) r^{-3/2} H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r)\,]$$
$$u'(r_0-\varepsilon) = C\,[r^{-1/2}\sqrt{\lambda}\, J'_{n+1/2}(\sqrt{\lambda}\, r) - (1/2) r^{-3/2} J(\sqrt{\lambda}\, r)\,]\,[r_0^{-1/2} H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_0)]$$

$$u'(r_0+\varepsilon) = C/\sqrt{r r_0}\ [\, J_{n+1/2}(\sqrt{\lambda}\, r_0)]\,[\sqrt{\lambda}\, H^{(1)\prime}_{n+1/2}(\sqrt{\lambda}\, r) - (1/2) r^{-1} H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r)\,]$$
$$u'(r_0-\varepsilon) = C/\sqrt{r r_0}\ [\sqrt{\lambda}\, J'_{n+1/2}(\sqrt{\lambda}\, r) - (1/2) r^{-1} J(\sqrt{\lambda}\, r)\,]\,[H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_0)]$$

Now we can take the difference

$$u'(r_0+\varepsilon) - u'(r_0-\varepsilon) = C/\sqrt{r r_0}\ *$$

$$\{\ \ J_{n+1/2}(\sqrt{\lambda}\, r_0)\,[\sqrt{\lambda}\, H^{(1)\prime}_{n+1/2}(\sqrt{\lambda}\, r) - (1/2) r^{-1} H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r)\,]$$
$$H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_0)\,[\sqrt{\lambda}\, J'_{n+1/2}(\sqrt{\lambda}\, r) - (1/2) r^{-1} J_{n+1/2}(\sqrt{\lambda}\, r)\,]\,\}$$

But we want this difference evaluated at $r = r_0$ so get

$$[u'(r_0+\varepsilon)- u'(r_0- \varepsilon)]|^{r=r0} = (C/r_0) *$$

$$\{ \quad J_{n+1/2}(\sqrt{\lambda}\, r_0)\,[\sqrt{\lambda}\, H^{(1)}{}'_{n+1/2}(\sqrt{\lambda}\, r_0) - (1/2) r_0^{-1}\, H^{(1)}{}_{n+1/2}(\sqrt{\lambda}\, r_0)\,]$$
$$H^{(1)}{}_{n+1/2}(\sqrt{\lambda}\, r_0)\,[\sqrt{\lambda}\, J_{n+1/2}'(\sqrt{\lambda}\, r_0) \quad - \quad (1/2) r_0^{-1}\, J_{n+1/2}(\sqrt{\lambda}\, r_0) \quad ]\,\}$$

Now the two -(1/2) type terms cancel and we get

$$\text{jump in } u' = (C/r_0)\sqrt{\lambda}\ \{\, J_{n+1/2}(\sqrt{\lambda}\, r_0)\, H^{(1)}{}'_{n+1/2}(\sqrt{\lambda}\, r_0) - \sqrt{\lambda}\, J_{n+1/2}'(\sqrt{\lambda}\, r_0)\, H^{(1)}{}_{n+1/2}(\sqrt{\lambda}\, r_0)\,\}$$

$$(C/r_0)\sqrt{\lambda}\ W[J_{n+1/2}(\sqrt{\lambda}\, r_0),\, H^{(1)}{}_{n+1/2}(\sqrt{\lambda}\, r_0)]$$

and we go look up in AS2010

**10.5.3**

$$\mathscr{W}\{J_\nu(z), H_\nu^{(1)}(z)\} = J_{\nu+1}(z)\, H_\nu^{(1)}(z) - J_\nu(z)\, H_{\nu+1}^{(1)}(z)$$
$$= 2i/(\pi z),$$

and conclude then that

$$\text{jump in } u' = (C/r_0)\sqrt{\lambda}\ \ 2i/(\pi\sqrt{\lambda}\, r_0) = (C)\ 2i/(\pi r_0^2) = C\ (2i/\pi r_0^2)$$

But from above we had

$$\text{jump in } u' = [\ (2n+1)/4\pi\ ]\ /(-r_0^2)$$

so comparison says

$$C\ (2i/\pi r_0^2) = [\ (2n+1)/4\pi\ ]\ /(-r_0^2)$$

$$C\ (2i/\pi) = [\ (2n+1)/4\pi\ ]\ /(-1)$$

$$C\ (-2i) = [\ (2n+1)/4\ ]$$

$$C = i\ [\ (2n+1)/8\ ]$$

Then **our 1D Green's function is this:**

$$a_n(r) = C\ [r_<^{-1/2}\, J_{n+1/2}(\sqrt{\lambda}\, r_<)][r_>^{-1/2} H^{(1)}{}_{n+1/2}(\sqrt{\lambda}\, r_>)]$$

$$= i\ [\ (2n+1)/[8\sqrt{r r_0}\ ]\ \ J_{n+1/2}(\sqrt{\lambda}\, r_<)\, H^{(1)}{}_{n+1/2}(\sqrt{\lambda}\, r_>)$$

Now we install this into our initial expansion to get

$$u(r,z) = \Sigma_{n=0}^{\infty} a_n(r)\, P_n(z)$$

$$= (i/[8\sqrt{rr_0}\,]) \Sigma_{n=0}^{\infty} P_n(z)\,(2n+1)\, J_{n+1/2}(\sqrt{\lambda}\, r_<)\, H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_>)$$

and we have therefore shown that

$$\exp(i\sqrt{\lambda}\, R)/4\pi R \;=\; (i/[8\sqrt{rr_0}\,]) \Sigma_{n=0}^{\infty} P_n(\cos\theta)\,(2n+1)\, J_{n+1/2}(\sqrt{\lambda}\, r_<)\, H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_>)$$

and this agrees with Stak's result 7.244 on page 290 ! Remember that this is the special case that $\mathbf{r}_0$ lies on the + z axis and **r** can be anywhere.

**Aside on Green's Functions.** We just showed above that for

$$-(r^2 a_n'(r))' - [\lambda r^2 - n(n+1)]\, a_n(r) \;=\; \delta(r-r_0)\,(2n+1)/4\pi$$

The Green's function on (a,∞) was this

$$a_n(r) = i\,[\,(2n+1)/[8\sqrt{rr_0}\,]\;\; J_{n+1/2}(\sqrt{\lambda}\, r_<)\, H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_>)$$

Therefore, if we were given this ODE

$$-(r^2 g'(r))' - [\lambda r^2 - n(n+1)]\, g(r) \;=\; \delta(r-r_0)$$

we could conclude that

$$g = 4\pi/(2n+1) * a_n(r)$$

$$= 4\pi/(2n+1) * i\,[\,(2n+1)/[8\sqrt{rr_0}\,]\;\; J_{n+1/2}(\sqrt{\lambda}\, r_<)\, H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_>)$$

$$= (i\pi/2\sqrt{rr_0}\,)\;\; J_{n+1/2}(\sqrt{\lambda}\, r_<)\, H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_>)$$

`

**Take the λ→0 limit:**

Get some z→0 limits from AS2010

**10.7.3** $$J_\nu(z) \sim (\tfrac{1}{2}z)^\nu / \Gamma(\nu+1), \quad \nu \neq -1, -2, -3, \ldots,$$

**10.7.7**
$$H_\nu^{(1)}(z) \sim -H_\nu^{(2)}(z) \sim -(i/\pi)\,\Gamma(\nu)(\tfrac{1}{2}z)^{-\nu}, \quad \Re\nu > 0.$$

Then

$$J_{n+1/2}(\sqrt{\lambda}\, r_<) \rightarrow (\sqrt{\lambda}\, r_</2)^{n+1/2} / \Gamma(n+3/2)$$

$$H^{(1)}{}_{n+1/2}(\sqrt{\lambda}\, r_>) \rightarrow (-i/\pi)\Gamma(n+1/2)\, (\sqrt{\lambda}\, r_>/2)^{-(n+1/2)}$$

$$J_{n+1/2}(\sqrt{\lambda}\, r_<)\, H^{(1)}{}_{n+1/2}(\sqrt{\lambda}\, r_>) \rightarrow (-i/\pi)\, [\Gamma(n+1/2)/\, \Gamma(n+3/2)]\, (\sqrt{\lambda}\, r_</2)^{n+1/2}(\sqrt{\lambda}\, r_>/2)^{-(n+1/2)}$$

$$= (-i/\pi)\, [\Gamma(n+1/2)/\, \{(n+1/2)\Gamma(n+1/2)]\, (r_</r_>)^{n+1/2}$$

$$= (-i/[\pi(n+1/2)])\, (r_</r_>)^{n+1/2}$$

Then we get

$$1/4\pi R = (i/[8\sqrt{rr_0}\,])\, \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, (2n+1)\, \{(-i/[\pi(n+1/2)])\, (r_</r_>)^{n+1/2}\}$$

$$= r_<^{-1/2}\, r_>^{-1/2} (i/8)\, \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, (2n+1)\, \{(-i/[\pi(n+1/2)])\, (r_</r_>)^{n+1/2}\}$$

$$= r_<^{-1/2}\, r_>^{-1/2} (i/8)\, \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, (2n+1)\, \{(-2i/[\pi(2n+1)])\, (r_<)^{n+1/2}\, (r_>)^{-n-1/2}\}$$

$$= (i/8)\, \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, \{(-2i/[\pi])\, (r_<)^n\, (r_>)^{-n-1}\}$$

$$= (i/8)\, (-2i/[\pi])\, \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, \{(r_<)^n\, (r_>)^{-n-1}\}$$

$$= (1/4\pi)\, \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, \{(r_<)^n\, (r_>)^{-n-1}\}$$

$$1/R = \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, \{(r_<)^n\, (r_>)^{-n-1}\}$$

Now from my curvilinear / spherical 1/R doc, I know that

$$1/R = \Sigma_{n=0}^{\infty}\, \Sigma_{m=0}^{\infty}\, \varepsilon_m\, f(n,-m)\, (r_<)^n (r_>)^{-n-1}\, P_n^m(z)\, P_n^m(z_0)\, \cos[m(\varphi-\varphi_0)] \qquad (**)$$

when $\mathbf{r}_0$ is at $(r_0, \theta_0, \varphi_0)$. If $\theta_0 = 0$ so that $z_0 = 1$, we know $P_n^m(1) = \delta^m{}_0$ and $\varepsilon_0 = 1$ so

$$1/R = \Sigma_{n=0}^{\infty}\, (r_<)^n (r_>)^{-n-1}\, P_n(z)$$

and this agrees with our limit above.

**Write in the other form valid when $\lambda = -k^2$.**

We are concerned about $J_{n+1/2}(\sqrt{\lambda}\, r_<)$ when $\lambda = -k^2$ and $\sqrt{\lambda} = +ik$ in which case we have $J_{n+1/2}(ikr_<)$. We call upon AS2010

**10.27.6**

$$I_\nu(z) = e^{\mp\nu\pi i/2}\, J_\nu\left(ze^{\pm\pi i/2}\right), \quad -\pi \le \pm \operatorname{ph} z \le \tfrac{1}{2}\pi,$$

**10.27.8**

$$K_\nu(z) = \begin{cases} \frac{1}{2}\pi i e^{\nu\pi i/2}\, H_\nu^{(1)}\left(ze^{\pi i/2}\right), & -\pi \le \operatorname{ph} z \le \frac{1}{2}\pi, \\ -\frac{1}{2}\pi i e^{-\nu\pi i/2}\, H_\nu^{(2)}\left(ze^{-\pi i/2}\right), & -\frac{1}{2}\pi \le \operatorname{ph} z \le \pi. \end{cases}$$

This tells us that (assume z has positive imaginary part)

$$I_\nu(z)K_\nu(z) = e^{-i\pi\nu/2}\, J_\nu(iz)\ \ (i\pi/2)e^{i\pi\nu/2}\, H^{(1)}(iz) \ = \ (i\pi/2)\, J_\nu(iz)\, H^{(1)}(iz)$$

So we know that

$$J_\nu(iz)\, H^{(1)}(iz) \ = (2/i\pi)\, I_\nu(z)K_\nu(z)$$

and we let z = kr and we get

$$J_\nu(ikr_<)\, H^{(1)}(ikr_>) \ = (2/i\pi)\, I_\nu(kr_<)K_\nu(kr_>)$$

Then our expansion is

$$\exp(i\sqrt{\lambda}\, R)/4\pi R \ = \ (i/[8\sqrt{rr_0}\,])\, \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, (2n+1)\, J_{n+1/2}(\sqrt{\lambda}\, r_<)\, H^{(1)}{}_{n+1/2}(\sqrt{\lambda}\, r_>)$$

$$\exp(-kR)/4\pi R \ = \ (i/[8\sqrt{rr_0}\,])\, \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, (2n+1)\, (2/i\pi) I_{n+1/2}(kr_<)\, K_{n+1/2}(kr_>)$$

$$= (i/[8\sqrt{rr_0}\,])\, (2/i\pi)\, \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, (2n+1)\, (2/i\pi) I_{n+1/2}(kr_<)\, K_{n+1/2}(kr_>)$$

$$= (1/[4\pi\sqrt{rr_0}\,])\, \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, (2n+1)\, (2/i\pi) I_{n+1/2}(kr_<)\, K_{n+1/2}(kr_>)$$

and this agrees with (7.225).

**Take the $r_0 \to 0$ limit of the first result**

We move our point down the z axis to the origin. Let's use the first form

$$\exp(i\sqrt{\lambda}\, R)/4\pi R \ = \ (i/[8\sqrt{rr_0}\,])\, \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, (2n+1)\, J_{n+1/2}(\sqrt{\lambda}\, r_<)\, H^{(1)}{}_{n+1/2}(\sqrt{\lambda}\, r_>)$$

In this limit, $r_<$ is $r_0$ so

$$\text{RHS} = \ (i/[8\sqrt{rr_0}\,])\, \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, (2n+1)\, J_{n+1/2}(\sqrt{\lambda}\, r_0)\, H^{(1)}{}_{n+1/2}(\sqrt{\lambda}\, r)$$

I quoted the $z \to 0$ limit above

**10.7.3** $J_\nu(z) \sim (\frac{1}{2}z)^\nu / \Gamma(\nu+1), \quad \nu \neq -1, -2, -3, \ldots,$

so

$J_{n+1/2}(\sqrt{\lambda}\, r_0) \rightarrow (\sqrt{\lambda}\, r_0/2)^{n+1/2}/\Gamma(n+3/2)$

so the claim then is that

$\exp(i\sqrt{\lambda}\, r)/4\pi r = (i/[8\sqrt{r r_0}\,]) \Sigma_{n=0}^{\infty} P_n(\cos\theta)\,(2n+1)\,(\sqrt{\lambda}\, r_0/2)^{n+1/2}/\Gamma(n+3/2)\, H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r)$

or

$\exp(i\sqrt{\lambda}\, r)/4\pi r = (i/[8\sqrt{r r_0}\,]) \Sigma_{n=0}^{\infty} P_n(\cos\theta)\,(2n+1)\,(\sqrt{\lambda}\, r_0/2)^{n+1/2}/\Gamma(n+3/2)\, H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r)$

in the limit. The RHS has factor $(r_0)^n$ so only the n=0 term survives and we claim then

$\exp(i\sqrt{\lambda}\, r)/4\pi r = (i/[8\sqrt{r r_0}\,])\; 1\; (1)\,(\sqrt{\lambda}\, r_0/2)^{1/2}/\Gamma(3/2)\, H^{(1)}_{1/2}(\sqrt{\lambda}\, r)$

$= (i/[8\sqrt{r}\,])\,(\sqrt{\lambda}/2)^{1/2}/\Gamma(3/2)\, H^{(1)}_{1/2}(\sqrt{\lambda}\, r)$

$= (i/[8\sqrt{r}\,])\,(\sqrt{\lambda}/2)^{1/2}\,(2/\sqrt{\pi}\,)\, H^{(1)}_{1/2}(\sqrt{\lambda}\, r)$

$= i\, 2^{-3}\, r^{-1/2}\, \lambda^{1/4}\, 2^{-1/2}\, 2\;\; \pi^{-1/2}\;\; H^{(1)}_{1/2}(\sqrt{\lambda}\, r)$

$= i\, 2^{-2}\, r^{-1/2}\, \lambda^{1/4}\, 2^{-1/2}\;\; \pi^{-1/2}\;\; H^{(1)}_{1/2}(\sqrt{\lambda}\, r)$

$= (i/4)\,(\sqrt{\lambda}/2\pi r)^{1/2}\;\; H^{(1)}_{1/2}(\sqrt{\lambda}\, r)$

But according to p 54 5.118 for n=3 , this is exactly right for $E_3$ with source at origin.

**Exercise 7.32 Ring source expansion (290) 8.2.11**

How do we describe this little ring of charge at $r = r_0$ and $\theta = \theta_0$? I know how to do a point charge

point at $(r_0, \theta_0, \varphi_0)$ $= \delta(\mathbf{r}-\mathbf{r}_0) = \delta(r-r_0)/r^2\; \delta(\theta-\theta_0)/\sin\theta\; \delta(\varphi-\varphi_0)$

Integrate this around the ring and get

ring at $(r_0, \theta_0)$ $= K\, \delta(r-r_0)/r^2\; \delta(\theta-\theta_0)/\sin\theta$

To get the constant, we want the integral of the above to be 1. So integrate over an upper cone of larger angle than $\theta_0$ and set to 1:

$$\int_0^{r_0+\varepsilon} r^2 dr \int_0^{\theta_0+\varepsilon} d\theta \sin\theta \int_0^{2\pi} d\varphi \{ K \delta(r-r_0)/r^2 \delta(\theta-\theta_0)/\sin\theta \} = 1$$

$$\int_0^{r_0+\varepsilon} dr \int_0^{\theta_0+\varepsilon} d\theta \int_0^{2\pi} d\varphi \{ K \delta(r-r_0) \delta(\theta-\theta_0) \} = 1$$

$K 2\pi = 1$ $K = 1/2\pi$

Just practicing this stuff. So our desired source is

source = $(1/2\pi) \delta(r-r_0)/r^2 \delta(\theta-\theta_0)/\sin\theta$

But this is exactly the same source as in problem 7.31 but $\theta_0$ was 0 there. So we should be able to crib large parts of the previous exercise.

In the first section, only the RHS is different under transformation. We get

The transformed RHS is this

$$(n+1/2) \int_{-1}^{1} dz P_n(z) \{ \delta(r-r_0) \delta(\theta-\theta_0)/[2\pi\sin\theta] \}$$

$$= (n+1/2) \int_0^{\pi} d\theta \sin\theta P_n(\cos\theta) \{ \delta(r-r_0) \delta(\theta-\theta_0)/[2\pi\sin\theta] \}$$

$$= (n+1/2) \int_0^{\pi} d\theta P_n(\cos\theta) \{ \delta(r-r_0) \delta(\theta-\theta_0)/[2\pi] \}$$

$$= (n+1/2) P_n(\cos\theta_0) \{ \delta(r-r_0) /[2\pi] \}$$

$$= \{ (n+1/2) P_n(\cos\theta_0) \delta(r-r_0) /[2\pi] \} = \delta(r-r_0) P_n(\cos\theta_0) (n+1/2) /[2\pi] = \delta(r-r_0) P_n(\cos\theta_0) (2n+1)/4\pi$$

so we just gain a new factor of $P_n(\cos\theta_0)$ on our RHS. Our transformed equation is now

$$- (r^2 a_n'(r))' - [\lambda r^2 - n(n+1)] a_n(r) = \delta(r-r_0) (2n+1) P_n(\cos\theta_0)/4\pi$$

In our "second method" this same factor appears when we write

$$\Sigma_{n=0}^{\infty} (n+1/2) P_n(\cos\theta) P_n(\cos\theta_0) = \delta(\cos\theta-\cos\theta_0) = \delta(\theta-\theta_0)/|\sin\theta|$$

Our 1D Green's function form is the same, C will be different. From our ODE we will get

jump in $u' = [ (2n+1) P_n(\cos\theta_0)/4\pi ] /(-r_0^2)$

and this will lead to

$$C = i\,[\,(2n+1)\,P_n(\cos\theta_0)/8\,]$$

and

$$a_n(r) = i\,[\,(2n+1)\,P_n(\cos\theta_0)/[8\sqrt{rr_0}\,]\;J_{n+1/2}(\sqrt{\lambda}\,r_<)\,H^{(1)}{}_{n+1/2}(\sqrt{\lambda}\,r_>)$$

So our expansion will then be

$$u(r,\theta;\lambda) = (i/[8\sqrt{rr_0}\,])\,\Sigma_{n=0}^{\infty}\,P_n(\cos\theta)\,P_n(\cos\theta_0)\,(2n+1)\,J_{n+1/2}(\sqrt{\lambda}\,r_<)\,H^{(1)}{}_{n+1/2}(\sqrt{\lambda}\,r_>)$$

and this shows the explicit symmetry were we to do $\mathbf{r} \leftrightarrow \mathbf{r}_0$. And of course we will also get

$$u(r,\theta;\lambda=-k^2) == (1/[4\pi\sqrt{rr_0}\,])\,\Sigma_{n=0}^{\infty}\,P_n(\cos\theta)\,P_n(\cos\theta_0)\,(2n+1)\,(2/i\pi)I_{n+1/2}(kr_<)\,K_{n+1/2}(kr_>\,)$$

Of course since we have a ring of source, the LHS is no longer the $E_3$ fundy $\exp(i\sqrt{\lambda}\,R)/4\pi R$ . It is just the solution to this PDE in 3D:

$$(-\nabla^2-\lambda)u(r,\theta) = \delta(r-r_0)\delta(\theta-\theta_0)/[2\pi r^2\sin\theta]$$

with BCs of being finite at r=0 and bounded at r=∞. It is a "fundy like" solution.