Exercise 7.33: Heat flow with spherical delta shell initial condition. (290)

Exercise 7.33: Heat flow with spherical delta shell initial condition. (290) **1**
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Overview from Meta Notes

Exercise 7.33 (p 290) Problem Statement: Consider an infinite 3D uniform heat conducting medium with a spherical hole of radius r = a at the origin. The hole contains vacuum which we assume does not conduct heat. The "boundary" here then is the sphere at r = a and the great sphere at r = ∞ and we assume that u=0 on the boundary. We take the entire medium to be at u = 0 for time t< 0. At time t=0 we turn on a spherical shell initial condition at radius r_0 > a lying inside the medium. This is not a continuous heat source, it is just an initial condition, like saying the left inch of a frozen rod is at u=1 at t = 0+ε. The heat energy in this initial condition region is going to spread out (with inwards and outwards), and that is our problem. This problem has both θ and φ symmetry, so is a 1D problem in variable r. Recall how a heat source initial condition turns into a Helmholtz equation driving source, and that happens here.

To solve this problem, we first do a time Laplace to replace t by s, and we then get a 1D Helmholtz Green's problem in r for U(r,s): $-(r^2U')' - \lambda r^2 U = \delta(r-r_0)/4\pi$ where λ = -s. We realize that this is just the special case n=0 of a Green's equation we encountered in Ex 7.31 (whose function was called $a_n(r)$) so at this point we know the general atomic form of the solution must be $Z_{1/2}(\sqrt{\lambda}\, r)/\sqrt{r}$. Also, we have certain BC's on U(r,s) at r=a and r=∞ (0 in both places). We then solve the Green's problem in r to get

$$U(r,s) = (i/8\sqrt{r r_0}\,)\,[\,H_{1/2}^{(1)}(\sqrt{\lambda}\,a)J_{1/2}(\sqrt{\lambda}\,r_<) - J_{1/2}(\sqrt{\lambda}\,a)\,H_{1/2}^{(1)}(\sqrt{\lambda}\,r_<)]\,[H_{1/2}^{(1)}(\sqrt{\lambda}\,r_>)/\,H_{1/2}^{(1)}(\sqrt{\lambda}\,a)]$$

But then since λ = -s this gets rewritten tat

$$U(r,s) = (1/4\pi\sqrt{r r_0}\,)[\;K_{1/2}^{(1)}(\sqrt{s}\,a)I_{1/2}(\sqrt{s}\,r_<) - I_{1/2}(\sqrt{s}\,a)\,K_{1/2}^{(1)}(\sqrt{s}\,r_<)][K_{1/2}^{(1)}(\sqrt{s}\,r_>)/\,K_{1/2}^{(1)}(\sqrt{s}\,a)]$$

But these simple K functions are in fact just elementary functions, allowing the above to be written as

$$U(r,s) = -(1/[8\pi r\, r_0])\{\quad \exp[\sqrt{s}\,(2a-r-r_0)\,] - \exp[-\sqrt{s}\,|r-r_0|]\quad\}\;/\sqrt{s}$$

I verify at this point that this meets both BC's. Each term above has a trivial Schaum inverse Laplace, and we then get this very simple result

$$u(r,t) = (1/8\pi r r_0)\{\exp(-|r-r_0|^2/4t)/\sqrt{\pi t} \;-\exp(-(2a-r-r_0)^2/4t)/\sqrt{\pi t}\;\}$$

I then take limit t→0 to verify that we get the initial condition shell δ. I plot the solution in Maple.

Consider an infinite uniform 3D heat conducting medium with a spherical hole of radius r = a at the origin. The hole contains vacuum which we assume does not conduct heat. The "boundary" here is the sphere at r = a and the great sphere at r = ∞ and we assume that u=0 on the boundary. We take the entire medium to be at u = 0 for time t< 0. At time t=0 we turn on a spherical shell initial condition at radius r_0 > a lying inside the medium. This is not a continuous heat source, it is just an initial condition, like saying the left inch of a frozen rod is at u=1 at t = 0+ε. The heat energy in this initial condition region is going to spread out (with inwards and outwards), and that is our problem.

I suspect this will convert into a Helmholtz problem with a shell source on the RHS, but let's give a shot.

The heat equation system is this

$(\partial_t - \nabla^2)\, u = 0$ t>0

boundary condition: $u(a,t) = 0$ $u(\infty,t) = 0$

initial condition: $u(r,0) = f(r) = \delta(r-r_0)/4\pi r^2$

We know from symmetry that $u(\mathbf{r},t) = u(r,t)$, so only one spatial coordinate to worry about.

Let's check on the form of this source. If we integrate it over a thick shell containing the thin shell, we will get 4π from the dΩ and an r^2 from the radial integral, and we will get 1 as the integral, so this is a "unit" initial condition in some sense. Notice that $\nabla^2 = \nabla^2_r$ only.

The first step is to do our Laplace time transform

$$U(r,s) = \int_0^\infty dt\, e^{-st} u(r,t)$$

We just apply $\int_0^\infty dt\, e^{-st}$ to both terms of our homo PDE and get

$$\int_0^\infty dt\, e^{-st}\, \partial_t u(r,t) - \int_0^\infty dt\, e^{-st} \nabla^2 u(r,t) = 0$$

$$\int_0^\infty dt\, e^{-st}\, \partial_t u(r,t) - \nabla^2 \int_0^\infty dt\, e^{-st} u(r,t) = 0$$

$$\int_0^\infty dt\, e^{-st}\, \partial_t u(r,t) - \nabla^2\, U(r,s) = 0$$

Now we do parts integration on the first term,

$$\int_0^\infty dt\, e^{-st}\, \partial_t u(r,t) = - \int_0^\infty dt\, \partial_t e^{-st}\, u(r,t) + [e^{-st}\, u(r,t)]^\infty_0$$

$$= + s \int_0^\infty dt\, e^{-st}\, u(r,t) + [\, 0 - u(r,0)]$$

$$= s\, U(r,s) - u(r,0)$$

So our transformed equation is now

$$s\, U(r,s) - u(r,0) - \nabla^2 U(r,s) = 0$$

or

$$s\, U(r,s) - \nabla^2 U(r,s) = u(r,0)$$

or

$$(-\nabla^2 - (-s)\,)\, U(r,s) = u(r,0) = \delta(r-r_0)/4\pi r^2$$

So this has the standard Helmholtz equation form with $\lambda = -s$ and we see our source as expected. So we end up with this 1D Green's Function problem:

$$-\nabla^2 f(r) = -\, r^{-2}\partial_r(r^2\partial_r f(r)) = -\, r^{-2}\,(r^2 f')'$$

We think of s as a fixed parameter and U=U(r) and we then have

$$-\, r^{-2}(r^2 U')' + sU = \delta(r-r_0)/4\pi r^2$$

$$-\,(r^2 U')' + sr^2 U = \delta(r-r_0)/4\pi \qquad \text{// this is our 1D Green's Equation}$$

Now let's use $\lambda = -s$ and stay in "standard form" so we write

$$-\,(r^2 U')' - \lambda r^2 U = \delta(r-r_0)/4\pi$$

It is certainly useful to compare this to the 1D Green's equation we got in Ex 7.31, which was

$$-\,(r^2 a_n'(r))' - [\lambda r^2 - n(n+1)]\, a_n(r) = \delta(r-r_0)\,(2n+1)/4\pi$$

Wrong Turn (ignore this section)

Therefore, this Green's problem is the n=0 case of the problem we already solved! So we can just read off the solution

$$W(r,\lambda) = i\,[\,(2n+1)/[8\sqrt{rr_0}\,]\; J_{n+1/2}(\sqrt{\lambda}\, r_<)\, H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_>) = U(r,s)$$

Now if we had $\lambda = -k^2$ we would write, again from Ex 7.31, that $\sqrt{\lambda} = +ik$ and

$$J_\nu(ikr_<)\, H^{(1)}(ikr_>) = (2/i\pi)\, I_\nu(kr_<) K_\nu(kr_>)$$

and in our present case we just set k = $\sqrt{s}$, so here is our solution

$U(r,s) = i\,[\,(2n+1)/[8\sqrt{rr_0}\,]\,(2/i\pi)\,I_{n+1/2}(kr_<)K_{n+1/2}(kr_>)$ // but with n=0 and k=$\sqrt{s}$

$=(\,i\,/[8\sqrt{rr_0}\,])(2/i\pi)\,I_{1/2}(\sqrt{s}\,r_<)K_{1/2}(\sqrt{s}\,r_>)$

$=(\,1\,/[4\pi\sqrt{rr_0}\,])\,I_{1/2}(\sqrt{s}\,r_<)K_{1/2}(\sqrt{s}\,r_>)$

So this is the s-space solution to our problem. How it happens that we know

Elementary Functions

10.39.1

$$I_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sinh z, \quad I_{-\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cosh z,$$

10.39.2

$$K_{\frac{1}{2}}(z) = K_{-\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}.$$

So we can write

$I_{1/2}(\sqrt{s}\,r_<)K_{1/2}(\sqrt{s}\,r_>) = (2/\pi)^{1/2}\ (\sqrt{s}\,r_<)^{-1/2}\,\mathrm{sh}(\sqrt{s}\,r_<)\ (2/\pi)^{-1/2}\,(\sqrt{s}\,r_>)^{-1/2}\exp(-\sqrt{s}\,r_>)$

$= (r_<)^{-1/2}\,\mathrm{sh}(\sqrt{s}\,r_<)\ (r_>)^{-1/2}\exp(-\sqrt{s}\,r_>)/\sqrt{s}$ // note that $(\sqrt{s}\,)^{-1/2}(\sqrt{s}\,)^{-1/2} = 1/\sqrt{s}$

$= (rr_0)^{-1/2}\,\mathrm{sh}(\sqrt{s}\,r_<)\exp(-\sqrt{s}\,r_>)/\sqrt{s}$

Therefore our s-space solution may be written as

$U(r,s) =(\,1\,/[4\pi\sqrt{rr_0}\,])(rr_0)^{-1/2}\,\mathrm{sh}(\sqrt{s}\,r_<)\exp(-\sqrt{s}\,r_>)/\sqrt{s}$

$= (1/4\pi rr_0)\,\mathrm{sh}(\sqrt{s}\,r_<)\exp(-\sqrt{s}\,r_>)/\sqrt{s}$

In hopes of doing a lookup, we would do well to expand

$\mathrm{sh}(\sqrt{s}\,r_<) = (1/2)\,[\exp(\sqrt{s}\,r_<) - \exp(-\sqrt{s}\,r_<)\,]$

and then we have

$U(r,s) =(1/8\pi rr_0)\{\exp(\sqrt{s}\,r_<)\exp(-\sqrt{s}\,r_>) - \exp(-\sqrt{s}\,r_<)\exp(-\sqrt{s}\,r_>)\}/\sqrt{s}$

$=(1/8\pi rr_0)\{\ \exp(-\sqrt{s}\,(r_> - r_<)) - \exp(-\sqrt{s}\,(r_> + r_<))\ \}/\sqrt{s}$

Now $(r_> - r_<)$ = a positive number = $|r-r_0|$ and $r_> + r_< = r+r_0$ so

$$U(r,s) = (1/8\pi r r_0) \{ \exp(-|r-r_0|\sqrt{s})/\sqrt{s} - \exp(-(r+r_0)\sqrt{s})/\sqrt{s} \}$$

Now from Schaum p 169 we have that

$$\exp(-b\sqrt{s})/\sqrt{s} \leftrightarrow \exp(-b^2/4t)/\sqrt{\pi t}$$

So our solution should then be

$$u(r,t) = (1/8\pi r r_0) \{ \exp(-|r-r_0|^2/4t)/\sqrt{\pi t} - \exp(-(r+r_0)^2/4t)/\sqrt{\pi t} \}$$

But how does that respect the BC ? Oooooo.. I think I made a big mistake. I used the wrong left end boundary condition on the 1D Green's function! My solution might be right in the limit $a \to 0$. This is going to make things very much messier because in the low region we can have a mix of J and H1.

Back on the Track

We had our ODE for U = U(r,s) correctly I think obtained as

$$-(r^2U')' - \lambda r^2 U = \delta(r-r_0)/4\pi$$

And it is certainly useful to compare this to the 1D Green's equation we got in Ex 7.31, which was

$$-(r^2 a_n'(r))' - [\lambda r^2 - n(n+1)]\, a_n(r) = \delta(r-r_0)\,(2n+1)/4\pi$$

and with n=0 this gives our current situation including the RHS stuff (but not the BC's). We know that our homo equation solutions will be of this form

$$J_{1/2}(\sqrt{\lambda}\, r)/\sqrt{r}$$

and we know this because our homo is the Ex 7.31 homo with n=0. But, we have now to state our boundary conditions on U(s,t). That is where I went wrong above. We had:

boundary condition: $u(a,t) = 0$ $u(\infty,t) = 0$

and if we apply $\int_0^\infty dt\, e^{-st}$ to both these we find that

$$U(a,s) = 0 \text{ and } U(\infty,s) = 0$$

The section between the two lines is replaced at the end with a much smaller section!

We know on the right side we still need $H_{1/2}^{(1)}(\sqrt{\lambda}\, r)/\sqrt{r}$. But on the left we can have

$f_{left}(r) = [\ A\ J_{1/2}(\sqrt{\lambda}\ r) + B\ H_{1/2}^{(1)}(\sqrt{\lambda}\ r)]\ /\sqrt{r}$

If we want this to vanish at a, we require that

$[\ A\ J_{1/2}(\sqrt{\lambda}\ a) + B\ H_{1/2}^{(1)}(\sqrt{\lambda}\ a)]\ /\sqrt{a}\ \ = 0$

$\Rightarrow\ \ A\ J_{1/2}(\sqrt{\lambda}\ a) + B\ H_{1/2}^{(1)}(\sqrt{\lambda}\ a)$

$\Rightarrow\ B/A = -\ J_{1/2}(\sqrt{\lambda}\ a)/\ H_{1/2}^{(1)}(\sqrt{\lambda}\ a)$

So we can frame things in terms only of constant A by writing

$f_{left}(r) = A\ [\ \ J_{1/2}(\sqrt{\lambda}\ r) + B/A\ H_{1/2}^{(1)}(\sqrt{\lambda}\ r)]\ /\sqrt{r}$

$= A\ [\ \ J_{1/2}(\sqrt{\lambda}\ r) -\ [J_{1/2}(\sqrt{\lambda}\ a)/\ H_{1/2}^{(1)}(\sqrt{\lambda}\ a)]\ H_{1/2}^{(1)}(\sqrt{\lambda}\ r)]\ /\sqrt{r}$

But I know (from experience, thinking Smythian forms) that we should redefine this constant and write

$= C\ [\ \ H_{1/2}^{(1)}(\sqrt{\lambda}\ a)J_{1/2}(\sqrt{\lambda}\ r) -\ \ J_{1/2}(\sqrt{\lambda}\ a)\ H_{1/2}^{(1)}(\sqrt{\lambda}\ r)]\ /\sqrt{r}$

and now it is obvious that this vanishes at r = a, so we then have

$U(r,s) = C\ (rr_0)^{-1/2}\ [\ \ H_{1/2}^{(1)}(\sqrt{\lambda}\ a)J_{1/2}(\sqrt{\lambda}\ r_<) -\ \ J_{1/2}(\sqrt{\lambda}\ a)\ H_{1/2}^{(1)}(\sqrt{\lambda}\ r_<)]\ H_{1/2}^{(1)}(\sqrt{\lambda}\ r_>)$

which is not so bad really. I won't bother to do the jump from scratch. we know

$jump = [\ (2n+1)/4\pi\ ]\ /(-r_0^2)\ = -1/(4\pi r_0^2)$ // since n=0 for our current problem

Our task now is to compute this jump from the U(r,s) form shown above, which will take some time and require a more compact notation perhaps. // Next day 8.3.11: // As before we write

$U(r+,s) = C\ (rr_0)^{-1/2}\ [\ \ H_{1/2}^{(1)}(\sqrt{\lambda}\ a)J_{1/2}(\sqrt{\lambda}\ r_0) -\ \ J_{1/2}(\sqrt{\lambda}\ a)\ H_{1/2}^{(1)}(\sqrt{\lambda}\ r_0)]\ H_{1/2}^{(1)}(\sqrt{\lambda}\ r)$
$U'(r+,s) = C\sqrt{\lambda}\ \ (rr_0)^{-1/2}\ [\ \ H_{1/2}^{(1)}(\sqrt{\lambda}\ a)J_{1/2}(\sqrt{\lambda}\ r_0) -\ \ J_{1/2}(\sqrt{\lambda}\ a)\ H_{1/2}^{(1)}(\sqrt{\lambda}\ r_0)]\ H_{1/2}^{(1)}{}'(\sqrt{\lambda}\ r)$
$U'(r+,s)|^{r0} = C\sqrt{\lambda}\ \ (r_0)^{-1}\ [\ \ H_{1/2}^{(1)}(\sqrt{\lambda}\ a)J_{1/2}(\sqrt{\lambda}\ r_0) -\ \ J_{1/2}(\sqrt{\lambda}\ a)\ H_{1/2}^{(1)}(\sqrt{\lambda}\ r_0)]\ H_{1/2}^{(1)}{}'(\sqrt{\lambda}\ r_0)$

$U(r-,s) = C\ (rr_0)^{-1/2}\ [\ \ H_{1/2}^{(1)}(\sqrt{\lambda}\ a)J_{1/2}(\sqrt{\lambda}\ r) -\ \ J_{1/2}(\sqrt{\lambda}\ a)\ H_{1/2}^{(1)}(\sqrt{\lambda}\ r)]\ H_{1/2}^{(1)}(\sqrt{\lambda}\ r_0)$
$U'(r-,s) = C\sqrt{\lambda}\ \ (rr_0)^{-1/2}\ [\ \ H_{1/2}^{(1)}(\sqrt{\lambda}\ a)J_{1/2}'(\sqrt{\lambda}\ r) -\ \ J_{1/2}(\sqrt{\lambda}\ a)\ H_{1/2}^{(1)}{}'(\sqrt{\lambda}\ r)]\ H_{1/2}^{(1)}(\sqrt{\lambda}\ r_0)$
$U'(r-,s)\ )|^{r0} = C\sqrt{\lambda}\ \ (r_0)^{-1}\ [\ \ H_{1/2}^{(1)}(\sqrt{\lambda}\ a)J_{1/2}'(\sqrt{\lambda}\ r_0) -\ \ J_{1/2}(\sqrt{\lambda}\ a)\ H_{1/2}^{(1)}{}'(\sqrt{\lambda}\ r_0)]\ H_{1/2}^{(1)}(\sqrt{\lambda}\ r_0)$

$jump = U'(r+,s)|^{r0} - U'(r-,s)\ )|^{r0}\ \ = C\sqrt{\lambda}\ \ (r_0)^{-1}\ *$
$\{\ \ [\ \ H_{1/2}^{(1)}(\sqrt{\lambda}\ a)J_{1/2}(\sqrt{\lambda}\ r_0) -\ \ J_{1/2}(\sqrt{\lambda}\ a)\ H_{1/2}^{(1)}(\sqrt{\lambda}\ r_0)]\ H_{1/2}^{(1)}{}'(\sqrt{\lambda}\ r_0)$

$$- [\ H_{1/2}^{(1)}(\sqrt{\lambda}\, a) J_{1/2}'(\sqrt{\lambda}\, r_0) - J_{1/2}(\sqrt{\lambda}\, a)\, H_{1/2}^{(1)}'(\sqrt{\lambda}\, r_0)]\, H_{1/2}^{(1)}(\sqrt{\lambda}\, r_0)\}$$

$$\text{jump} = U'(r+,s)|^{r0} - U'(r-,s)\ )|^{r0} \ = C\sqrt{\lambda}\ (r_0)^{-1} *$$
$$\{\ [\ H_a J - J_a H]\, H'$$
$$- [\ H_a J' - J_a H']\, H\ \}$$ // compact notation

Now we have

$$\{\ [\ H_a J - J_a H]\, H' - [\ H_a J' - J_a H']\, H\ \}$$
$$= H_a J\, H' - J_a H\, H' - H_a J'\, H + J_a H'\, H$$ // cleared brackets
$$= H_a J\, H' - H_a J'\, H$$ // two terms cancelled (as usual)
$$= H_a\, W[J,H]$$ // Wronskian form
$$= H_a 2i/[\pi\, (\sqrt{\lambda}\, r_0)]$$ // Wronskian taken from Ex 7.31 doc $z = \sqrt{\lambda}\, r_0$

So this tells us that

$$\text{jump} = C\sqrt{\lambda}\ (r_0)^{-1}\, H_a 2i/[\pi\, (\sqrt{\lambda}\, r_0)] \ = C\, (2i/\pi r_0^2)\, H_{1/2}^{(1)}(\sqrt{\lambda}\, a)$$

We now set this to $-1/(4\pi r_0^2)$ as per above to get

$$C\, (2i/\pi r_0^2)\, H_{1/2}^{(1)}(\sqrt{\lambda}\, a) = -1/(4\pi r_0^2)$$

$$C\, (2i)\, H_{1/2}^{(1)}(\sqrt{\lambda}\, a) = -1/(4)$$
$$C\, (8/i)\, H_{1/2}^{(1)}(\sqrt{\lambda}\, a) = 1$$

$$C = (i/8)\, [H_{1/2}^{(1)}(\sqrt{\lambda}\, a)]^{-1}$$

Our Green's function solution is therefore

$$U(r,s) = C\, (r r_0)^{-1/2} [\ H_{1/2}^{(1)}(\sqrt{\lambda}\, a) J_{1/2}(\sqrt{\lambda}\, r_<) - J_{1/2}(\sqrt{\lambda}\, a)\, H_{1/2}^{(1)}(\sqrt{\lambda}\, r_<)]\, H_{1/2}^{(1)}(\sqrt{\lambda}\, r_>)$$

$$= (i/8\sqrt{r r_0}\ ) [\ H_{1/2}^{(1)}(\sqrt{\lambda}\, a) J_{1/2}(\sqrt{\lambda}\, r_<) - J_{1/2}(\sqrt{\lambda}\, a)\, H_{1/2}^{(1)}(\sqrt{\lambda}\, r_<)]\, [H_{1/2}^{(1)}(\sqrt{\lambda}\, r_>)/\, H_{1/2}^{(1)}(\sqrt{\lambda}\, a)]$$

This solves our Helmholtz equation where $\lambda = -s$, and meets the boundary conditions specified. Now we can use the idea that when $\lambda = -k^2$ we chose $\sqrt{\lambda} = +ik$ so here we will use $\sqrt{\lambda} = i\sqrt{s}$. So we could put that in everywhere above. Then we want to use our facts quoted ni Ex 7.31

10.27.6

$$I_\nu(z) = e^{\mp\nu\pi i/2} J_\nu\left(z e^{\pm\pi i/2}\right), \quad -\pi \le \pm \operatorname{ph} z \le \tfrac{1}{2}\pi,$$

10.27.8

$$K_\nu(z) = \begin{cases} \frac{1}{2}\pi i e^{\nu\pi i/2} H_\nu^{(1)}\left(ze^{\pi i/2}\right), & -\pi \le \operatorname{ph} z \le \frac{1}{2}\pi, \\ -\frac{1}{2}\pi i e^{-\nu\pi i/2} H_\nu^{(2)}\left(ze^{-\pi i/2}\right), & -\frac{1}{2}\pi \le \operatorname{ph} z \le \pi. \end{cases}$$

which we write as

$I_\nu(z) = e^{-i\pi\nu/2} J_\nu(iz)$ z above the cut

$K_\nu(z) = (i\pi/2)\, e^{+i\pi\nu/2} H_\nu^{(1)}(iz)$ z above the cut

$J_\nu(iz) = e^{+i\pi\nu/2} I_\nu(z)$

$H_\nu^{(1)}(iz) = (2/i\pi)\, e^{-i\pi\nu/2} K_\nu(z)$

$H_\nu^{(1)}(iz)\, J_\nu(iz') = (2/i\pi)\, K_\nu(z)\, I_\nu(z')$

$H_\nu^{(1)}(iz)/\, H_\nu^{(1)}(iz') = K_\nu(z)/\, K_\nu(z')$

Applying these last two rules to our U(r,s) above we get

(for example, $iz = \sqrt{\lambda}\, a = (+\sqrt{s})ia = i\sqrt{s}\, a$, so then $z = \sqrt{s}\, a$)

$$U(r,s) = (i/8\sqrt{rr_0})\,[\, H_{1/2}^{(1)}(\sqrt{\lambda}\, a) J_{1/2}(\sqrt{\lambda}\, r_<) - J_{1/2}(\sqrt{\lambda}\, a)\, H_{1/2}^{(1)}(\sqrt{\lambda}\, r_<)] * [H_{1/2}^{(1)}(\sqrt{\lambda}\, r_>)/\, H_{1/2}^{(1)}(\sqrt{\lambda}\, a)]$$

$$U(r,s) = (i/8\sqrt{rr_0})\,(2/i\pi)\,[\, K_{1/2}^{(1)}(\sqrt{s}\, a) I_{1/2}(\sqrt{s}\, r_<) - I_{1/2}(\sqrt{s}\, a)\, K_{1/2}^{(1)}(\sqrt{s}\, r_<)] * [K_{1/2}^{(1)}(\sqrt{s}\, r_>)/\, K_{1/2}^{(1)}(\sqrt{s}\, a)]$$

$$= (1/4\pi\sqrt{rr_0})[\, K_{1/2}^{(1)}(\sqrt{s}\, a) I_{1/2}(\sqrt{s}\, r_<) - I_{1/2}(\sqrt{s}\, a)\, K_{1/2}^{(1)}(\sqrt{s}\, r_<)] * [K_{1/2}^{(1)}(\sqrt{s}\, r_>)/\, K_{1/2}^{(1)}(\sqrt{s}\, a)]$$

and we have everything now in terms of s.

Next we note that these functions are just elementary functions

Elementary Functions

10.39.1

$$I_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sinh z, \quad I_{-\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cosh z,$$

10.39.2

$$K_{\frac{1}{2}}(z) = K_{-\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}.$$

so that that last factor in U becomes

$$[K_{1/2}^{(1)}(\sqrt{s}\, r_>)/ K_{1/2}^{(1)}(\sqrt{s}\, a)] = (\sqrt{s}\, r_>)^{-1/2} \exp(-\sqrt{s}\, r_>)/ [(\sqrt{s}\, a)^{-1/2} \exp(-\sqrt{s}\, a)]$$

$$= (r_>/a)^{-1/2} \exp[-(r_> - a)\sqrt{s}\,]$$

and the first factor becomes

$$[\ K_{1/2}^{(1)}(\sqrt{s}\, a) I_{1/2}(\sqrt{s}\, r_<) - I_{1/2}(\sqrt{s}\, a)\, K_{1/2}^{(1)}(\sqrt{s}\, r_<)]$$

$$= (\sqrt{s}\, a \sqrt{s}\, r_<)^{-1/2} [\ \exp(-\sqrt{s}\, a)\, \mathrm{sh}(\sqrt{s}\, r_<) - \exp(-\sqrt{s}\, r_<)\, \mathrm{sh}(\sqrt{s}\, a)]$$

$$= (\sqrt{s}\, a \sqrt{s}\, r_<)^{-1/2} *$$
$$[\ \exp(-\sqrt{s}\, a)\, \mathrm{sh}(\sqrt{s}\, r_<) - \exp(-\sqrt{s}\, r_<)\, \mathrm{sh}(\sqrt{s}\, a)]$$

$$= (\sqrt{s}\, a \sqrt{s}\, r_<)^{-1/2}\ (1/2)^*$$
$$[\ \exp(-\sqrt{s}\, a)\ \{\exp(\sqrt{s}\, r_<) - \exp(-\sqrt{s}\, r_<)\} - \exp(-\sqrt{s}\, r_<)\ \{\exp(\sqrt{s}\, a) - \exp(-\sqrt{s}\, a)\}]$$

$$= (\sqrt{s}\, a \sqrt{s}\, r_<)^{-1/2}\ (1/2)^*$$
$$[\ \exp(-\sqrt{s}\, a)\ \{\exp(\sqrt{s}\, r_<) - \exp(-\sqrt{s}\, r_<)\}$$
$$- \exp(-\sqrt{s}\, r_<)\ \{\exp(\sqrt{s}\, a) - \exp(-\sqrt{s}\, a)\}]$$

$$= (\sqrt{s}\, a \sqrt{s}\, r_<)^{-1/2}\ (1/2)^*$$
$$[\ \exp(-\sqrt{s}\, a) \exp(\sqrt{s}\, r_<) - \exp(-\sqrt{s}\, a) \exp(-\sqrt{s}\, r_<)\}$$
$$- \exp(-\sqrt{s}\, r_<) \exp(\sqrt{s}\, a) + \exp(-\sqrt{s}\, r_<) \exp(-\sqrt{s}\, a)\}]$$

$$= (\sqrt{s}\, a \sqrt{s}\, r_<)^{-1/2}\ (1/2)^*$$
$$[\ \exp(-\sqrt{s}\, (a-r_<) - \exp(-\sqrt{s}\, (a+r_<)\}$$
$$- \exp(+\sqrt{s}\, (a-r_<) + \exp(-\sqrt{s}\, (a+r_<)\}\ ]$$

$$= (\sqrt{s}\, a \sqrt{s}\, r_<)^{-1/2}\ (1/2)^*\ [\ \exp(-\sqrt{s}\, (a-r_<) - \exp(+\sqrt{s}\, (a-r_<)\ ]$$

$$= - (\sqrt{s}\, a \sqrt{s}\, r_<)^{-1/2}\, \mathrm{sh}[\sqrt{s}\, (a-r_<)]$$

$$= - (a r_<)^{-1/2}\, \mathrm{sh}[\sqrt{s}\, (a-r_<)] / \sqrt{s}$$

To summarize, we have just evaluated our two factors:

$$[\ K_{1/2}^{(1)}(\sqrt{s}\, a) I_{1/2}(\sqrt{s}\, r_<) - I_{1/2}(\sqrt{s}\, a)\, K_{1/2}^{(1)}(\sqrt{s}\, r_<)] = - (a r_<)^{-1/2}\, \mathrm{sh}[\sqrt{s}\, (a-r_<)] / \sqrt{s}$$

$$[K_{1/2}^{(1)}(\sqrt{s}\, r_>)/ K_{1/2}^{(1)}(\sqrt{s}\, a)] = (r_>/a)^{-1/2} \exp[-(r_> - a)\sqrt{s}\,]$$

If we install these now into our U expression we get

$$U(r,s) = (i/8\sqrt{rr_0}\,)\,(2/i\pi)\,[\;K_{1/2}^{(1)}(\sqrt{s}\,a)I_{1/2}(\sqrt{s}\,r_<) - I_{1/2}(\sqrt{s}\,a)\,K_{1/2}^{(1)}(\sqrt{s}\,r_<)]$$
$$* [K_{1/2}^{(1)}(\sqrt{s}\,r_>)/\,K_{1/2}^{(1)}(\sqrt{s}\,a)]$$

$$= (i/8\sqrt{rr_0}\,)\,(2/i\pi)\,\{-(ar_<)^{-1/2}\,sh[\sqrt{s}\,(a-r_<)]/\sqrt{s}\;\}\,\{(r_>/a)^{-1/2}\exp[-(r_>-a)\sqrt{s}\,]\}$$

$$= (i/8\sqrt{rr_0}\,)\,(2/i\pi)\,\{-(r_<)^{-1/2}\,sh[\sqrt{s}\,(a-r_<)]/\sqrt{s}\;\}\,\{(r_>)^{-1/2}\exp[-(r_>-a)\sqrt{s}\,]\}$$

$$= (i/8\sqrt{rr_0r_<r_>}\,)\,(2/i\pi)\,\{-\;sh[\sqrt{s}\,(a-r_<)]/\sqrt{s}\;\}\,\{\exp[-(r_>-a)\sqrt{s}\,]\}$$

$$= (1/[8r\,r_0)]\,(2/\pi)\,\{-\;sh[\sqrt{s}\,(a-r_<)]/\sqrt{s}\;\}\,\{\exp[-(r_>-a)\sqrt{s}\,]\}$$

$$= -(1/[4\pi r\,r_0)])\,sh[\sqrt{s}\,(a-r_<)]\;\exp[-(r_>-a)\sqrt{s}\,]/\sqrt{s}$$

$$= -(1/[8\pi r\,r_0)])\{\exp[\sqrt{s}\,(a-r_<)] - \exp[-\sqrt{s}\,(a-r_<)]\}\;\exp[-(r_>-a)\sqrt{s}\,]/\sqrt{s}$$

$$= -(1/[8\pi r\,r_0)])\{\exp[\sqrt{s}\,(a-r_<)]\exp[-(r_>-a)\sqrt{s}\,] - \exp[-\sqrt{s}\,(a-r_<)]\exp[-(r_>-a)\sqrt{s}\,]\}\;/\sqrt{s}$$

$$= -(1/[8\pi r\,r_0)])\{\exp[\sqrt{s}\,(a-r_<)]\exp[-\sqrt{s}\;(r_>-a)] - \exp[-\sqrt{s}\,(a-r_<)]\exp[-\sqrt{s}\;(r_>-a)]\}\;/\sqrt{s}$$

$$= -(1/[8\pi r\,r_0)])\{\quad\exp[\sqrt{s}\,\{(a-r_<)-(r_>-a)\}\,] - \exp[-\sqrt{s}\,\{(a-r_<)+(r_>-a)\}]\quad\}\;/\sqrt{s}$$

$$= -(1/[8\pi r\,r_0)])\{\quad\exp[\sqrt{s}\,(2a-r_<-r_>)\,] - \exp[-\sqrt{s}\,(r_>-r_<)]\quad\}\;/\sqrt{s}$$

$$= -(1/[8\pi r\,r_0)])\{\quad\exp[\sqrt{s}\,(2a-r-r_0)\,] - \exp[-\sqrt{s}\,|r-r_0|]\quad\}\;/\sqrt{s}$$

Comment: in the limit $a\to 0$, you would expect this result to match our "wrong turn" result above. If I take that limit on the above I get

$$= -(1/[8\pi rr_0)])\{\quad\exp[\sqrt{s}\,(-r-r_0)\,] - \exp[-\sqrt{s}\,|r-r_0|]\quad\}\;/\sqrt{s}$$

And this agrees exactly with my "wrong turn" result. [I ferreted out three errors by doing this check and they are all fixed above so the reader does not know they ever existed.] [Doing wrong turn paid off.] Notice how small the difference is between the general a result and the a=0 result!

Now here is our U(r,s) one more time with terms reversed

$$U(r,s) = (1/8\pi rr_0)\;\{\exp[-\sqrt{s}\,|r-r_0|]/s\;-\;\exp[\sqrt{s}\,(2a-r-r_0)\,]/\sqrt{s}\;\}$$

Let's check our BC's here. First,

$$U(a,s) = (1/8\pi r r_0)\ \{\ \exp[-\sqrt{s}\ |a-r_0|]/s\ \ -\ \ \exp[\sqrt{s}\ (2a-a-r_0)\ ]/\ \sqrt{s}\ \ \}$$
$$= (1/8\pi r r_0)\ \{\ \exp[-\sqrt{s}\ |a-r_0|]/s\ \ -\ \ \exp[\sqrt{s}\ (a-r_0)\ ]/\ \sqrt{s}\ \ \}$$

But we know that r_0 where the initial condition was set is > a, the hole radius, so r_0>a so r_0-a > 0 so

$$= (1/8\pi r r_0)\ \{\ \exp[-\sqrt{s}\ (r_0-a)\ ]/\ \sqrt{s}\ \ -\ \ \exp[-\sqrt{s}\ (r_0-a))\ ]/\ \sqrt{s}\ \ \}\ = 0 \qquad \text{// correct}$$

Next we can see that as $r\to\infty$, both expos decay and again we get 0. So BC's are met!

Now, once again, from Schaum p 169 we have that

$$\exp(-b\sqrt{s}\ )/\sqrt{s}\ \ \leftrightarrow\ \ \exp(-b^2/4t)/\sqrt{\pi t}$$

So our solution should then be

$$U(r,s) = (1/8\pi r r_0)\ \{\ \exp[-\sqrt{s}\ |r-r_0|]/\sqrt{s}\ \ -\ \ \exp[\sqrt{s}\ (2a-r-r_0)\ ]/\ \sqrt{s}\ \ \}$$

$$u(r,t) =(1/8\pi r r_0)\{\ \exp(-|r-r_0|^2/4t)/\sqrt{\pi t}\ \ -\exp(-\ (2a-r-r_0)^2/4t)/\sqrt{\pi t}\ \ \}$$

If we take $a\to 0$, this replicates our "wrong turn" result. So this thing looks good.

Now as $t\to\infty$, both terms approach $e^0/\sqrt{\infty}\ = 0$. This seems reasonable since our finite initial energy pulse in the shell gets absorbed by the infinite size medium.

Now what about $t\to 0$? The second term $\to \exp(-k\infty)/\sqrt{\infty}\ = 0$ since k>0 for all r including $r=r_0$. Note that we cannot have both r and r_0 = a since r_0 > a. So we have

$$\text{Lim}_{t\to 0}\ u(r,t) =\ (1/8\pi r r_0)\ \text{Lim}_{t\to 0}\ \{\ \exp(-|r-r_0|^2/4t)/\sqrt{\pi t}\ \ \}$$

As long as $r \neq r_0$, we get 0, so it looks like a delta function. We then have to integrate and see what happens

$$\int_{r_0-\varepsilon}^{r_0+\varepsilon}\ dr\ \ \lim_{t\to 0}\ [u(r,t)]\ \ = (1/8\pi r r_0)\ \int_{r_0-\varepsilon}^{r_0+\varepsilon}\ dr\ \text{Lim}_{t\to 0}\ \{\ \exp(-|r-r_0|^2/4t)/\sqrt{\pi t}\ \ \}$$

We keep ε and a small but finite value, then we can interchange order

$$= (1/8\pi r r_0)\ \text{Lim}_{t\to 0}\ \ \int_{r_0-\varepsilon}^{r_0+\varepsilon}\ dr\ \{\ \exp(-|r-r_0|^2/4t)/\sqrt{\pi t}\ \ \}$$

$$= (1/8\pi r r_0)\ \text{Lim}_{t\to 0}\ \ \{\ (1/\sqrt{\pi t}\ )\ \int_{r_0-\varepsilon}^{r_0+\varepsilon}\ dr\ \exp(-|r-r_0|^2/4t)$$

Change variable to $s = r-r_0$ to get

$= (1/8\pi r r_0)\ \mathrm{Lim}_{t\to 0}\ \{ (1/\sqrt{\pi t}\ ) \int_{-\varepsilon}^{\varepsilon} ds\ \exp(-|s|^2/4t) \}$

$= (1/4\pi r r_0)\ \mathrm{Lim}_{t\to 0}\ \{ (1/\sqrt{\pi t}\ ) \int_{0}^{\varepsilon} ds\ \exp(-s^2/4t) \}$

$= (1/4\pi r r_0)\ \mathrm{Lim}_{t\to 0}\ \{ (1/\sqrt{\pi t}\ ) [\mathrm{erf}(\varepsilon/2\sqrt{t}\ )] \sqrt{\pi t}\ \}$ // according to Maple

$= (1/4\pi r r_0)\ \mathrm{Lim}_{t\to 0}\ \{ \mathrm{erf}(\varepsilon/2\sqrt{t}\ )\}$

$= (1/4\pi r r_0)\ \mathrm{erf}(\infty)\}$ $= (1/4\pi r r_0) * 1$

Therefore we have shown that

$\mathrm{Lim}_{t\to 0}\ u(r,t) = (1/4\pi r r_0)\ \delta(r-r_0) = (1/4\pi r_0^2)\ \delta(r-r_0)$

and this agrees with our initial condition. So I think my final answer is correct which I repeat here:

$U(r,s) = (1/8\pi r r_0)\ \{ \exp[-\sqrt{s}\ |r-r_0|]/\sqrt{s}\ \ -\ \exp[\sqrt{s}\ (2a-r-r_0)\ ]/\ \sqrt{s}\ \}$

$u(r,t) = (1/8\pi r r_0)\{ \exp(-(r-r_0)^2/4t)/\sqrt{\pi t}\ \ -\exp(-\ (2a-r-r_0)^2/4t)/\sqrt{\pi t}\ \ \}$ ****

Here is a little Maple treatment

```
u := (1/(8*Pi*r*r0))*(exp(-(r-r0)^2/(4*t)) - exp(-(2*a-r-r0)^2/(4*t)));
```

$$u := \frac{1}{8}\frac{e^{\left(-\frac{1}{4}\frac{(r-r0)^2}{t}\right)} - e^{\left(-\frac{1}{4}\frac{(2a-r-r0)^2}{t}\right)}}{\pi r r0}$$

```
r0 := 2;
```

$$r0 := 2$$

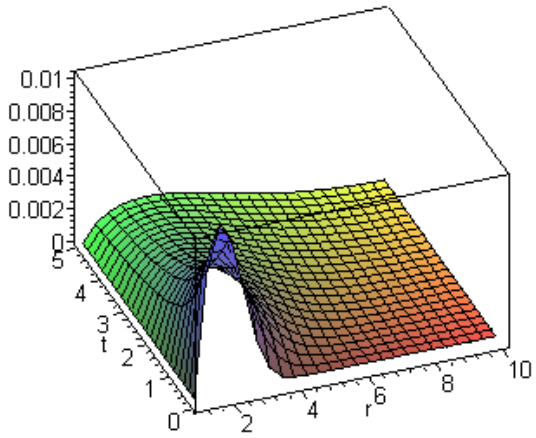
```
a := 1;
```

$$a := 1$$

```
u;
```

$$\frac{1}{16}\frac{e^{\left(-\frac{1}{4}\frac{(r-2)^2}{t}\right)} - e^{\left(-\frac{1}{4}\frac{r^2}{t}\right)}}{\pi r}$$

```
plot3d(u,r=10..1,t=0..5, axes=BOXED);
```



The left edge of the cube is r = 1. You see that the solution is glued down to 0 at r=1 and r=∞ as required. At t = 0 we start with a delta at r=2 and over time it spreads out. Heat solutions always look like this.

Acrobat OCR Exercise

By the way, I pasted the above Maple equation into Acrobat 8 and asked for OCR and got this

u : = (1/(8*Pi *r*r0)) *(exp(-(r-r0)^2/(4*t)) - exp(-(2* a-r-r0)^2/(4*t))) ;

As Marilyn said, the OCR is not very good. I highlight all the errors in red.

Redo a part of this problem after "properties of Bessel" written.

Above we arrive at this point

$- (r^2U')' - \lambda r^2 U = \delta(r-r_0)/4\pi \qquad \lambda = -s$

We apply the result of Example 1A of the properties doc with $\nu = 1/2$ to conclude that

$$U = (1/4\pi\sqrt{rr_0})[\, I_{1/2}(kr_<)\, K_{1/2}(ka)) - K_{1/2}(kr_<)\, I_{1/2}(ka)]\, [\, K_{1/2}(kr_>)/\, K_{1/2}(ka)]$$

where $k=\sqrt{s}$ and this agrees with the result above just above the second line.