**Exercise 7.34: The 2D Exterior Helmholtz Green's Function for a circle.**

**Exercise 7.34: The 2D Exterior Helmholtz Green's Function for a circle. ..... 1**
Part (a) : Use the Addition Theorem to get g. ..... 2
Part (b): Use the 1D Green's method to get g. ..... 4
Take the electrostatics λ→0 limit of this g. ..... 6

---

**Overview from Meta Notes.**

**Exercise 7.34 ( p 290)** The problem is to find the 2D Helmholtz exterior Green's function outside a circle on which g=0, and we solve the problem by two different methods.

The first method is my "Green's Function by Dirichlet" method where we then get a homo PDE with a certain BC for v where $g = E_2 + v$. We solve the homo by first selecting $e^{in\theta}$ and having unknown $v_n(r)$. The Green's ODE then for $v_n(r)$ is something we already figured out near page 269B with homo solution $Z_n(\sqrt{\lambda}\, r)$, but in this simple exterior case we must have $Z_n(\sqrt{\lambda}\, r) = H^{(1)}_n(\sqrt{\lambda}\, r)$ so now $v_n(r) = V_n H^{(1)}_n(\sqrt{\lambda}\, r)$ and we have only to find the constants $V_n$. The trick is that we can look at r=a where we know $v_n(a) = -E_2$ and this (with an addition theorem for $E_2$) then tells us the $V_n$ and we get our solution which is

$$g(r,\theta|r_0,\theta_0) = (i/4)\ \Sigma_n\, e^{in(\theta-\theta 0)} * \qquad r_< = \min(r,r_0)$$
$$\{\ J_n(\sqrt{\lambda}\, r_<)H^{(1)}_n(\sqrt{\lambda}\, a) - J_n(\sqrt{\lambda}\, a)\, H^{(1)}_n(\sqrt{\lambda}\, r_<)\ \}\ [H^{(1)}_n(\sqrt{\lambda}\, r_>) / H^{(1)}_n(\sqrt{\lambda}\, a)]$$

The second method is more conventional. We start with the δδ Green's, select the same $e^{in\theta}$ eigenfunctions, and get then our usual 1D ODE Green's in r which we then just solve in the usual manner and the same result is obtained as shown above.

Then just for fun I took the "electrostatics limit" λ→0 of the above result and got agreement with 6.100 which was the solution we got eons ago back in potential theory.

---

Consider a circle of radius a. We have BC that u=0 on the boundary of the circle, and we want a 2D Helmholtz Green's Function for a point source outside this circle. We surely want u=0 at r=∞ as well.

Just out of interest, what problems like this were done in the text? Stak first listed off the fundies in this section. He then did some weird expansions for $E_2$ and then did the infinite wedge problem. Then one of the $E_2$ expansions by another method. Then came the "ring source" example. Here the ring source is the driver of the H equation, so it is "like" a Green's Function problem, but instead of a point source, you have this delta ring thing. The main idea is that you have a distribution as your driving function. Stak solves this two ways, doing partial eigenfunctions in r and in z. Then he is off into the half plane stuff, another H Green's problem in 2D. Then finally he pulsed source at t=0 in this half-plane problem. That is it.

So no, he did *not* do this exterior to disk H Green's Function problem.

So here is the problem. We have in 2D

$$(-\nabla^2 - \lambda)g(r,\theta) = \delta^{(2)}(\mathbf{r}-\mathbf{r}_0) = \delta(r-r_0)/r * \delta(\theta-\theta_0) \qquad \text{// Green's point charge at } r_0,\theta_0$$

$g(a,\theta) = 0 \qquad g(\infty,\theta) = 0 \qquad$ // boundary conditions

**Part (a) : Use the Addition Theorem to get g.**

His suggestion first off is to write $g = E_2+v$, one of my standard methods ("the Dirichlet method of computing Green's Functions"). Then since $E_2$ satisfies the PDE, we will get homo for v, but with different BC's:

$(-\nabla^2 -\lambda)v(r,\theta) = 0$
$v(a,\theta) = -E_2(a,\theta|r_0,\theta_0) \qquad v(\infty,\theta) = 0 \qquad$ // boundary conditions

We did something like this back on page 274-5. That problem was a 2D Green's for the half-line boundary, and he used the KL transform to replace r by $\gamma$ and got a simple equation in $\theta$. Hold on that.

Now, what can we say about the atoms for this Helmholtz problem? Looking back at p 268 where we considered this same PDE (but driven by a point source with no BC's), we assume

$$v(r,\theta) = \Sigma_n\, v_n(r) e^{in\theta}$$

and we can steal from p 269B to get the ODE for $v_n(r)$ except we have no driver. On that page we saw that the form of the homo solution is $J_n(\sqrt{\lambda}\, r)$. For our exterior problem, we can only have $H^{(1)}_n(\sqrt{\lambda}\, r)$, so we know that $v_n(r)$ = constant * $H^{(1)}_n(\sqrt{\lambda}\, r)$ . If we call this constant $v_n$, we get

$$v(r,\theta) = \Sigma_n\, v_n(r) e^{in\theta} = \Sigma_n\, v_n\, H^{(1)}_n(\sqrt{\lambda}\, r)\, e^{in\theta}$$

where we have all integer sum, and thus, we have done part (a) of this problem!

Now how can we obtain the $v_n$ values? At r = a we do have this fact

$$v(a,\theta) = \Sigma_n\, v_n\, H^{(1)}_n(\sqrt{\lambda}\, a)\, e^{in\theta} = -E_2(a,\theta|r_0,\theta_0) \qquad (*)$$

But now we can write $E_2$ using the expansion 7.163 (his "addition theorem" or "sum rule"):

$$E_2(r,\theta|r_0,\theta_0) = (i/4)\, \Sigma_n\, J_n(\sqrt{\lambda}\, r_<)\, H^{(1)}_n(\sqrt{\lambda}\, r_>) e^{in(\theta-\theta 0)}$$

where the < > refer to a,$r_0$. But since $r_0 > a$ for sure, we then have

$$E_2(a,\theta|r_0,\theta_0) = (i/4)\, \Sigma_n\, J_n(\sqrt{\lambda}\, a)\, H^{(1)}_n(\sqrt{\lambda}\, r_0) e^{in(\theta-\theta 0)}$$

Therefore, installing this in (*) above we find

$$\Sigma_n\, v_n\, H^{(1)}_n(\sqrt{\lambda}\, a)\, e^{in\theta} = -(i/4)\, \Sigma_n\, J_n(\sqrt{\lambda}\, a)\, H^{(1)}_n(\sqrt{\lambda}\, r_0) e^{in(\theta-\theta 0)}$$

Since the $e^{in\theta}$ form a complete set, we may identify

$$v_n H^{(1)}{}_n(\sqrt{\lambda}\, a) = -(i/4)\, J_n(\sqrt{\lambda}\, a)\, H^{(1)}{}_n(\sqrt{\lambda}\, r_0)\, e^{-in\theta 0}$$

and then we have our $v_n$

$$v_n = -(i/4)\, [J_n(\sqrt{\lambda}\, a)/ H^{(1)}{}_n(\sqrt{\lambda}\, a)\,]\, H^{(1)}{}_n(\sqrt{\lambda}\, r_0)\, e^{-in\theta 0}$$

and so this coefficient is really $v_n(a, \lambda, r_0, \theta_0)$ as you would expect. This concludes part a. Our solution of course is then this

$$v(r,\theta|r_0,\theta_0) = \Sigma_n\, v_n H^{(1)}{}_n(\sqrt{\lambda}\, r)\, e^{in\theta}$$

$$= -(i/4)\, \Sigma_n\, [J_n(\sqrt{\lambda}\, a)/ H^{(1)}{}_n(\sqrt{\lambda}\, a)\,]\, H^{(1)}{}_n(\sqrt{\lambda}\, r_0)\, H^{(1)}{}_n(\sqrt{\lambda}\, r)\, e^{in(\theta-\theta 0)}$$

and we see our expected $r,r_0$ symmetry. Now if we want g, we have to do $g = E_2+v$ so

$$g(r,\theta|r_0,\theta_0) = -(i/4)\, \Sigma_n\, [J_n(\sqrt{\lambda}\, a)/ H^{(1)}{}_n(\sqrt{\lambda}\, a)\,]\, H^{(1)}{}_n(\sqrt{\lambda}\, r_0)\, H^{(1)}{}_n(\sqrt{\lambda}\, r)\, e^{in(\theta-\theta 0)}$$

$$+ (i/4)\ \Sigma_n\ J_n(\sqrt{\lambda}\, r_<)\, H^{(1)}{}_n(\sqrt{\lambda}\, r_>) e^{in(\theta-\theta 0)}$$

To keep this result general, we cannot do much with the $r_<$ type variables, so all we can do is this:

$$g(r,\theta|r_0,\theta_0) = (i/4)\ \Sigma_n\, e^{in(\theta-\theta 0)} * \qquad r_< = \min(r,r_0)$$

$$\{\ J_n(\sqrt{\lambda}\, r_<)\, H^{(1)}{}_n(\sqrt{\lambda}\, r_>) - [J_n(\sqrt{\lambda}\, a)/ H^{(1)}{}_n(\sqrt{\lambda}\, a)\,]\, H^{(1)}{}_n(\sqrt{\lambda}\, r_0)\, H^{(1)}{}_n(\sqrt{\lambda}\, r)\ \}$$

If we set $r = a$, then we know that $r_< = a$ and $r_> = r_0$ and this becomes

$$g(a,\theta|r_0,\theta_0) = (i/4)\ \Sigma_n\, e^{in(\theta-\theta 0)} *$$

$$\{\ J_n(\sqrt{\lambda}\, a)\, H^{(1)}{}_n(\sqrt{\lambda}\, r_0) - [J_n(\sqrt{\lambda}\, a)/ H^{(1)}{}_n(\sqrt{\lambda}\, a)\,]\, H^{(1)}{}_n(\sqrt{\lambda}\, r_0)\, H^{(1)}{}_n(\sqrt{\lambda}\, a)\ \}$$

$$= (i/4)\ \Sigma_n\, e^{in(\theta-\theta 0)} *$$

$$\{\ J_n(\sqrt{\lambda}\, a)\, H^{(1)}{}_n(\sqrt{\lambda}\, r_0) - J_n(\sqrt{\lambda}\, a)\, H^{(1)}{}_n(\sqrt{\lambda}\, r_0)\ \} = 0$$

so we verify that g=0 on the circle at r=a. For large r we get the desired decay in r.

Let's go for a cleaner way to write g. Just process it a bit

$$g(r,\theta|r_0,\theta_0) = (i/4)\ \Sigma_n\, e^{in(\theta-\theta 0)} * \qquad r_< = \min(r,r_0)$$
$$\{\ J_n(\sqrt{\lambda}\, r_<)\, H^{(1)}{}_n(\sqrt{\lambda}\, r_>) - [J_n(\sqrt{\lambda}\, a)/ H^{(1)}{}_n(\sqrt{\lambda}\, a)\,]\, H^{(1)}{}_n(\sqrt{\lambda}\, r_0)\, H^{(1)}{}_n(\sqrt{\lambda}\, r)\ \}$$

$$g(r,\theta|r_0,\theta_0) = (i/4)\ \Sigma_n\, e^{in(\theta-\theta 0)} * \qquad r_< = \min(r,r_0)$$

$$\{\ \ J_n(\sqrt{\lambda}\, r_<)\, H^{(1)}_n(\sqrt{\lambda}\, r_>)\, H^{(1)}_n(\sqrt{\lambda}\, a) - J_n(\sqrt{\lambda}\, a)\, H^{(1)}_n(\sqrt{\lambda}\, r_0)\, H^{(1)}_n(\sqrt{\lambda}\, r)\ \ \}/\, H^{(1)}_n(\sqrt{\lambda}\, a)$$

$$g(r,\theta|r_0,\theta_0) = (i/4)\ \Sigma_n\, e^{in(\theta-\theta 0)} * \qquad r_< = \min(r,r_0)$$
$$\{\ \ J_n(\sqrt{\lambda}\, r_<)\, H^{(1)}_n(\sqrt{\lambda}\, r_>)\, H^{(1)}_n(\sqrt{\lambda}\, a) - J_n(\sqrt{\lambda}\, a)\, H^{(1)}_n(\sqrt{\lambda}\, r_<)\, H^{(1)}_n(\sqrt{\lambda}\, r_>)\ \ \}/\, H^{(1)}_n(\sqrt{\lambda}\, a)$$

$$g(r,\theta|r_0,\theta_0) = (i/4)\ \Sigma_n\, e^{in(\theta-\theta 0)} * \qquad r_< = \min(r,r_0)$$
$$\{\ \ J_n(\sqrt{\lambda}\, r_<) H^{(1)}_n(\sqrt{\lambda}\, a) - J_n(\sqrt{\lambda}\, a)\, H^{(1)}_n(\sqrt{\lambda}\, r_<)\ \ \}\ [H^{(1)}_n(\sqrt{\lambda}\, r_>) / H^{(1)}_n(\sqrt{\lambda}\, a)]$$

Now when $r_< = a$, we see at once that $g = 0$, so this is a nicer form. Below I give this alternate form

$$g(r,\theta|r_0,\theta_0) = (i/4)\ \Sigma_n\ \varepsilon_n \cos(n[\theta-\theta_0]) *$$
$$\{\ J_n(\sqrt{\lambda}\, r_<) H_n^{(1)}(\sqrt{\lambda}\, a) - \ J_n(\sqrt{\lambda}\, a)\, H_n^{(1)}(\sqrt{\lambda}\, r_<)\ \}\quad [\ H_n^{(1)}(\sqrt{\lambda}\, r_>) / H_n^{(1)}(\sqrt{\lambda}\, a)]$$

where we have just folded things over. So these are the results for this problem!

**Part (b): Use the 1D Green's method to get g.**

Now Stak tells us to go solve this little ODE p 269 B in its homo form. Well, this will be very much like my Exercise 7.33 piece of work. The difference is that here we have a 2D $\nabla^2$ so the ODE. Our starting point here is this

$$(-\nabla^2 - \lambda)g(r,\theta) = \delta^{(2)}(\mathbf{r}-\mathbf{r_0}) = \delta(r-r_0)/r * \delta(\theta) \qquad \text{// Green's point charge at } r_0,0 \ \ (\theta_0 = 0 \text{ now})$$
$$g(a,\theta) = 0 \qquad g(\infty,\theta) = 0 \qquad \text{// boundary conditions}$$

so the PDE says

$$(-r^{-1}\partial_r(r\partial_r..) - r^{-2}\partial_\theta^2 - \lambda)g(r,\theta) = \delta(r-r_0)/r * \delta(\theta)$$

$$(-\partial_r(r\partial_r..) - r^{-1}\partial_\theta^2 - r\lambda)g(r,\theta) = \delta(r-r_0)\,\delta(\theta)$$

$$-\partial_r(r\partial_r g) \ \ - r^{-1}\partial_\theta^2 g - r\lambda\, g = \delta(r-r_0)\,\delta(\theta)$$

$$-(rg')' \ \ - r^{-1}\partial_\theta^2 g - r\lambda\, g = \delta(r-r_0)\,\delta(\theta) \qquad \text{// where prime means } \partial_r$$

This is equation 7.162 page 269 and we know that if we expand $g = \Sigma_n e^{in\theta} a_n(r)$ we get p 269 B which is

$$-(ra'_n)' - \lambda r a_n + n^2 a_n/r = \delta(r-r_0)/2\pi$$

So this is our 1D Green's problem, and we already know that solutions form is $J_n(\sqrt{\lambda}\, r)$. So, we know the form for our Green's function, and it is exactly like what we did in Exercise 7.33 with these two changes:

(1) we have general n instead of n = 1/2
(2) we have no $1/\sqrt{r}$ factors as part of the general solution form.

So I can just read off various results making these changes. For example

$$U(r,s) = C\,(rr_0)^{-1/2}\,[\;H_{1/2}^{(1)}(\sqrt{\lambda}\,a)J_{1/2}(\sqrt{\lambda}\,r_<) - J_{1/2}(\sqrt{\lambda}\,a)\,H_{1/2}^{(1)}(\sqrt{\lambda}\,r_<)]\,H_{1/2}^{(1)}(\sqrt{\lambda}\,r_>)$$

becomes for us here

$$a_n(r;\,a,r_0,\lambda) = C\;[\;H_n^{(1)}(\sqrt{\lambda}\,a)J_n(\sqrt{\lambda}\,r_<) - J_n(\sqrt{\lambda}\,a)\,H_n^{(1)}(\sqrt{\lambda}\,r_<)]\,H_n^{(1)}(\sqrt{\lambda}\,r_>)$$

If we compute the jump in this thing at $r = r_0$ we got before that

$$\text{jump} = C\,(2i/\pi r_0^2)\,H_{1/2}^{(1)}(\sqrt{\lambda}\,a)$$

Since we are missing the $(rr_0)^{-1/2}$ factor, our result here will be

$$\text{jump} = C\,(2i/\pi r_0)\,H_n^{(1)}(\sqrt{\lambda}\,a)$$

Now going back to our ODE

$$-(ra'_n)' - \lambda r a_n + n^2 a_n/r = \delta(r-r_0)/2\pi$$

which we write as

$$-ra''_n - a'_n - \lambda r a_n + n^2 a_n/r = \delta(r-r_0)/2\pi$$

from which we see that our famous jump "$a_0$" (overloaded!) is $-r_0$ so jump $= 1/a_0 = -1/r_0$ here. But we have the extra $1/2\pi$, so we have

$$\text{jump} = -1/(2\pi r_0)$$

whereas in our Ex 7.33 we had jump $= -1/(4\pi r_0^2)$. So we set things equal here to get

$$C\,(2i/\pi r_0)\,H_n^{(1)}(\sqrt{\lambda}\,a) = -1/(2\pi r_0)$$

$$C\,(4i)\,H_n^{(1)}(\sqrt{\lambda}\,a) = -1$$

$$C = -(1/4i)\;1/\,H_n^{(1)}(\sqrt{\lambda}\,a) \;=\; (i/4)\,[\,H_n^{(1)}(\sqrt{\lambda}\,a)]^{-1}$$

which we can compare to $C = (i/8)\,[H_{1/2}^{(1)}(\sqrt{\lambda}\,a)]^{-1}$ of our Ex 7.33, pretty close! So we now have our Green's Function for the current 1D problem

$$a_n(r;\,a,r_0,\lambda) = C\;[\;H_n^{(1)}(\sqrt{\lambda}\,a)J_n(\sqrt{\lambda}\,r_<) - J_n(\sqrt{\lambda}\,a)\,H_n^{(1)}(\sqrt{\lambda}\,r_<)]\,H_n^{(1)}(\sqrt{\lambda}\,r_>)$$

$$= (i/4)\,[\,H_n^{(1)}(\sqrt{\lambda}\,a)]^{-1}\;[\;H_n^{(1)}(\sqrt{\lambda}\,a)J_n(\sqrt{\lambda}\,r_<) - J_n(\sqrt{\lambda}\,a)\,H_n^{(1)}(\sqrt{\lambda}\,r_<)]\,H_n^{(1)}(\sqrt{\lambda}\,r_>)$$

$$= (i/4)\,[\ H_n^{(1)}(\sqrt{\lambda}\,a)J_n(\sqrt{\lambda}\,r_<) - J_n(\sqrt{\lambda}\,a)\,H_n^{(1)}(\sqrt{\lambda}\,r_<)]\,[\ H_n^{(1)}(\sqrt{\lambda}\,r_>)\,/H_n^{(1)}(\sqrt{\lambda}\,a)]$$

Now we install this into our expansion and we get our solution

$$g(a,\theta|r_0,0) = \Sigma_n e^{in\theta} a_n(r)$$

$$= (i/4)\,\Sigma_n e^{in\theta}\,[\ H_n^{(1)}(\sqrt{\lambda}\,a)J_n(\sqrt{\lambda}\,r_<) - J_n(\sqrt{\lambda}\,a)\,H_n^{(1)}(\sqrt{\lambda}\,r_<)]\,[\ H_n^{(1)}(\sqrt{\lambda}\,r_>)\,/H_n^{(1)}(\sqrt{\lambda}\,a)]$$

This agrees exactly with the result obtained in part (a).

**Take the electrostatics λ→0 limit of this g.**

Now I know that if we take λ→0, somehow this has to give the Green's function for a grounded disk where the point charge is in the plane of the disk?? NO, that is a 3D problem, here we have a 2D problem, so this limit does not give that result. The limiting electrostatics problem is a line charge external to a grounded infinite cylinder, or the 2D Green's function of for a circle. Here are small z limits

**10.7.3** $$J_\nu(z) \sim (\tfrac{1}{2}z)^\nu / \Gamma(\nu+1), \quad \nu \neq -1, -2, -3, \ldots,$$

**10.7.7**
$$H_\nu^{(1)}(z) \sim -H_\nu^{(2)}(z) \sim -(i/\pi)\,\Gamma(\nu)(\tfrac{1}{2}z)^{-\nu}, \quad \Re\nu > 0.$$

So you see that

$$J_n(z)\,H_n^{(1)}(z') = (z/2)^n / \Gamma(n+1) * (1/i\pi)\,\Gamma(n)\,(z'/2)^{-n} = (z/z')^n / n * (1/i\pi) = (1/i\pi n)\,(z/z')^n$$

Therefore we have

$$J_n(\sqrt{\lambda}\,r)\,H_n^{(1)}(\sqrt{\lambda}\,r') = (1/i\pi n)\,(r/r')^n$$

and λ goes away! Similarly.

$$H_n^{(1)}(\sqrt{\lambda}\,r) / H_n^{(1)}(\sqrt{\lambda}\,r') = (r/r')^{-n}$$

Thus we can just write our electrostatics limit of the above Green's Function

$$g(r,\theta|r_0,0) = \Sigma_n e^{in\theta} a_n(r)$$

$$= (i/4)\,\Sigma_n e^{in\theta}\,[\ H_n^{(1)}(\sqrt{\lambda}\,a)J_n(\sqrt{\lambda}\,r_<) - J_n(\sqrt{\lambda}\,a)\,H_n^{(1)}(\sqrt{\lambda}\,r_<)]\,[\ H_n^{(1)}(\sqrt{\lambda}\,r_>)\,/H_n^{(1)}(\sqrt{\lambda}\,a)]$$

$$= (i/4)\,\Sigma_n e^{in\theta}\,[\ (1/i\pi n)\,(r_</a)^n - (1/i\pi n)\,(a/r_<)^n]\,[(r_>/a)^{-n}]$$

$= (i/4)\,\Sigma_n e^{in\theta}\,(1/i\pi n)\,[\;(r_</a)^n - (a/r_<)^n]\,(r_>/a)^{-n}$

$= (1/4\pi)\,\Sigma_n e^{in\theta}\,(1/n)\,[\;(r_</a)^n - (a/r_<)^n]\,(r_>/a)^{-n}$

$= (1/4\pi)\;\Sigma_n e^{in\theta}\,(1/n)\,[\;(r_</a)^n (r_>/a)^{-n} - (a/r_<)^n (r_>/a)^{-n}]$

$= (1/4\pi)\;\Sigma_n e^{in\theta}\,(1/n)\,[\;(r_</r_>)^n - (a^2/r_<r_>)^n]$

$= (1/4\pi)\;\Sigma_n e^{in\theta}\,(1/n)\,[\;(r_</r_>)^n - (a^2/r_<r_>)^n]$ // wrong, corrected below

and probably this is a problem we did in Chapter 6 somewhere. This problem is addressed on page 148, but I don't think a solution in the above form was given. You can sort of see an image charge here and I could figure it out but won't do that right now. Something looks wrong here because the negative n terms will diverge at r=∞. The reason is that the AS limit for $H_n^{(1)}$ does not apply for negative n, so you have to mess with connection formulas first, that is

**10.4.7** $$H_\nu^{(1)}(z) = i\csc(\nu\pi)\left(e^{-\nu\pi i}J_\nu(z) - J_{-\nu}(z)\right)$$

**10.4.6** $$H_{-\nu}^{(1)}(z) = e^{\nu\pi i}H_\nu^{(1)}(z),$$

For ν = integer, each of these formulas just says you pick up $(-1)^n$ when you negate order. So a product JK or a ratio H/H will be symmetric in the order. Then we can say,

$g(a,\theta|r_0,0) = \Sigma_n e^{in\theta} a_n(r)$

$= (i/4)\,\Sigma_n e^{in\theta}\,[\;H_n^{(1)}(\sqrt{\lambda}\,a)J_n(\sqrt{\lambda}\,r_<) - J_n(\sqrt{\lambda}\,a)\,H_n^{(1)}(\sqrt{\lambda}\,r_<)]\,[\,H_n^{(1)}(\sqrt{\lambda}\,r_>)\,/H_n^{(1)}(\sqrt{\lambda}\,a)]$

$= (i/4)\,\Sigma_n e^{in\theta}\,f_n = (i/4)\,[\,f_0 + \Sigma_{n=1}^{\infty} e^{in\theta}\,f_n + \Sigma_{n=1}^{\infty} e^{-in\theta}\,f_n\,]$ since $f_{-n} = f_n$

$= (i/4)\,[\,f_0 + 2\Sigma_{n=1}^{\infty}\cos(n\theta)\,f_n\,] = (i/4)\,\Sigma_{n=0}^{\infty}\cos(n\theta)\,\varepsilon_n f_n$

$= (i/4)\,\Sigma_{n=0}^{\infty}\varepsilon_n\cos(n\theta)\,[\;H_n^{(1)}(\sqrt{\lambda}\,a)J_n(\sqrt{\lambda}\,r_<) - J_n(\sqrt{\lambda}\,a)\,H_n^{(1)}(\sqrt{\lambda}\,r_<)]\,[\,H_n^{(1)}(\sqrt{\lambda}\,r_>)\,/H_n^{(1)}(\sqrt{\lambda}\,a)]$

This is a good form for the result of this problem, by the way.

NOW we can install our λ→0 limits as per above to get

$g(r,\theta|r_0,0) = (1/4\pi)\,\Sigma_{n=0}^{\infty}\varepsilon_n\cos(n\theta)\,(1/n)\,[\;(r_</r_>)^n - (a^2/r_<r_>)^n]$

and now we are good as r→∞. The n=0 term is a little unusual and gives f(n)/g(n) = 0/0 so

$[\;(r_</r_>)^n - (a^2/r_<r_>)^n]\,/\,n = \{\;\exp[x\ln(r_</r_>)] - \exp[x\ln(a^2/r_<r_>)]\;\}/x$ as $x\to 0$

$$\{\ \exp[\alpha x] - \exp[\beta x]\ \}/x\ =\ (\alpha-\beta)\ = \ln(r_</r_>) - \ln\,(a^2/r_<r_>) = \ln[(r_</r_>)\,/(a^2/r_<r_>)\,]$$

$$= \ln[(r_<^2)\,/(a^2)\,]\ =\ 2\,\ln(r_</a)$$

If this is right, we then have

$$g(r,\theta|r_0,0) = (1/4\pi)\{\ 2\,\ln(r_</a) +\ \ 2\,\Sigma_{n=1}^{\infty}\cos(n\theta)\,(1/n)\,[\ \ (r_</r_>)^n -\ \ (a^2/r_<r_>)^n]\ \ \}$$

Now suppose we can apply the famous sum rule (6.22) to each term!

$$\Sigma_{n=1}^{\infty}\cos(n\theta)\,(1/n)\,[\ \ (r_</r_>)^n] \ = -(1/2)\,\ln(1 +\ \ (r_</r_>)^2 - 2(r_</r_>)\cos\theta)$$

$$\Sigma_{n=1}^{\infty}\cos(n\theta)\,(1/n)\,[(a^2/r_<r_>)^n] \ = -(1/2)\,\ln(1 +\ \ (a^2/r_<r_>)^2 - 2(a^2/r_<r_>)\cos\theta)$$

Then we have some horrible log answer:

$$g(r,\theta|r_0,0) = (1/2\pi)\,\ln(r_</a) + (1/2\pi)[\ -(1/2)\,\ln(1 +\ \ (r_</r_>)^2 - 2(r_</r_>)\cos\theta)]$$
$$- (1/2\pi)[\ -(1/2)\,\ln(1 +\ \ (a^2/r_<r_>)^2 - 2(a^2/r_<r_>)\cos\theta\ ]$$

$$g(r,\theta|r_0,0) = (1/2\pi)\,\ln(r_</a) -\ \ (1/4\pi)\,\ln(1 +\ \ (r_</r_>)^2\ \ \ - 2(r_</r_>)\ \ \cos\theta)$$
$$+(1/4\pi)\,\ln(1 +\ \ (a^2/r_<r_>)^2 - 2(a^2/r_<r_>)\cos\theta)$$

Now suppose we define a certain image distance

$$r_<^*\ \equiv a^2/r_<$$

Then we have

$$g(r,\theta|r_0,0) = (1/2\pi)\,\ln(r_</a) -\ \ (1/4\pi)\,\ln(1 +\ \ (r_</r_>)^2\ \ - 2(r_</r_>)\ \ \cos\theta)$$
$$+\ \ (1/4\pi)\,\ln(1 +\ \ (r_<^*/r_>)^2 - 2(r_<^*r_>)\cos\theta)$$

Suppose we add these terms, making no difference in so doing,

$$-\ \ (1/4\pi)\,\ln(r_>^2) +\ \ (1/4\pi)\,\ln(r_>^2)$$

and we absorb the first into the first log, and the second into the second, getting

$$g(r,\theta|r_0,0) = (1/2\pi)\,\ln(r_</a) -\ \ (1/4\pi)\,\ln(r_>^2 +\ \ r_<^2\ \ \ - 2r_<r_>\ \cos\theta)$$
$$+(1/4\pi)\,\ln(r_>^2 +\ \ r_<^{*2}\ \ - 2r_<^*r_>\ \cos\theta)$$

Now let's just imagine that the log arguments are $|r_>- r_<|^2$ and $|r_>- r_<^*|^2$ so we then have

$$g(r,\theta|r_0,0) = (1/2\pi)\,\ln(r_</a) -\ \ (1/2\pi)\,\ln\,|r_>- r_<|\ \ \ +(1/2\pi)\,\ln\,|r_>- r_<^*|$$

$$= (1/2\pi)\ln(r_</a) + E_2(r_>| r_<) - E_2(r_>| r_<^*)$$

Now suppose I assume that a = 1 and $r > r_0$. Then this result becomes

$$= (1/2\pi)\ln(r_0) + E_2(r| r_0) - E_2(r| r_0^*)$$

and this agrees exactly with 6.100. Very good my friend.