Stakgold Exercise 7.35 PhL 8.5.11

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Overview (3 pages, written 8.8.11)

I did a large amount of work on this somewhat fuzzy problem (31 p) and it is worth keeping and indexing. The problem is the same one treated in 7.34 and it is simply the 2D Helmholtz Green's function for a circle with the point charge outside the circle. Here we use radial r eigenfunctions instead of angular θ ones.

In **Setting Up** I write the 2D Helmholtz PDE and, assuming separation of coordinates with ν as the separation constant, I come up with the radial ODE on (a,∞) which agrees with 7.161 of Stak. I quickly replace the $\lambda_{\texttt{Helmholtz}}$ by $-k^2$ since I want to reserve the symbol λ for ν, which will be an eigenvalue variable with some spectrum.

In **Review of KL**, I review the KL transform, and go ahead to apply it to my problem, even though the KL has the wrong interval (0,∞) ! I was late realizing this error. The upshot was that I thought the functions $K_{i\gamma}(kr)$ would serve as a complete set and I could expand my desired problem solution as

$$g(r,\theta) = \int_0^\infty d\gamma\, a_\gamma(\theta)\, K_{i\gamma}(kr) \qquad \gamma = \text{the continuous spectrum of KL eigenvalues}$$

In **Method 1** I inserted the above expansion into the Green's PDE terms (point Green's source at $(r_0,\theta=0)$)

$-(rg')' \;\; -r^{-1}\partial_\theta^2 g + rk^2 g = \delta(r-r_0)\,\delta(\theta)$

and used the usual KL expansion for $\delta(r-r_0)$. By appealing to the completeness of the $K_{i\gamma}(kr)$, this led to an ODE BV problem in θ,

$a_\gamma''(\theta) + b^2 a_\gamma(\theta) = c\,\delta(\theta) \qquad a_\gamma(\pi) = a_\gamma(-\pi) \qquad a_\gamma'(\pi) = a_\gamma'(-\pi) \qquad$ // on $(-\pi,\pi)$

$b^2 = [\gamma^2 - 2k^2r^2] \qquad c = [\,-(2/\pi^2)\,\gamma \sinh(\pi\gamma)\, K_{i\gamma}(kr_0)]$

In **Sub Problem** I solved this 1D θ Green's Function problem. It is unusual because it has mixed boundary conditions instead of the simpler homogeneous ones at each endpoint. That is, each BC involves both endpoints. I found this solution with b and c as shown above

$a_\gamma(\theta) = (c/2b)\sin(b|\theta|) + (c/2b)\cot(b\pi)\cos(b\theta) = (c/[2b\sin(b\pi)])\cos[b(\pi-|\theta|)]$

In **Resume Method 1** I just install this $a_\gamma(\theta)$ into the expansion above to get a final solution (b as above)

$$g(r,\theta|r_0,\theta_0) = -(1/\pi^2)\int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\,[\sin(b|\theta-\theta_0|) + \cot(b\pi)\cos(b(\theta-\theta_0)\,]/b$$

But this is <u>the solution of the wrong problem</u>. It is the solution to the problem stated but with a = 0, so we have a tiny sphere at the origin with g=0 on it. This is NOT the free-space problem therefore.

In **Whopping Huge Mistake** I realize my error.

In **Start over and Try** I start again with the new eigenvalue λ (Helmholtz λ is now $-k^2$)

$-(rg')' + k^2 rg - \lambda\, g/r = \delta(r-r_0) \qquad Lg = -(rg')' + k^2 rg \qquad \lambda = \nu = -\mu^2$

$[L-(1/r)\lambda]g = \delta(r-r_0) \qquad$ so $s(r) = 1/r \qquad \mu = -i\sqrt{\lambda}$

I look up the Green's function solution in my Bessel document and find (write it either way)

$$g = [\,I_\mu(kr_<)K_\mu(ka) - K_\mu(kr_<)\,I_\mu(ka)]\,[\,K_\mu(kr_>)/\,K_\mu(ka)\,]$$

$$g(r|r_0;\lambda) = [I_{-i\sqrt{\lambda}}(kr_<)\,K_{-i\sqrt{\lambda}}(ka) - K_{-i\sqrt{\lambda}}(kr_<)\,I_{-i\sqrt{\lambda}}(ka)]\,[K_{-i\sqrt{\lambda}}(kr_>)/\,K_{-i\sqrt{\lambda}}(ka)\,]$$

I then roll out the General Theory

Completeness: $\quad -(1/2\pi i)\oint d\lambda\, g(r|r_0;\lambda) = \delta(r-r_0)/s(r) = "\Sigma_n\, \varphi_n(r)\,\bar{\varphi}_n(r_0)"$

Orthogonality: $\quad \langle\sqrt{s}\,\varphi_n, \sqrt{s}\,\varphi_m\rangle = K_n\,\delta_{n,m}$

and I am much surprised when I find that $g(r|r_0;\lambda)$ has no discontinuity across the positive real axis, despite the fact that $\mu(\lambda) = -i\sqrt{\lambda}$ has a very clear cut there. So this is a big difference between the current scheme and the regular KL where there *is* a discontinuity which then accounts for the continuous

spectrum of eigenvalues. After battling with this disc=0 situation several times and checking again on the above g to see if I omitted some λ-containing factor, it finally dawns on me that $K_\mu(ka)$ has zeros which cause poles in g, and these are going to make the discrete spectrum!

In **Does K_μ Have Zeros** in u, I take a look at Watson who quotes early McDonald work on this question, then I actually find from Wolfram plotting that $K_\mu(3)$ has a zero around μ = 8i, and I get the idea that there are poles on the imaginary axis. I then get confused by the issue that something as innocent as $K_{-i\sqrt{\lambda}}(ka)$ in the numerator of g might have cuts at zeros of K in λ and they might be a big mess. I won't resume this thread till a later section below.

In **Start Over Again** I digress and take a look at the Hankel transform, shifted for (a,∞), but then I realize that the ν on J_ν is just a bystander and not the true λ, not the transform conjugate to r, and also of course we have a continuous spectrum but Stak says it should be discrete. So I drop this path.

In **Return to KL** Idea I ask again about those numerator zero branch points, then stop for the day.

In **Resuming on Sunday** I have already cleared up this cut mystery by adding a section to my KL document. The easy way to see things is to change the great circle in λ to a right half plane half circle in μ, and this also makes clear that the spectrum is going to come from the poles of $1/K_\mu(ka)$ located in the right half plane! The residual vertical integral vanishes due to symmetry, just the way the disc[] vanished in the λ plane. I then just assume some poles at μ_i and I then convert the dλ g integral into a sum over these poles, and I finally end up with my normalized eigenfunctions on (a,∞),

$$\varphi_i(r) = \sqrt{(2\mu_i)\, I_{\mu i}(ka)\, \text{Res}[1/K_\mu(ka)]|^{\mu=\mu i}}\; K_{\mu i}(kr) \qquad \text{// μi means } \mu_i \text{ here}$$

I then offer seven comments on this result. One is that the general solution $\varphi_i(r) \sim K_{\mu i}(kr)$ is in retrospect completely obvious since it is the only thing it could be! Could have started with this and done a Smythian form analysis, similar to the Smythe cone problem. But I don't know the $\{\mu_i\}$, cannot find them listed anywhere for $K_\mu(z)$ for z > 0. Each zero is of course a function of z, so could say $\mu_i(z)$. So each zero is really a "trajectory", and I actually something on Regge poles while doing all this.

In **Summary to this Point** I review everything about this problem. Some signs may be wrong.

In **Finish Up** I install the above $\varphi_i(r)$ into $u(r,\theta) = \Sigma_i\, u_i(\theta)\, \varphi_i(r)$ and from the PDE obtain the 1D Green's for $u_i(\theta)$, which I already solved back in Sub Problem. I then obtain my full result which I then compare to the result obtained in problem 7.34 by expanding instead on the $e^{in\theta}$. The first is 7.35, second is 7.34:

$$u(r,\theta) = -\Sigma_i \quad K_{\mu i}(kr)\; \{K_{\mu i}(kr_0)\, I_{\mu i}(ka)\, \text{Res}[1/K_\mu(ka)]|^{\mu=\mu i}\}\; [\; \cot(\mu_i\pi)\cos(\mu_i\theta) + \sin(\mu_i|\theta|)\;]$$

$$u(r,\theta) = \Sigma_{n=0}^{\infty} K_n(kr) \qquad [\; A_n \cos(n\theta) + B_n \sin(n|\theta|)\;] \qquad r>r_0$$

$$A_n = -(\pi/8)\, \varepsilon_n\, [\; K_n(ka) I_n(kr_0) - I_n(ka)\, K_n^{(1)}(kr_0)]\; [\; 1/K_n(ka)] \qquad B_n = 0$$

In **Another Look** I ponder this zeros question more, and I find there is current activity on this question, and that the answer is not clearly known. Only for large z do I see estimates regarding those zeros in μ. I

have lots of references if I want to pursue this further. I found a nice German Digital PDF website that seems to be able to break the Elsevier strangle hold on information.

In **My Own Comments** I derive some obvious theorems about the location of the zeros μ_i. They come either in pairs on the imaginary axis, or in complex quads, and there are none on the real axis (μ plane).

Exercise 7.35: Repeat Problem 7.34 using Radial Eigenfunctions

Setting Up

In the previous problem in both parts we used $e^{in\theta}$ expansions to get our result. Now our task is to expand on the other variable r. I suspect this will involved either KL or Hankel basis functions. Our first order of business is to state our PDE and stare at it:

$(-\nabla^2 - \lambda)g(r,\theta) = \delta^{(2)}(\mathbf{r}-\mathbf{r}_0) = \delta(r-r_0)/r * \delta(\theta-\theta_0)$ // Green's point charge at r_0,θ_0
$g(a,\theta) = 0 \qquad g(\infty,\theta) = 0$ // boundary conditions

$-(rg')' - r^{-1}\partial_\theta^2 g - r\lambda\, g = \delta(r-r_0)\,\delta(\theta)$ // where prime means ∂_r

Next, we go off to our Hankel/KL doc and see what was learned there. Also, page 273 of the text applies. Stak was not fully clear on this subject, I will try to be clearer.

If we separate the above PDE, we get 7.160 and 7.161 where ν is the separation constant:

$-\Phi'' = -\nu\Phi \qquad \Phi''/\Phi = \nu$ // interval $(-\pi,\pi)$

$-(rR')' - \lambda rR = (\nu/r)R$
$-r(rR')' - \lambda r^2 R = \nu R \qquad [r(-rR')' - \lambda r^2 R]/R = \nu$ // interval (a,∞)
$-r(rR')' - \lambda r^2 R - \nu R = 0$
$-(rR')' - \lambda rR - \nu R/r = 0$
$-rR'' - R' - \lambda rR - \nu R/r = 0$ // all 6 equations here are the same

In our previous approaches, we started with $\Phi''/\Phi = \nu$ and got quantization of ν to integers due to the full azimuthal range of our problems (the main being all of free space in Stak's earlier sections where we developed various expansions for the free-space E_2 propagator).

Review of KL.

The task now is to start with the second equation above where again ν is regarded as the eigenvalue and λ, despite its symbol, is just a constant. To this end, let's rename things to reduce confusion. Write the Helmholtz parameter λ as $-k^2$, and write ν as λ (the true EV). Then our last equation above reads

$-(rR')' + k^2 rR - \lambda R/r = 0$ eigenfunctions: $K_{\pm i\sqrt{\lambda}}(kr)$

As my "comparison doc" shows, this ODE is in the KL form, not the Hankel form. Thus, our non-normalized eigenfunctions are those of the KL transform which are $K_{-i\sqrt{\lambda}}(kr)$. At this point, if we define $\lambda = \gamma^2$ (where λ is our true eigenvalue λ, not the Helmholtz λ) then the eigenfunctions are $K_{-i\gamma}(kr)$. But K has order symmetry, so $K_{-i\gamma}(kr) = K_{i\gamma}(kr)$, so we can also regard $K_{i\gamma}(kr)$ as the complete set of eigenfunctions.

So let's show these eigenfunctions in various modes:

$-(rR')' + k^2rR - \lambda R/r = 0$	eigenfunctions:	$K_{\pm i\sqrt{\lambda}}(kr)$
$-(rR')' + k^2rR - \gamma^2 R/r = 0$	eigenfunctions:	$K_{\pm i\gamma}(kr)$
$-(rR')' - \lambda rR - \gamma^2 R/r = 0$	eigenfunctions:	$K_{\pm i\gamma}(\sqrt{-\lambda}\, r)$
$-(rR')' - \lambda rR - \nu R/r = 0$	eigenfunctions:	$K_{\pm i\sqrt{\nu}}(\sqrt{-\lambda}\, r)$

The last line shows our "starting position" and you see that we have two ugly square roots in the eigenfunction form. A lot messier than, say, $e^{in\theta}$. For this reason, we ALWAYS change the parameter names so things are more tractable.

So starting with our radial equation

$-(rR')' - \lambda rR - \nu R/r = 0$ $\qquad$ EF = $K_{\pm i\sqrt{\nu}}(\sqrt{-\lambda}\, r)$

we replace $\lambda = -k^2$ (this is the H λ) and $\nu = \gamma^2$ to get

$-(rR')' + k^2rR - \gamma^2 R/r = 0$ $\qquad$ EF = $K_{\pm i\gamma}(kr)$

While here, we should take note of other properties:

$$r\,\delta(r-r') = (2/\pi^2) \int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma) K_{i\gamma}(kr)\, K_{i\gamma}(kr')$$

Now, in our former case with $e^{in\theta}$, we knew this was a complete set on θ, so we knew we could use these basis functions to expand our solution $u(r,\theta)$. Just so, as Homer would say, here we know $K_{i\gamma}(kr)$ are a complete set of eigenfunctions in r, so we can do this expansion

$$g(r,\theta) = \int_0^\infty d\gamma\, a_\gamma(\theta)\, K_{i\gamma}(kr)$$

If the K were not a complete set, this expansion would not be justified because you must have a complete set for your problem of interest. Notice that the symmetry removes $\gamma < 0$. I am usong γ only because Stak used γ.

Method 1: Insert the KL expansion into the PDE and see what we get.

A priori, we don't really know what ODE in θ we will arrive at if we do this expansion, so let's just try it.

$$- (rg')' \quad -r^{-1}\partial_\theta^2 g \; - r\lambda\, g = \delta(r-r_0)\,\delta(\theta)$$

$$- (rg')' \quad -r^{-1}\partial_\theta^2 g + rk^2\, g = \delta(r-r_0)\,\delta(\theta) \qquad \text{// get rid of } \lambda \text{ in favor of } k$$

Now insert the expansion, so we will need to compute

$$g' = k \int_0^\infty d\gamma\, a_\gamma(\theta)\, K_{i\gamma}'(kr)$$

$$rg' = kr \int_0^\infty d\gamma\, a_\gamma(\theta)\, K_{i\gamma}'(kr)$$

$$(rg')' = k \int_0^\infty d\gamma\, a_\gamma(\theta)\, K_{i\gamma}'(kr) + kr \int_0^\infty d\gamma\, a_\gamma(\theta)\, kK_{i\gamma}''(kr)$$

$$= \int_0^\infty d\gamma\, a_\gamma(\theta)\, [\, k\, K_{i\gamma}'(kr) + k^2\, r\, K_{i\gamma}''(kr)]$$

$$- (rg')' = \int_0^\infty d\gamma\, a_\gamma(\theta)\, [\, - k\, K_{i\gamma}'(kr) - k^2\, r\, K_{i\gamma}''(kr)]$$

Now we move onto the next two terms

$$-r^{-1}\partial_\theta^2 g = -r^{-1}\partial_\theta^2 \int_0^\infty d\gamma\, a_\gamma(\theta)\, K_{i\gamma}(kr) = -r^{-1} \int_0^\infty d\gamma\, [\partial_\theta^2 a_\gamma(\theta)]\, K_{i\gamma}(kr)$$

$$rk^2\, g = rk^2 \int_0^\infty d\gamma\, a_\gamma(\theta)\, K_{i\gamma}(kr)$$

$$\delta(r-r_0)\,\delta(\theta) = \delta(\theta)\,(2/\pi^2)(1/r) \int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma) K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)$$

If we insert all these results into our PDE, we get

$$- (rg')' \quad -r^{-1}\partial_\theta^2 g - r\lambda\, g = \delta(r-r_0)\,\delta(\theta)$$

$$\int_0^\infty d\gamma\, a_\gamma(\theta)\, [\, k\, K_{i\gamma}'(kr) + k^2\, r\, K_{i\gamma}''(kr)] \; -r^{-1} \int_0^\infty d\gamma\, [\partial_\theta^2 a_\gamma(\theta)]\, K_{i\gamma}(kr)$$

$$+ rk^2 \int_0^\infty d\gamma\, a_\gamma(\theta)\, K_{i\gamma}(kr) = \delta(\theta)\,(2/\pi^2)(1/r) \int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma) K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)$$

We need to get each term in the form $\int_0^\infty d\gamma\, K_{i\gamma}(kr)$ * something, and then we can use completeness to equate the somethings. The obvious problem child is the first term. But we do know that the K's solve our ODE, that is to say

$$-(rR')' + k^2rR - \gamma^2R/r = 0$$

$$-rR'' - R' + k^2rR - \gamma^2R/r = 0$$

and if we set $R(r) = K_{i\gamma}(kr)$ and don't forget picked up k factors, we have

$$-rk^2 K_{i\gamma}''(kr) - k K_{i\gamma}'(kr) + k^2r K_{i\gamma}(kr) - \gamma^2 K_{i\gamma}(kr)/r = 0$$

which we can write as

$$[\, k K_{i\gamma}'(kr) + k^2 r K_{i\gamma}''(kr)] = \quad k^2r K_{i\gamma}(kr) - \gamma^2 K_{i\gamma}(kr)/r$$

So this certainly gives an improvement in our problem child term, and our equation is now

$$\int_0^\infty d\gamma\, a_\gamma(\theta) [k^2r K_{i\gamma}(kr) - \gamma^2 K_{i\gamma}(kr)/r] \; -r^{-1} \int_0^\infty d\gamma\, [\partial_\theta^2 a_\gamma(\theta)] K_{i\gamma}(kr)$$

$$+ rk^2 \int_0^\infty d\gamma\, a_\gamma(\theta) K_{i\gamma}(kr) = \delta(\theta) (2/\pi^2)(1/r) \int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma) K_{i\gamma}(kr) K_{i\gamma}(kr_0)$$

Let's now rewrite each term in the form suggested above,

$$\int_0^\infty d\gamma\, K_{i\gamma}(kr) a_\gamma(\theta) [k^2r - \gamma^2/r] \; -r^{-1} \int_0^\infty d\gamma\, K_{i\gamma}(kr) [\partial_\theta^2 a_\gamma(\theta)]$$

$$+ rk^2 \int_0^\infty d\gamma\, K_{i\gamma}(kr) a_\gamma(\theta) = \delta(\theta) (2/\pi^2)(1/r) \int_0^\infty d\gamma\, K_{i\gamma}(kr)\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kr_0)$$

Now using the completeness of the $K_{i\gamma}(kr)$ functions, we write the above as

$$a_\gamma(\theta) [k^2r - \gamma^2/r] \; -r^{-1} [\partial_\theta^2 a_\gamma(\theta)]$$

$$+ rk^2 a_\gamma(\theta) = \delta(\theta) (2/\pi^2)(1/r)\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kr_0)$$

or

$$[2k^2r - \gamma^2/r] a_\gamma(\theta) -r^{-1} [\partial_\theta^2 a_\gamma(\theta)] = \delta(\theta) (2/\pi^2)(1/r)\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kr_0)$$

or

$$-r^{-1}\,[a_\gamma''(\theta)] + [2k^2 r - \gamma^2/r]\,a_\gamma(\theta) = [\,(2/\pi^2)(1/r)\,\gamma\sinh(\pi\gamma)\,K_{i\gamma}(kr_0)]\;\delta(\theta)$$

or

$$-\,[a_\gamma''(\theta)] + [2k^2 r^2 - \gamma^2]\,a_\gamma(\theta) = [\,(2/\pi^2)\,\gamma\sinh(\pi\gamma)\,K_{i\gamma}(kr_0)]\;\delta(\theta)$$

or

$$a_\gamma''(\theta) + [\gamma^2 - 2k^2 r^2]\,a_\gamma(\theta) = [\,-(2/\pi^2)\,\gamma\sinh(\pi\gamma)\,K_{i\gamma}(kr_0)]\;\delta(\theta)$$

and this then is our ODE in variable θ. From the θ point of view, this equation says

$$a_\gamma''(\theta) + b^2\,a_\gamma(\theta) = c\,\delta(\theta) \qquad \text{// on } (-\pi,\pi)$$

$$b^2 = [\gamma^2 - 2k^2 r^2] \qquad c = [\,-(2/\pi^2)\,\gamma\sinh(\pi\gamma)\,K_{i\gamma}(kr_0)]$$

The boundary conditions on this equation are

$$a_\gamma(\pi) = a_\gamma(-\pi)$$
$$a_\gamma'(\pi) = a_\gamma'(-\pi)$$

just as in p 273 B. What is the solution to this 1D Green's function equation? Homos are form sin(bθ). Notice that we don't have "homo BC's" here because each BC involves both endpoints.

Sub Problem: Solve our 1D θ Green's equation which has inhomo BC's.

Here is our problem

$$a_\gamma''(\theta) + b^2\,a_\gamma(\theta) = c\,\delta(\theta) \qquad \text{// on } (-\pi,\pi)$$
$$b^2 = [\gamma^2 - 2k^2 r^2] \qquad c = [\,-(2/\pi^2)\,\gamma\sinh(\pi\gamma)\,K_{i\gamma}(kr_0)]$$

The boundary conditions on this equation are

$$a_\gamma(\pi) = a_\gamma(-\pi)$$
$$a_\gamma'(\pi) = a_\gamma'(-\pi)$$

The first BC says that $a_\gamma(\theta)$ must be periodic with period 2π, and that means b must be quantized in our *homo* solutions, and we must then have

$$b = n = \text{any integer.}$$

But it is not the homo solutions we seek here. Rather, we need the 1D Green's functions solutions. So let's try to find a Green's Function by brute force

$$a_\gamma(\theta) = A\sin(b\theta) + B\cos(b\theta) \qquad (0,\pi)$$
$$a_\gamma(\theta) = A'\sin(b\theta) + B'\cos(b\theta) \qquad (-\pi,0)$$

Continuity at θ = 0 tells us that B' = B since the sines vanish.

Continuity at ±π tells us that A = -A' since we have

$a_\gamma(\pi) = A\sin(b\pi) + B\cos(b\pi)$ $(0,\pi)$
$a_\gamma(-\pi) = -A'\sin(b\pi) + B\cos(b\pi)$ $(-\pi,0)$

So we now have this "Smythian form" so far

$a_\gamma(\theta) = A\sin(b\theta) + B\cos(b\theta)$ $(0,\pi)$
$a_\gamma(\theta) = -A\sin(b\theta) + B\cos(b\theta)$ $(-\pi,0)$

Next comes continuity of the derivative. So

$a_\gamma'(\theta) = Ab\cos(b\theta) - Bb\sin(b\theta)$ $(0,\pi)$
$a_\gamma'(\theta) = -Ab\cos(b\theta) - Bb\sin(b\theta)$ $(-\pi,0)$

$a_\gamma'(\pi) = Ab\cos(b\pi) - Bb\sin(b\pi)$ $(0,\pi)$
$a_\gamma'(-\pi) = -Ab\cos(b\pi) + Bb\sin(b\pi)$ $(-\pi,0)$

Setting these equal says

$Ab\cos(b\pi) - Bb\sin(b\pi) = -Ab\cos(b\pi) + Bb\sin(b\pi)$
$A\cos(b\pi) - B\sin(b\pi) = -A\cos(b\pi) + B\sin(b\pi)$
$2A\cos(b\pi) = 2B\sin(b\pi)$
$B = A\cot(b\pi)$

Now we come to the jump condition :

$a_\gamma''(\theta) + b^2 a_\gamma(\theta) = c\,\delta(\theta)$ // on $(-\pi,\pi)$
$b^2 = [\gamma^2 - 2k^2r^2]$ $c = [\,-(2/\pi^2)\,\gamma\sinh(\pi\gamma)\,K_{i\gamma}(kr_0)]$

jump = c // jump = c * 1/"a_0" = c*1/1 = c

We evaluate

$a_\gamma'(\theta) = Ab\cos(b\theta) - Bb\sin(b\theta)$ $(0,\pi)$
$a_\gamma'(\theta) = -Ab\cos(b\theta) - Bb\sin(b\theta)$ $(-\pi,0)$

$a_\gamma'(0+) = Ab$ $(0,\pi)$
$a_\gamma'(0-) = -Ab$ $(-\pi,0)$

jump = 2Ab = c

so

A = (c/2b)

we then of course have $B = A\cot(b\pi) = (c/2b)\cot(b\pi)$, and here then is our solution

$a_\gamma(\theta) = (c/2b)\sin(b\theta) + (c/2b)\cot(b\pi)\cos(b\theta)$ $\quad (0,\pi)$
$a_\gamma(\theta) = -(c/2b)\sin(b\theta) + (c/2b)\cot(b\pi)\cos(b\theta)$ $\quad (-\pi,0)$

$b^2 = [\gamma^2 - 2k^2r^2]$
$c = [\, -(2/\pi^2)\,\gamma \sinh(\pi\gamma)\, K_{i\gamma}(kr_0)]$

We can combine our two solution parts into one as follows

$a_\gamma(\theta) = (c/2b)[\, \sin(b|\theta|) + \cot(b\pi)\cos(b\theta)]$ $\quad (-\pi,\pi)$

And we can check now on our BC's. First

$a_\gamma(\pm\pi) = (c/2b)[\, \sin(b|\pi|) + \cot(b\pi)\cos(b\pi)]$ // they are the same

and then

$a_\gamma'(\theta) = (c/2)[\, \cos(b|\theta|)\,\mathrm{sign}(\theta) - \cot(b\pi)\sin(b\theta)]$

$a_\gamma'(\pi) = (c/2)[\, \cos(b\pi) - \cot(b\pi)\sin(b\pi)] = (c/2)[\, \cos(b\pi) - \cos(b\pi)] = 0$
$a_\gamma'(-\pi) = (c/2)[\, -\cos(b\pi) - \cot(b\pi)\sin(-b\pi)] = (c/2)[\, -\cos(b\pi) + \cos(b\pi)] = 0$

So we meet the BC's Why are the derivatives 0? The reason is that $a_\gamma(\theta)$ is necessarily even in θ from the problem symmetry. Notice also that $a_\gamma(\theta=0) = (c/2b)\cot(b\pi)$, just some finite number which is the same as you approach from either side, showing that our solution is indeed continuous at $\theta = 0$. Finally, we can check the jump condition

$a_\gamma'(\theta=0+) = (c/2)[\, 1\,]$
$a_\gamma'(\theta=0-) = (c/2)[\, -1\,]$
jump = c

which is correct for $a_\gamma''(\theta) + b^2\, a_\gamma(\theta) = c\,\delta(\theta)$. So I did all this to boost confidence in our solution:

$a_\gamma''(\theta) + b^2\, a_\gamma(\theta) = c\,\delta(\theta)$ // on $(-\pi,\pi)$
$a_\gamma(\pi) = a_\gamma(-\pi)$, $a_\gamma'(\pi) = a_\gamma'(-\pi)$
$a_\gamma(\theta) = (c/2b)[\, \sin(b|\theta|) + \cot(b\pi)\cos(b\theta)]$

But now I belatedly see a better way to write this answer:

$\sin(b|\theta|) + \cot(b\pi)\cos(b\theta) = \sin(b|\theta|) + \cos(b\pi)\cos(b\theta)/\sin(b\pi)$

$= [\, \sin(b\pi)\sin(b|\theta|) + \cos(b\pi)\cos(b\theta)] / \sin(b\pi)$

$= \cos[b(\pi - |\theta|)] / \sin(b\pi)$

And our summary then becomes

$a_\gamma''(\theta) + b^2\, a_\gamma(\theta) = c\,\delta(\theta)$ // on $(-\pi,\pi)$
$a_\gamma(\pi) = a_\gamma(-\pi)$, $a_\gamma'(\pi) = a_\gamma'(-\pi)$
$a_\gamma(\theta) = (c/[2b\sin(b\pi)])\cos[b(\pi-|\theta|)]$

$a_\gamma'(\theta) = (c/[2b\sin(b\pi)])\sin[b(\pi-|\theta|)]\,(-b)\,\mathrm{sign}(\theta)$

It is now obvious by inspection that the two BC's are met and the second is 0 and solution is even in θ.

Resume Method 1 solution

We then have

$$g(r,\theta) = \int_0^\infty d\gamma\, a_\gamma(\theta)\, K_{i\gamma}(kr)$$

$$= \int_0^\infty d\gamma\, K_{i\gamma}(kr)\,(c/2b)\,[\sin(b|\theta|)+ \cot(b\pi)\cos(b\theta)\,]$$

$$b = \sqrt{\gamma^2 - 2k^2r^2} \qquad c = [\,-(2/\pi^2)\,\gamma\,\sinh(\pi\gamma)\,K_{i\gamma}(kr_0)]$$

If we insert b explicitly, we get

$$g(r,\theta) = (1/2)\int_0^\infty d\gamma\, K_{i\gamma}(kr)\, c\,[\sin(b|\theta|)+ \cot(b\pi)\cos(b\theta)\,]/b$$

$$= -(2/\pi^2)\,(1/2)\int_0^\infty d\gamma\, K_{i\gamma}(kr)\,\gamma\,\sinh(\pi\gamma)\,K_{i\gamma}(kr_0)\,[\sin(b|\theta|)+ \cot(b\pi)\cos(b\theta)\,]/b$$

$$= -(1/\pi^2)\int_0^\infty d\gamma\,\gamma\,\sinh(\pi\gamma)\,K_{i\gamma}(kr)\,K_{i\gamma}(kr_0)\,[\sin(b|\theta|)+ \cot(b\pi)\cos(b\theta)\,]/b$$

where $b = \sqrt{\gamma^2 - 2k^2r^2}$

This then is the "Method 1" solution to our original problem. Notice symmetry $r \leftrightarrow r_0$, and notice even in θ. Our "original problem" was to find the 2D external Helmholtz Green's Function where we put the point source at $\theta_0 = 0$ and radius r_0. Thus, we expect the solution to be even in θ, which it is. We could also write

$$g(r,\theta|r_0,\theta_0) = -(1/\pi^2)\int_0^\infty d\gamma\,\gamma\,\sinh(\pi\gamma)\,K_{i\gamma}(kr)\,K_{i\gamma}(kr_0)\,[\sin(b|\theta-\theta_0|)+ \cot(b\pi)\cos(b(\theta-\theta_0)\,]/b$$

$$b = \sqrt{\gamma^2 - 2k^2r^2}$$

Now if we swap $r,\theta \leftrightarrow r_0,\theta_0$, the result is exactly the same.

Whopping Huge Mistake Discovered

I noticed that a was "missing" from my solution. Then I realized that the KL interval is $(0,\infty)$, not (a,∞). So here is the problem I solved

$(-\nabla^2 -\lambda)g(r,\theta) = \delta^{(2)}(\mathbf{r}-\mathbf{r_0}) = \delta(r-r_0)/r * \delta(\theta-\theta_0)$ // Green's point charge at r_0,θ_0

$g(0,\theta) = 0$ $g(\infty,\theta) = 0$ // boundary conditions

So the problem I have solved is the Green's Function for a point boundary at the origin held at 0 field. Note that this is NOT the free-space solution since it would not vanish at that point. There is some induced source on that point, though this is hard to picture. Think of it as a tiny sphere I guess.

Start Over and try to do the KL on interval (a,∞)

So do I know a SL problem on the interval (a,∞)? Our radial equation has not changed, it is still

$-(rR')' - \lambda rR - \nu R/r = 0$ (a,∞)

So this is going to be an exercise in "how do you DO a SL problem? " Here is one way I know to do it. We start by finding the Green's function for this equation

$-(rR')' - \lambda rR - \nu R/r = \delta(r-r_0)$ ν = sep constant λ= Helm param

Since ν is the separation constant here (whose spectrum I want to know), let's call $\nu = \lambda_{\mathbf{true}}$ so we have

$-(rR')' - \lambda rR - \lambda_{\mathbf{true}} R/r = \delta(r-r_0)$

which we interpret this way

$[L-\lambda_{\mathbf{true}}(1/r))]g = \delta(r-r_0)$ so $s(r) = 1/r$

and we are clearly on the KL side of my KL comparison docket (see doc). Let's set the Helmholtz λ to its usual $-k^2$ value, and then remove the subscript from our λ to get

$-(rR')' +k^2rR - \lambda R/r = \delta(r-r_0)$

$[L-\lambda(1/r))]g = \delta(r-r_0)$ so $s(r) = 1/r$

and this matches our KL forms exactly now. So the plan is to find g, and then use that to get the eigenfunctions for our KL situation on (a,∞). Let's write the Green's 1D again,

$-(rR')' +k^2rR - \lambda R/r = \delta(r-r_0)$ $\lambda = \lambda_{\mathbf{true}} = \nu = -\mu^2$

so $\sqrt{\lambda} = i\mu$ (as required) and thus $\mu = -i\sqrt{\lambda}$. Our 1D Green's is then

$$-(rR')' + k^2 rR + \mu^2 R/r = \delta(r-r_0)$$

Now I know from "prop Bessel" how to solve this 1D Green's function problem. Cribbing from Example 2A, the answer is

$$R = [\, I_\mu(kr_<) K_\mu(ka) - K_\mu(kr_<) I_\mu(ka)] \, [\, K_\mu(kr_>)/ K_\mu(ka) \,]$$

and now I will refer to this as g and insert $\mu = -i\sqrt{\lambda}$ to get

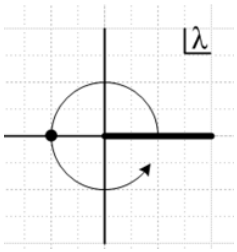
$$g(r|r_0; \lambda) = [I_{-i\sqrt{\lambda}}(kr_<) K_{-i\sqrt{\lambda}}(ka) - K_{-i\sqrt{\lambda}}(kr_<) I_{-i\sqrt{\lambda}}(ka)] \, [K_{-i\sqrt{\lambda}}(kr_>)/ K_{-i\sqrt{\lambda}}(ka) \,]$$

Now we can use our standard method to obtain the eigenfunctions from g (take from KL doc)

Completeness: $-(1/2\pi i) \oint d\lambda \, g(r|r_0; \lambda) = \delta(r-r_0)/s(r) = "\Sigma_n \varphi_n(r) \, \bar{\varphi}_n(r_0)"$

Orthogonality: $\langle \sqrt{s}\, \varphi_n, \sqrt{s}\, \varphi_m \rangle = K_n \, \delta_{n,m}$

At this point, things are very similar to our KL derivation doc. There, since we were on $(0,\infty)$, we had a simpler $g = I_{-i\sqrt{\lambda}}(kx_<) \; K_{-i\sqrt{\lambda}}(kx_>)$ (converted to our present variable names, μ there is k^2 here). We then repeat our argument concerning what we now call $\mu(\lambda) = -i\sqrt{\lambda}$. We take the λ cut here to be on the positive real axis, so on the negative real axis of λ we will have $\lambda = e^{i\pi}|\lambda|$ and $\sqrt{\lambda} = i\sqrt{|\lambda|}$ and then we end up with $\mu = -i(i\sqrt{|\lambda|}) = \sqrt{|\lambda|} > 0$, as we assumed in the first place, ie, $\mu > 0$.



Our great circle $d\lambda$ integral is going to pick up the discontinuity across this cut which we now compute

$$\text{disc}[g] = g(\lambda = \lambda e^{i0}) - g(\lambda = \lambda e^{2i\pi})$$

$$= g(\sqrt{\lambda} = \sqrt{\lambda}) - g(\sqrt{\lambda} = -\sqrt{\lambda})$$

$$= [I_{-i\sqrt{\lambda}}(kr_<) K_{-i\sqrt{\lambda}}(ka) - K_{-i\sqrt{\lambda}}(kr_<) I_{-i\sqrt{\lambda}}(ka)] \, [K_{-i\sqrt{\lambda}}(kr_>)/ K_{-i\sqrt{\lambda}}(ka) \,]$$
$$- [I_{i\sqrt{\lambda}}(kr_<) K_{i\sqrt{\lambda}}(ka) - K_{i\sqrt{\lambda}}(kr_<) I_{i\sqrt{\lambda}}(ka)] \quad [K_{i\sqrt{\lambda}}(kr_>)/ K_{i\sqrt{\lambda}}(ka) \,]$$

Since $K_\nu = K_{-\nu}$ we will use + wherever we can so we then get

$$\text{disc}[g] = [I_{-i\sqrt{\lambda}}(kr_<) K_{i\sqrt{\lambda}}(ka) - K_{i\sqrt{\lambda}}(kr_<) I_{-i\sqrt{\lambda}}(ka)] [K_{i\sqrt{\lambda}}(kr_>)/ K_{i\sqrt{\lambda}}(ka)]$$
$$- [I_{i\sqrt{\lambda}}(kr_<) K_{i\sqrt{\lambda}}(ka) - K_{i\sqrt{\lambda}}(kr_<) I_{i\sqrt{\lambda}}(ka)] [K_{i\sqrt{\lambda}}(kr_>)/ K_{i\sqrt{\lambda}}(ka)]$$

$$= \{ [I_{-i\sqrt{\lambda}}(kr_<) K_{i\sqrt{\lambda}}(ka) - K_{i\sqrt{\lambda}}(kr_<) I_{-i\sqrt{\lambda}}(ka)] - [I_{i\sqrt{\lambda}}(kr_<) K_{i\sqrt{\lambda}}(ka) - K_{i\sqrt{\lambda}}(kr_<) I_{i\sqrt{\lambda}}(ka)] \}$$
$$* [K_{i\sqrt{\lambda}}(kr_>)/ K_{i\sqrt{\lambda}}(ka)]$$

$$= \{ I_{-i\sqrt{\lambda}}(kr_<) K_{i\sqrt{\lambda}}(ka) - K_{i\sqrt{\lambda}}(kr_<) I_{-i\sqrt{\lambda}}(ka) - I_{i\sqrt{\lambda}}(kr_<) K_{i\sqrt{\lambda}}(ka) + K_{i\sqrt{\lambda}}(kr_<) I_{i\sqrt{\lambda}}(ka)\}$$
$$* [K_{i\sqrt{\lambda}}(kr_>)/ K_{i\sqrt{\lambda}}(ka)]$$

Now we try to get rid of the I- factors using

$$I_{-\nu}(z) = I_\nu(z) + (2/\pi)\sin(\pi\nu) K_\nu(z)$$

$$I_{-i\sqrt{\lambda}}(kr_<) = I_{i\sqrt{\lambda}}(kr_<) + (2/\pi)\sin(\pi i\sqrt{\lambda}) K_{i\sqrt{\lambda}}(kr_<)$$
$$I_{-i\sqrt{\lambda}}(ka) = I_{i\sqrt{\lambda}}(ka) + (2/\pi)\sin(\pi i\sqrt{\lambda}) K_{i\sqrt{\lambda}}(ka)$$

The curly bracket shown above them becomes

$$\{ [I_{i\sqrt{\lambda}}(kr_<) + (2/\pi)\sin(\pi i\sqrt{\lambda}) K_{i\sqrt{\lambda}}(kr_<)]K_{i\sqrt{\lambda}}(ka)$$
$$- [I_{i\sqrt{\lambda}}(ka) + (2/\pi)\sin(\pi i\sqrt{\lambda}) K_{i\sqrt{\lambda}}(ka)]K_{i\sqrt{\lambda}}(kr_<)$$
$$- I_{i\sqrt{\lambda}}(kr_<) K_{i\sqrt{\lambda}}(ka) + K_{i\sqrt{\lambda}}(kr_<) I_{i\sqrt{\lambda}}(ka) \}$$

$$= \{ I_{i\sqrt{\lambda}}(kr_<) K_{i\sqrt{\lambda}}(ka) + (2/\pi)\sin(\pi i\sqrt{\lambda}) K_{i\sqrt{\lambda}}(kr_<) K_{i\sqrt{\lambda}}(ka)$$
$$- I_{i\sqrt{\lambda}}(ka) K_{i\sqrt{\lambda}}(kr_<) - (2/\pi)\sin(\pi i\sqrt{\lambda}) K_{i\sqrt{\lambda}}(ka) K_{i\sqrt{\lambda}}(kr_<)$$
$$- I_{i\sqrt{\lambda}}(kr_<) K_{i\sqrt{\lambda}}(ka) + K_{i\sqrt{\lambda}}(kr_<) I_{i\sqrt{\lambda}}(ka) \}$$

$$= \{ I_{i\sqrt{\lambda}}(kr_<) K_{i\sqrt{\lambda}}(ka)$$
$$- I_{i\sqrt{\lambda}}(ka) K_{i\sqrt{\lambda}}(kr_<)$$
$$- I_{i\sqrt{\lambda}}(kr_<) K_{i\sqrt{\lambda}}(ka) + K_{i\sqrt{\lambda}}(kr_<) I_{i\sqrt{\lambda}}(ka) \}$$

$$= 0$$

so I end up with zero discontinuity (second time this has happened). What is going on here?

Start again on the discontinuity.

The constant on g is one because I computed it, but I might check it soon. [see next section]. So:

$$-(rg')' + k^2 rg - \lambda g/r = \delta(r-r_0)$$

$$g(r|r_0;\lambda) = [I_{-i\sqrt{\lambda}}(kr_<) K_{-i\sqrt{\lambda}}(ka) - K_{-i\sqrt{\lambda}}(kr_<) I_{-i\sqrt{\lambda}}(ka)] [K_{-i\sqrt{\lambda}}(kr_>)/ K_{-i\sqrt{\lambda}}(ka)]$$

Rewrite

$$g(r|r_0;\lambda) = [I_{-i\sqrt{\lambda}}(kr_<) K_{i\sqrt{\lambda}}(ka) - K_{i\sqrt{\lambda}}(kr_<) I_{-i\sqrt{\lambda}}(ka)] [K_{i\sqrt{\lambda}}(kr_>)/ K_{i\sqrt{\lambda}}(ka)]$$

The function $K_{i\sqrt{\lambda}}(z)$ of λ has no discontinuity, even though $\sqrt{\lambda}$ does. So $K_{i\sqrt{\lambda}}(z)$ is analytic in λ and acts here just as a constant. On the other hand, I does have a discontinuity,

$$I_{-i\sqrt{\lambda}}(kr_<) = I_{i\sqrt{\lambda}}(kr_<) + (2/\pi)\sin(\pi i\sqrt{\lambda}) K_{i\sqrt{\lambda}}(kr_<)$$

$$\text{disc}[I_{i\sqrt{\lambda}}(kr_<)] = I_{i\sqrt{\lambda}}(kr_<) - I_{-i\sqrt{\lambda}}(kr_<) = -(2/\pi)\sin(\pi i\sqrt{\lambda}) K_{i\sqrt{\lambda}}(kr_<)$$
$$\text{disc}[I_{-i\sqrt{\lambda}}(kr_<)] = I_{-i\sqrt{\lambda}}(kr_<) - I_{i\sqrt{\lambda}}(kr_<) = +(2/\pi)\sin(\pi i\sqrt{\lambda}) K_{i\sqrt{\lambda}}(kr_<)$$

Let's forget about the analytic K/K ratio for now and just look at the core stuff:

$$g = [I_{-i\sqrt{\lambda}}(kr_<) K_{i\sqrt{\lambda}}(ka) - K_{i\sqrt{\lambda}}(kr_<) I_{-i\sqrt{\lambda}}(ka)]$$

$$\text{disc}(g) = K_{i\sqrt{\lambda}}(ka)\, \text{disc}[I_{-i\sqrt{\lambda}}(kr_<)] - K_{i\sqrt{\lambda}}(kr_<)\, \text{disc}[I_{-i\sqrt{\lambda}}(ka)]$$

$$= K_{i\sqrt{\lambda}}(ka) \{ + (2/\pi)\sin(\pi i\sqrt{\lambda}) K_{i\sqrt{\lambda}}(kr_<)\} - K_{i\sqrt{\lambda}}(kr_<) \{ + (2/\pi)\sin(\pi i\sqrt{\lambda}) K_{i\sqrt{\lambda}}(ka))\}$$

$$= (2/\pi)\sin(\pi i\sqrt{\lambda}) [K_{i\sqrt{\lambda}}(ka) K_{i\sqrt{\lambda}}(kr_<) - K_{i\sqrt{\lambda}}(ka) K_{i\sqrt{\lambda}}(kr_<)] = 0$$

So our whole g theory of SL problems is breaking down here, hard to believe it. Time to go check that constant in front of g again.

Check the constant on g.

$$g(r|r_0;\lambda) = [I_{-i\sqrt{\lambda}}(kr_<) K_{-i\sqrt{\lambda}}(ka) - K_{-i\sqrt{\lambda}}(kr_<) I_{-i\sqrt{\lambda}}(ka)] [K_{-i\sqrt{\lambda}}(kr_>)/ K_{-i\sqrt{\lambda}}(ka)]$$

$$g'^{+} = [I_{-i\sqrt{\lambda}}(kr_0) K_{-i\sqrt{\lambda}}(ka) - K_{-i\sqrt{\lambda}}(kr_0) I_{-i\sqrt{\lambda}}(ka)] \{ [K_{-i\sqrt{\lambda}}(kr)]' / K_{-i\sqrt{\lambda}}(ka)] \}$$

$$= [I_{-i\sqrt{\lambda}}(kr_0) K_{-i\sqrt{\lambda}}(ka) - K_{-i\sqrt{\lambda}}(kr_0) I_{-i\sqrt{\lambda}}(ka)] [k K_{-i\sqrt{\lambda}}'(kr)/ K_{-i\sqrt{\lambda}}(ka)]$$

$$g'^{-} = [k I_{-i\sqrt{\lambda}}'(kr) K_{-i\sqrt{\lambda}}(ka) - k K_{-i\sqrt{\lambda}}'(kr_<) I_{-i\sqrt{\lambda}}(ka)] [K_{-i\sqrt{\lambda}}(kr_0)/ K_{-i\sqrt{\lambda}}(ka)]$$

Restate these:

$$g'^{+} = k\,[I_{-i\sqrt{\lambda}}(kr_0)\,K_{-i\sqrt{\lambda}}(ka) - K_{-i\sqrt{\lambda}}(kr_0)\,I_{-i\sqrt{\lambda}}(ka)]\;[K_{-i\sqrt{\lambda}}'(kr)/\,K_{-i\sqrt{\lambda}}(ka)\,]$$
$$g'^{-} = k\,[\,I_{-i\sqrt{\lambda}}'(kr)\,K_{-i\sqrt{\lambda}}(ka) - K_{-i\sqrt{\lambda}}'(kr)\,I_{-i\sqrt{\lambda}}(ka)]\,[K_{-i\sqrt{\lambda}}(kr_0)/\,K_{-i\sqrt{\lambda}}(ka)\,]$$

There are only four terms to worry about. We can set $r = r_0$ now to get

$$g'^{+} = [I_{-i\sqrt{\lambda}}(kr_0)\,K_{-i\sqrt{\lambda}}(ka) - K_{-i\sqrt{\lambda}}(kr_0)\,I_{-i\sqrt{\lambda}}(ka)]\;[k\,K_{-i\sqrt{\lambda}}'(kr_0)/\,K_{-i\sqrt{\lambda}}(ka)\,]$$
$$g'^{-} = [\,I_{-i\sqrt{\lambda}}'(kr_0)\,K_{-i\sqrt{\lambda}}(ka) - K_{-i\sqrt{\lambda}}'(kr_0)\,I_{-i\sqrt{\lambda}}(ka)]\,[k\,K_{-i\sqrt{\lambda}}(kr_0)/\,K_{-i\sqrt{\lambda}}(ka)\,]$$

For now, just ignore the common k 1/K factor, and we can put it back in later. So then

$$g'^{+} = [I_{-i\sqrt{\lambda}}(kr_0)\,K_{-i\sqrt{\lambda}}(ka) - K_{-i\sqrt{\lambda}}(kr_0)\,I_{-i\sqrt{\lambda}}(ka)]\,K_{-i\sqrt{\lambda}}'(kr_0)$$
$$g'^{-} = [\,I_{-i\sqrt{\lambda}}'(kr_0)\,K_{-i\sqrt{\lambda}}(ka) - K_{-i\sqrt{\lambda}}'(kr_0)\,I_{-i\sqrt{\lambda}}(ka)]\,K_{-i\sqrt{\lambda}}(kr_0)$$

Take the difference

$$\text{jump} = I_{-i\sqrt{\lambda}}(kr_0)\,K_{-i\sqrt{\lambda}}(ka)\,K_{-i\sqrt{\lambda}}'(kr_0) - K_{-i\sqrt{\lambda}}(kr_0)\,I_{-i\sqrt{\lambda}}(ka)\,K_{-i\sqrt{\lambda}}'(kr_0)$$
$$- I_{-i\sqrt{\lambda}}'(kr_0)\,K_{-i\sqrt{\lambda}}(ka)\,K_{-i\sqrt{\lambda}}(kr_0) + K_{-i\sqrt{\lambda}}'(kr_0)\,I_{-i\sqrt{\lambda}}(ka)\,K_{-i\sqrt{\lambda}}(kr_0)$$

$$= K_{-i\sqrt{\lambda}}(ka)\,\{\;I_{-i\sqrt{\lambda}}(kr_0)\,K_{-i\sqrt{\lambda}}'(kr_0) - I_{-i\sqrt{\lambda}}'(kr_0)\,K_{-i\sqrt{\lambda}}(kr_0)\,\}$$

$$+ I_{-i\sqrt{\lambda}}(ka)\,\{\,K_{-i\sqrt{\lambda}}'(kr_0)\,K_{-i\sqrt{\lambda}}(kr_0) - K_{-i\sqrt{\lambda}}(kr_0)\,K_{-i\sqrt{\lambda}}'(kr_0)\;\}$$

This second line is 0, no question about it. We are left with

$$\text{jump} = K_{-i\sqrt{\lambda}}(ka)\;W[I_{-i\sqrt{\lambda}}(kr_0).\,K_{-i\sqrt{\lambda}}(kr_0)\,] = K_{-i\sqrt{\lambda}}(ka)\,(-1/kr_0)$$

Now I will reinstall the k 1/K factor I left out

$$\text{jump} = K_{-i\sqrt{\lambda}}(ka)\,(-1/kr_0)\;*\;k/\,K_{-i\sqrt{\lambda}}(ka) = -1/r_0$$

But this agrees with

$$-(rg')' + k^2rg - \lambda\,g/r = \delta(r-r_0) \qquad\qquad \text{jump} = -1/r_0$$

Therefore the constant really is 1 out front! So that is not the problem.

Conclusion: The g theory is failing with this ODE and this interval. Do I have the Z form right?

$$-(xu')' + \nu^2u/x - k^2xu = 0 \qquad Z_\nu(kx) \qquad (*)$$
$$-(xu')' + \nu^2u/x + k^2xu = 0 \qquad K_\nu(kx) \qquad (*)$$

Let $\nu^2 = -\lambda$, then $\nu = \sqrt{-\lambda} = \pm i\sqrt{\lambda}$

$$-(xu')' - \lambda \qquad u/x + k^2 xu = 0 \qquad K_{-i\sqrt{\lambda}}(kx) \qquad (*)$$

So the form seems right. The BC's at the endpoints are met. Each side is a solution. The jump condition is met. What else is there? What about continuity at the jump point?

$$g(r|r_0;\lambda) = [I_{-i\sqrt{\lambda}}(kr_<) K_{-i\sqrt{\lambda}}(ka) - K_{-i\sqrt{\lambda}}(kr_<) I_{-i\sqrt{\lambda}}(ka)] [K_{-i\sqrt{\lambda}}(kr_>)/ K_{-i\sqrt{\lambda}}(ka)]$$

This form automatically is continuous there. Just set $r_< = r_> = r_0$ to get

$$g(r_0|r_0;\lambda) = [I_{-i\sqrt{\lambda}}(kr_0) K_{-i\sqrt{\lambda}}(ka) - K_{-i\sqrt{\lambda}}(kr_0) I_{-i\sqrt{\lambda}}(ka)] [K_{-i\sqrt{\lambda}}(kr_0)/ K_{-i\sqrt{\lambda}}(ka)]$$

The same approaching from either way.

Does $K_{-i\sqrt{\lambda}}(ka)$ have zeroes in λ which cause poles in g ???

If so, then we might have a discrete spectrum here, and that would be like the Hankel situation. We have

$$K_{-i\sqrt{\lambda}}(ka) = (i\pi/2)\, i^{\nu}\, H^{(1)}_{-i\sqrt{\lambda}}(ika) \qquad \nu = -i\sqrt{\lambda}$$

It is not easy to find data on the zeros, and surely they are not at any regular locations, so you would end up with a series that is a bit ugly, similar to ones I used for Laplace in a cone from Smythe, etc.

Even **Watson** has not much clarity on "the zeros of $K_\nu(z)$" see page 511 of the treatise. He quotes work by **McDonald**, but the latter assume ν was real and positive. I guess I could imagine $\lambda = -h^2$ as we usually do (k is used up!) so then we are talking $h = -i\sqrt{\lambda}$ and $\sqrt{\lambda} = ih$, $h>0$, so we then really want to know the zeros of $K_h(ka)$ for $h>0$, just as McDonald looked into. But McDonald is looking at zeros in argument, not zeros in the order!

I think this is something that is just no known! Look at Bateman p 265 vol 1 for

$$K_\nu(x) = \sqrt{\pi}\, e^{-x}(2x)^\nu \Psi(\nu+1/2,\, 1+2\nu;\, 2x)$$

The Ψ is that second kind confluent, so like a Q function, not clear where zeros are located.

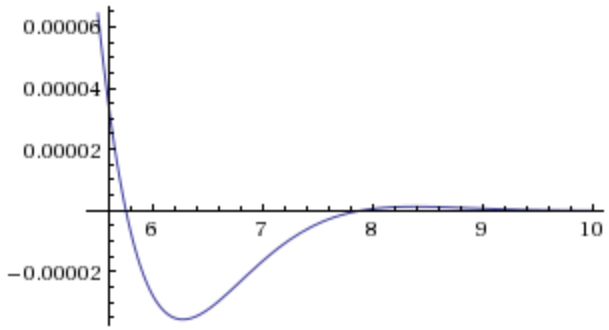
Conclusion: When I tried to generalize the KL SL problem to one on (a,∞), I found that there is not cut at all in the λ plane, and that there are likely singularities in λ caused the places where $K_{i\sqrt{\lambda}}(ka) = 0$. If we put $\sqrt{\lambda}$ just above the real axis, then we are talking zeros of $K_{iy}(z)$ as in the KL transform. I am sure this thing does have zeros, but Maple does not plot very fast. Wolfram plots plenty fast and here are some zeros

Input interpretation:

plot	$K_{i\,x}(3)$	$x = 5.5$ to 10

$K_n(x)$ is the modified Bessel function of the second kind »
i is the imaginary unit »

Plots of the real and imaginary parts:



(x from 5.5 to 10)

I think the KL function is purely real and you see a zero around $K_{i8}(3)$. In other words, this suggests that the function $K_\mu(3)$ has a zero at $\mu = 8i$! So suppose there is a simple 0 here as it appears (and cannot be cut since K_ν analytic in ν). Then that means near this zero we have

$$K_\nu \approx a(\nu-\nu_0) \qquad \text{so} \qquad K_{i\sqrt{\lambda}} \approx a(\sqrt{\lambda}-\sqrt{\lambda_0})$$

ouch! This means that every zero of this thing really causes a cut in λ, so the λ plane is smothered in nasty cuts! For this one for example we get $g \approx \text{stuff} / [\sqrt{\lambda}-\sqrt{\lambda_0}\,]$. [I later resolved this issue, so let's stay focused on the idea that $K_\mu(z)$ for $z>0$ seems to have zeros on the imaginary μ axis!]

So I guess we can throw the idea of generalizing KL in this way into the trash can, it is a horrible mess.

Start over again on this problem.

What is Stak asking us to do here? What kind of "radial expansion" can I make on (a,∞) ? I need some kind of complete set of radial functions for this interval, and I don't know how to do that, I have never done an external problem like this. The Fourier-Bessel Series Transform works on $(0,a)$, so there ought to be some version of it that works on the other part of the interval. I feel pretty stupid after 2 years of Stak not even able to think of some options here. Can we modify the Hankel Transform? Can I just shift it somehow? Here is that transform

$$f(\rho) = \int_0^\infty dk\, k\, J_\nu(k\rho)\, F_\nu(k) \qquad \text{// expansion}$$

$$F_\nu(k) = \int_0^\infty d\rho\, \rho\, J_\nu(k\rho)\, f(\rho) \qquad \text{// projection}$$

$$\int_0^\infty d\rho\, \rho\, J_\nu(k\rho)\, J_\nu(k'\rho) = \delta(k-k')/k \qquad \text{// orthogonality}$$

$$\int_0^\infty dk\, k\, J_\nu(k\rho)\, J_\nu(k\rho') = \delta(\rho-\rho')/\rho \qquad \text{// completeness}$$

Suppose I define $\sigma = \rho + a$ then we have

$$F_\nu(k) = \int_a^\infty d\sigma\, (\sigma-a)\, J_\nu(k[\sigma-a])\, g(\sigma) \qquad g(\sigma) = f(\sigma-a)$$

$$g(\sigma) = f(\sigma-a) = \int_0^\infty dk\, k\, J_\nu(k[\sigma-a])\, F_\nu(k)$$

$$\delta(\sigma-\sigma') = (\sigma-a) \int_0^\infty dk\, k\, J_\nu(k[\sigma-a])\, J_\nu(k[\sigma'-a])$$

Looks fine to me. Start with some $g(\sigma)$ on (a,∞) and project it and recover it. The problem here is that this approach does not have the ν of J_ν connected with the eigenvalue! And it is continuous.

Return to the KL idea again

What is this

$$\text{disc}[1/K_{i\sqrt{\lambda}}(kr_>)]$$ along a cut going from some λ_i to $+\infty$, where λ_i is a zero.

Question: I said that $K_\nu(z)$ was analytic in ν, but it could have linear zeros, say. For example, the function $f_\nu(z) = \nu - 3$ has a linear zero. That means that $f_{i\sqrt{\lambda}}(z) = i\sqrt{\lambda} - 3$ and this has a branch point and a cut and is therefore NOT analytic in λ. So I guess $K_\nu(z)$ is not analytic in λ as I thought.

I need to have a linear variable like $K_{i\kappa}(z)$ as I had in the actual KL derivation. So resume tomorrow with this in mind. Imitate the KL more closely maybe.

Resuming Sunday Aug 7, 2011.

I just added a section to my KL derivation which clarified things quite a bit. One good method I learned is to replace the full great circle with some kind of half circle in the other variable, and then deform the contour there, and then go back to λ space. This is the simplest way to handle things! I showed that in the λ space, the cuts induced by zeros do exist, but they have zero discontinuity and thus we can forget about them.

So now we can start afresh on our great circle integral above. Here is the situation we had above:

"Let's write the Green's 1D again,

$$-(rR')' + k^2 rR - \lambda R/r = \delta(r-r_0) \qquad \lambda = \lambda_{true} = \nu = -\mu^2$$

so $\sqrt{\lambda} = i\mu$ (as required) and thus $\mu = -i\sqrt{\lambda}$. Our 1D Green's is then

$$-(rR')' + k^2 rR + \mu^2 R/r = \delta(r-r_0)$$

Now I know from "prop Bessel" how to solve this 1D Green's function problem. Cribbing from Example 2A, the answer is

$$R(\mu) = [\, I_\mu(kr_<)K_\mu(ka) - K_\mu(kr_<)\, I_\mu(ka)]\, [\, K_\mu(kr_>)/\, K_\mu(ka)\,] \quad "$$

So let's go ahead and use variables λ and μ as shown here. If we had $\mu = \sqrt{\lambda}$, we could claim that the mapping of the CCW great circle in λ is a CCW upper half great circle in μ space. The extra factor of -i which appears in $\mu = -i\sqrt{\lambda}$ merely rotates this half great circle CW by a quarter turn, so it is a CCW half contour around the right half μ plane. No problem. We have

$$\mu = -i\sqrt{\lambda} \qquad d\mu = (-i)(1/2)\, d\lambda/\sqrt{\lambda} \quad \text{so } d\lambda = 2\, i\, \sqrt{\lambda}\, d\mu = (-2)(-\, i\, \sqrt{\lambda}\,)d\mu = -2\mu\, d\mu.$$

So our great circle integral is this

$$GCI \equiv \int_{GC} d\lambda\, R(-i\sqrt{\lambda}\,) = \int_{RHC} [-2\mu d\mu]\, R(\mu) = -2 \int_{RHC} d\mu\, \mu\, R(\mu)$$

where RHS means right half great circle, in the CCW sense. So then we have

$$GCI = -2 \int_{RHC} d\mu\, \mu\, [\, I_\mu(kr_<)K_\mu(ka) - K_\mu(kr_<)\, I_\mu(ka)]\, [\, K_\mu(kr_>)/\, K_\mu(ka)\,]$$

Now we know that the integrand is analytic everywhere EXCEPT $K_\mu(ka)$ may have some zeros in μ in the right half plane. Let's assume there are some zeros there somewhere, call them μ_i. Let's assume for simplicity's sake that there are no multiple zeros, they are all simple zeros. We want to deform our RHC so that it becomes an up-going vertical line just to the right of the imaginary axis. In doing so, we will pick up the residues of those poles μ_i in the RH plane.

What do we know about this up-going vertical line integral? Well, in fact we know it is going to be zero, because we know in the λ plane it is that discontinuity integral which we kept finding above, much to our surprise, was 0 ! But let's show this now directly in the μ plane. That integral is a multiple of this integral (we ignore the constant

$$\int_{-\infty}^{\infty} ds\, s\, [\, I_{is}(kr_<)K_{is}(ka) - K_{is}(kr_<)\, I_{is}(ka)]\, [\, K_{is}(kr_>)/\, K_{is}(ka)\,]$$

and I think we want to show that this

$$f(s) \equiv R(is) = [\, I_{is}(kr_<)K_{is}(ka) - K_{is}(kr_<)\, I_{is}(ka)]\, [\, K_{is}(kr_>)/\, K_{is}(ka)\,]$$

is an even function of s, and thus s f(s) is odd, so the integral vanishes! Well, we know it is going to work because we did it above. We know we can write

$$f(-s) \equiv R(is) = [\, I_{-is}(kr_<)K_{is}(ka) - K_{is}(kr_<)\, I_{-is}(ka)]\, [\, K_{is}(kr_>)/\, K_{is}(ka)\,]$$

where I have already used $K_\nu = K_{-\nu}$. So we then make use of

$$I_{-\nu}(z) = I_{\nu}(z) + (2/\pi)\sin(\pi\nu)\, K_{\nu}(z)$$
$$I_{-is}(z) = I_{is}(z) + (2/\pi)\sin(\pi is)\, K_{is}(z)$$

and we then have

$$f(-s)_{num} = I_{-is}(kr_<)K_{is}(ka) - K_{is}(kr_<)\, I_{-is}(ka)$$

$$= \{ I_{is}(kr_<) + (2/\pi)\sin(\pi is)\, K_{is}(kr_<)\}K_{is}(ka) - K_{is}(kr_<)\, \{ I_{is}(ka) + (2/\pi)\sin(\pi is)\, K_{is}(ka)\}$$

$$= I_{is}(kr_<)\, K_{is}(ka) + (2/\pi)\sin(\pi is)\, K_{is}(kr_<)\, K_{is}(ka)$$
$$- K_{is}(kr_<)I_{is}(ka) - (2/\pi)\sin(\pi is)\, K_{is}(kr_<)K_{is}(ka)\}$$

$$= I_{is}(kr_<)\, K_{is}(ka)$$
$$- K_{is}(kr_<)I_{is}(ka)$$

$$= [I_{is}(kr_<)\, K_{is}(ka) - K_{is}(kr_<)I_{is}(ka)]$$

$$= f(s)_{num}$$

Therefore, since $f(-s)_{den} = f(s)_{den}$, we know that f(-s) = f(s) and our vertical line integral vanishes, QED.

So all we have is those pole residues! Let's now throw in our $1/2\pi i$ factor and say

$$(1/2\pi i) \int d\mu\, H(\mu) = \Sigma_i\, \mathrm{Res}[H(\mu)]^{\mu=\mu i}$$

But we have

$$(1/2\pi i) \int_{GC} d\lambda\, R(-i\sqrt{\lambda}\,) = -2\,(1/2\pi i) \int_{RHC} d\mu\, \mu\, [\, I_{\mu}(kr_<)K_{\mu}(ka) - K_{\mu}(kr_<)\, I_{\mu}(ka)]\, [\, K_{\mu}(kr_>)/\, K_{\mu}(ka)\,]$$

$$= -2\, \Sigma_i\, \mu_i\, [\, I_{\mu i}(kr_<)K_{\mu i}(ka) - K_{\mu i}(kr_<)\, I_{\mu i}(ka)]\, [\, K_{\mu i}(kr_>)]\, \mathrm{Res}[1/\, K_{\mu}(ka)]|^{\mu=\mu i}$$

But (a point I missed until late in the game as well), this simplifies -- since we know $K_{\mu i}(ka) = 0$ -- to

$$= +2\, \Sigma_i\, \mu_i\, K_{\mu i}(kr_<)\, I_{\mu i}(ka)\, K_{\mu i}(kr_>)\, \mathrm{Res}[1/\, K_{\mu}(ka)]|^{\mu=\mu i}$$

Now we recall our general rules

Completeness: $-(1/2\pi i) \oint d\lambda\, g(r|r_0;\lambda) = \delta(r-r_0)/s(r) = "\Sigma_n\, \varphi_n(r)\, \bar{\varphi}_n(r_0)"$

Orthogonality: $\langle \sqrt{s}\, \varphi_n, \sqrt{s}\, \varphi_m \rangle = K_n\, \delta_{n,m}$

Since s(x) = 1/x, our first conclusion is this:

$$r_0\delta(r-r_0) = -2\,\Sigma_i\,\mu_i\,K_{\mu i}(kr_<)\,I_{\mu i}(ka)\,K_{\mu i}(kr_>)\,\mathrm{Res}[1/K_\mu(ka)]|^{\mu=\mu i}$$

which is the completeness relation. Secondly, we conclude that

$$\Sigma_i\,\varphi_i(r)\,\bar{\varphi}_i(r_0) = -2\,\Sigma_i\,\mu_i\,K_{\mu i}(kr_<)\,I_{\mu i}(ka)\,K_{\mu i}(kr_>)\,\mathrm{Res}[1/K_\mu(ka)]|^{\mu=\mu i}$$

$$= -\,\Sigma_i\,2\mu_i\,I_{\mu i}(ka)\,\mathrm{Res}[1/K_\mu(ka)]|^{\mu=\mu i}\;K_{\mu i}(kr_<)\,K_{\mu i}(kr_>)$$

and that in fact (subscript μi means μ_i). Let's suppose for the moment that the minus sign here is wrong. Then suppose we really have

$$\Sigma_i\,\varphi_i(r)\,\bar{\varphi}_i(r_0) = +\,\Sigma_i\,2\mu_i\,I_{\mu i}(ka)\,\mathrm{Res}[1/K_\mu(ka)]|^{\mu=\mu i}\;K_{\mu i}(kr_<)\,K_{\mu i}(kr_>)$$

Next, let's suppose that somehow the RHS here is real (to be discussed later). Then we can identify

$$\varphi_i(r)\,\varphi_i(r_0) = (2\mu_i)\,I_{\mu i}(ka)\,\mathrm{Res}[1/K_\mu(ka)]|^{\mu=\mu i}\;K_{\mu i}(kr_<)\,K_{\mu i}(kr_>)$$

Then it seems pretty clear that

$$\varphi_i(r) = \sqrt{(2\mu_i)\,I_{\mu i}(ka)\,\mathrm{Res}[1/K_\mu(ka)]|^{\mu=\mu i}}\;K_{\mu i}(kr) = \alpha(\mu_i, k, a)\,K_{\mu i}(kr)$$

At this point, several observations:

(1) this vanishes at r=a since $K_{\mu i}(ka) = 0$;

(2) this vanishes at r=∞ since it is a K;

(3) We know $\varphi_i(r)$ has to be a linear combination of this K and its I partner, and it is!

(4) If we just sat down and tried to deduce $\varphi_i(r)$, we would conclude that it had to be $\varphi_i(r) = \alpha\,K_{\mu i}(kr)$ where μ_i is a zero for $K_{\mu i}(ka)$, because that is the only possibly solution at r=∞.

(5) This is very similar to Smythe's solution of the electrostatics cone Green's Function where we used the atom $P_{ni}(\cos\theta)$ where the non-integer values of n_i were determined by $P_{ni}(\cos\alpha) = 0$ where α was the cone angle, so this makes V vanish on the conical surface.

(6) Since K has expo decay, we expect no trouble with the orthogonality integral

$$\langle\sqrt{s}\,\varphi_n, \sqrt{s}\,\varphi_m\rangle = K_n\,\delta_{n,m}$$

$$\int_a^\infty dr\,(1/r)\,\varphi_n^*(r)\varphi_m(r) = \int_a^\infty dr\,(1/r)\,K_{\mu n}(kr)\,K_{\mu m}(kr) = K_n\,\delta_{m,n}$$

Note carefully that m and n are indices of the zeros μ_n and this integral is only valid for those μvalues.

(7) I think if we look up the above integral, we might take a limit and learn something about K_n.

Summary to this Point:

We started off with this radial equation and BC:

$$-rR'' - R' - \lambda rR - \nu R/r = 0 \qquad R(a) = R(\infty) = 0 \qquad \text{interval } (a,\infty)$$

where λ was the Helmholtz λ, and ν is the separation constant which becomes "eigenvalue". We made these replacements

$$\lambda = -k^2 \text{ (Helmholtz } \lambda) \qquad \nu = \lambda$$

and then our new system is this

$$-rR'' - R' + k^2 rR - \lambda R/r = 0 \qquad R(a) = R(\infty) = 0 \qquad \text{interval } (a,\infty)$$

We recognized this as a KL type equation with $s(x) = 1/x$ (see comparison doc). We knew that solutions of this homo ODE had the form $K_{\pm i\sqrt{\lambda}}(kr)$. We then considered this Green's Function problem

$$-(rg')' + k^2 rg - \lambda g/r = \delta(r-r_0) \qquad g(a) = 0 \qquad R(\infty) = 0 \qquad \text{interval } (a,\infty)$$

and we found that

$$g(r|r_0;\lambda) = [I_{-i\sqrt{\lambda}}(kr_<) K_{-i\sqrt{\lambda}}(ka) - K_{-i\sqrt{\lambda}}(kr_<) I_{-i\sqrt{\lambda}}(ka)] [K_{-i\sqrt{\lambda}}(kr_>)/ K_{-i\sqrt{\lambda}}(ka)]$$

which certainly had a nice overall constant of 1. We then used our general rule

$$-(1/2\pi i) \oint d\lambda\, g(r|r_0;\lambda) = \delta(r-r_0)/s(r) = \Sigma_i \varphi_i(r) \bar{\varphi}_i(r_0)$$

and we were able to obtain our normalized eigenfunctions which form a complete set on (a,∞):

$$\varphi_i(r) = \sqrt{(2\mu_i) I_{\mu i}(ka) \text{ Res}[1/ K_\mu(ka)]|^{\mu=\mu i}}\ K_{\mu i}(kr) \qquad \equiv \alpha_i(ka) K_{\mu i}(kr)$$

where the μ_i are the zeros of the function $K_\mu(ka)$. Recall that $k^2 = -\lambda_{Helmholtz}$ so k = constant, and of course a = constant, so talking about these zeros is a well defined matter.

What is our spectrum here? We have $\lambda = \nu$ and $\mu = -i\sqrt{\lambda}$ and $\mu^2 = -\lambda$. So in the λ plane, our spectral points of interest are $\lambda_i = -\mu_i^2$. These are supposedly the eigenvalues of this problem:

$$[L-\lambda_i(1/r))]\varphi_i = 0 \qquad Lu = -(ru')' + k^2 ru \qquad r \text{ in } (a,\infty) \qquad L\,\varphi_i(r) = \lambda_i(1/r)\,\varphi_i(r)$$

If the μ_i are all real and > 0 (remember, we only capture those in the RH μ plane), then these eigenvalues are on the negative real λ axis which seems a bit strange and probably wrong. Perhaps the poles of the function $1/ K_\mu(ka)$ are just to the right of the imaginary axis somehow, so μ_i are imaginary, and then we would have $\lambda_i = -\mu_i^2 > 0$ and we get the expected picture. If they are exactly on the imaginary axis, then we would pick up a half residue at each such pole.

Finish Up: Find the 1D Green's Equation in θ, Solve it, and state Full 2D Solution

Let's imagine that our $\varphi_i(r)$ are as shown above and form a complete set on (a,∞). Then we can do our "radial expansion" as instructed by Stak for our circle exterior problem, and we have

$u(r,\theta) = \Sigma_i\, u_i(\theta)\, \varphi_i(r)$ // general form of a solution

Our PDE from the top of this doc is (exterior to circle Green's Function problem, find u!)

$- (ru')' \quad -r^{-1}\partial_\theta^2 u + k^2 r\, u = \delta(r-r_0)\,\delta(\theta)$ $\qquad u(a,\theta) = u(\infty,\theta) = 0$ $\qquad k^2 = -\lambda_{Helmholtz}$

or

$Lu \quad - r^{-1}\partial_\theta^2 u = \delta(r-r_0)\,\delta(\theta)$

with L as given above. Now we install from above to get

$L\, \Sigma_i\, u_i(\theta)\, \varphi_i(r) \; - r^{-1}\partial_\theta^2\, \Sigma_i\, u_i(\theta)\, \varphi_i(r) = \delta(r-r_0)\,\delta(\theta)$

$\Sigma_i\, u_i(\theta)\, L\, \varphi_i(r) \; - r^{-1}\, \Sigma_i\, (\partial_\theta^2 u_i(\theta))\, \varphi_i(r) = \delta(r-r_0)\,\delta(\theta)$

$\Sigma_i\, u_i(\theta)\, r^{-1}\lambda_i \varphi_i(r) \; - r^{-1}\, \Sigma_i\, (\partial_\theta^2 u_i(\theta))\, \varphi_i(r) = \delta(r-r_0)\,\delta(\theta)$

$\Sigma_i\, u_i(\theta)\, \lambda_i \varphi_i(r) \; - \Sigma_i\, (\partial_\theta^2 u_i(\theta))\, \varphi_i(r) = r_0\; \delta(r-r_0)\,\delta(\theta)$

Now we expand

$\delta(r-r_0) r_0 \; = \Sigma_i\, \varphi_i(r)\, \bar{\varphi}_i(r_0)$

and we then have

$\Sigma_i\, u_i(\theta)\, \lambda_i \varphi_i(r) \; - \Sigma_i\, (\partial_\theta^2 u_i(\theta))\, \varphi_i(r) = \Sigma_i\, \varphi_i(r)\, \bar{\varphi}_i(r_0)\, \delta(\theta)$

Appealing to the completeness of our set of radial functions, we get

$u_i(\theta)\, \lambda_i \; - \; (\partial_\theta^2 u_i(\theta)) \; = \; \bar{\varphi}_i(r_0)\, \delta(\theta)$

$- \; (\partial_\theta^2 u_i(\theta)) + \lambda_i\, u_i(\theta) \; = \; \bar{\varphi}_i(r_0)\, \delta(\theta)$

$- \; (\partial_\theta^2 u_i(\theta)) + \lambda_i\, u_i(\theta) \; = \; \bar{\varphi}_i(r_0)\, \delta(\theta)$

$$-u_i'' + \lambda_i u_i = \bar{\varphi}_i(r_0)\,\delta(\theta)$$

$$u_i'' - \lambda_i u_i = -\bar{\varphi}_i(r_0)\,\delta(\theta)$$

Now this last problem I solved in my "Sub Problem" section above,

$$a_\gamma''(\theta) + b^2 a_\gamma(\theta) = c\,\delta(\theta) \qquad \text{// on } (-\pi,\pi)$$
$$a_\gamma(\pi) = a_\gamma(-\pi)\,,\quad a_\gamma'(\pi) = a_\gamma'(-\pi)$$
$$a_\gamma(\theta) = (c/[2b\sin(b\pi)])\cos[b(\pi-|\theta|)]$$

and here we have

$$b^2 = -\lambda_i = \mu_i^2 \;\Rightarrow\; b = \mu_i \qquad c = -\bar{\varphi}_i(r_0)$$

and the solution is

$$u_i = (c/[2b\sin(b\pi)])\cos[b(\pi-|\theta|)] \qquad \theta \text{ in } (-\pi,\pi)$$

$$= -(\bar{\varphi}_i(r_0)/[2\mu_i\sin(\mu_i\pi)])\cos[\mu_i(\pi-|\theta|)]$$

Then our exterior to circle problem solution is this

$$u(r,\theta) = \Sigma_i\, u_i(\theta)\,\varphi_i(r)$$

where
$$u_i(\theta) = -(\bar{\varphi}_i(r_0)/[2\mu_i\sin(\mu_i\pi)])\cos[\mu_i(\pi-|\theta|)]$$

$$\varphi_i(r) = \alpha_i(ka)\,K_{\mu i}(kr) \qquad \alpha_i(ka) = \sqrt{(2\mu_i)\,I_{\mu i}(ka)\,\mathrm{Res}[1/K_\mu(ka)]|^{\mu=\mu i}}$$

Inserting the pieces, and again assuming the φ_i are real, we get

$$u(r,\theta) = \Sigma_i\, u_i(\theta)\,\varphi_i(r)$$

$$= -\Sigma_i(\bar{\varphi}_i(r_0)/[2\mu_i\sin(\mu_i\pi)])\cos[\mu_i(\pi-|\theta|)]\,\varphi_i(r)$$

$$= -\Sigma_i\,\varphi_i(r)\,\varphi_i(r_0)\,[2\mu_i\sin(\mu_i\pi)]^{-1}\cos[\mu_i(\pi-|\theta|)]$$

$$= -\Sigma_i\,K_{\mu i}(kr)\,K_{\mu i}(kr_0)\,(2\mu_i)\,I_{\mu i}(ka)\,\mathrm{Res}[1/K_\mu(ka)]|^{\mu=\mu i} * [2\mu_i\sin(\mu_i\pi)]^{-1}\cos[\mu_i(\pi-|\theta|)]$$

$$= -\Sigma_i\,K_{\mu i}(kr)\,K_{\mu i}(kr_0)\,I_{\mu i}(ka)\,\mathrm{Res}[1/K_\mu(ka)]|^{\mu=\mu i}\,[\sin(\mu_i\pi)]^{-1}\cos[\mu_i(\pi-|\theta|)]$$

We can compare this to our solution done "the other way" in problem 7.34

$$g(a,\theta|r_0,0) = \Sigma_n e^{in\theta} a_n(r)$$

$$= (i/4)\,\Sigma_n e^{in\theta}\,[\;H_n^{(1)}(\sqrt{\lambda}\,a)J_n(\sqrt{\lambda}\,r_<) - J_n(\sqrt{\lambda}\,a)\,H_n^{(1)}(\sqrt{\lambda}\,r_<)]\;[\;H_n^{(1)}(\sqrt{\lambda}\,r_>)\,/H_n^{(1)}(\sqrt{\lambda}\,a)]$$

where we could if we wanted use $I_\nu(z)\,K_\nu(z') = (i\pi/2)\,J_\nu(iz)\,H_\nu^{(1)}(iz')$ and H/H = K/K and then that 7.34 result would read

$$g(a,\theta|r_0,0) = \Sigma_n e^{in\theta} a_n(r)$$

$$= (i/4)\,(i\pi/2)\,\Sigma_n e^{in\theta}\,[\;K_n(ka)I_n(kr_<) - I_n(ka)\,K_n^{(1)}(kr_<)]\;[\;K_n(kr_>)\,/K_n(ka)]$$

$$= -(\pi/8)\;\Sigma_n e^{in\theta}\,[\;K_n(ka)I_n(kr_<) - I_n(ka)\,K_n^{(1)}(kr_<)]\;[\;K_n(kr_>)\,/K_n(ka)]$$

$$= -(\pi/8)\;\Sigma_{n=0}^{\infty}\;\varepsilon_n\cos(n\theta)\,[\;K_n(ka)I_n(kr_<) - I_n(ka)\,K_n^{(1)}(kr_<)]\;[\;K_n(kr_>)\,/K_n(ka)]$$

And once again, here is a side by side comparison

$$u(r,\theta) = -\,\Sigma_i\;K_{\mu i}(kr)\,K_{\mu i}(kr_0)\,I_{\mu i}(ka)\,\mathrm{Res}[1/\,K_\mu(ka)]|^{\mu=\mu i}\;\;[\sin(\mu_i\pi)]^{-1}\cos[\mu_i(\pi-|\theta|)]$$

$$u(r,\theta) = -(\pi/8)\;\Sigma_{n=0}^{\infty}\;\varepsilon_n\,[\;K_n(ka)I_n(kr_<) - I_n(ka)\,K_n^{(1)}(kr_<)]\;[\;K_n(kr_>)\,/K_n(ka)]\cos(n\theta)$$

where n sums over all integers, while i sums over the right half plane μ zeros of $K_\mu(ka)$. We know how to add θ_0 into both equations, and then we would see that both are symmetric under $\mathbf{r} \leftrightarrow \mathbf{r}_0$ which is true for any Green's Function like this.

Now suppose we just started with Stak result p 291 A converted to K which would read

$$u(r,\theta) = \Sigma_n\,u_n\,K_n(kr)e^{in\theta} = \Sigma_{n=0}^{\infty}\,\varepsilon_n\,u_n\,K_n(kr)\cos(n\theta)$$

where this is really a Smythian form using a known atomic form. We don't have a sin(n|θ|) term here because the set of functions cos(nθ) with n integers forms a complete set for even functions of θ. But suppose, anticipating the next move, we were to add such terms anyway with a dummy coefficient, so our Smythian form answer would then be

$$u(r,\theta) = \Sigma_{n=0}^{\infty}\,K_n(kr)\,[\;A_n\cos(n\theta) + B_n\sin(n|\theta|)\;] \qquad r > r_0$$

$$u(r,\theta) = \Sigma_{n=0}^{\infty}\,[\;K_n(ka)I_n(kr) - I_n(ka)\,K_n^{(1)}(kr)]\;\;[\;A_n'\cos(n\theta) + B_n'\sin(n|\theta|)\;] \qquad r < r_0$$

We can regard this as the "form" of our problem 7.34 solution where $B_n = B_n' = 0$. We might conjecture that this same Smythian form gives the result of our current problem at least for $r > r_0$

$$u(r,\theta) = \Sigma_i\,K_{\mu i}(kr)\,[\;A_{\mu i}\cos(\mu_i\theta) + B_{\mu i}\sin(\mu_i|\theta|)\;]$$

and indeed this is exactly the form our solution has (and it happens to have it for $r>r_0$ and $r<r_0$)

$$u(r,\theta) = -\,\Sigma_i \qquad K_{\mu i}(kr)\;\;\{K_{\mu i}(kr_0)\,I_{\mu i}(ka)\,\mathrm{Res}[1/\,K_\mu(ka)]|^{\mu=\mu i}\;\}\,[\;\cot(\mu_i\pi)\cos(\mu_i\theta)\;+\sin(\mu_i|\theta|)\;]$$

$u(r,\theta) = \Sigma_{n=0}^{\infty} K_n(kr) \quad [A_n \cos(n\theta) + B_n \sin(n|\theta|)] \quad r>r_0$

$A_n = -(\pi/8)\, \varepsilon_n [K_n(ka) I_n(kr_0) - I_n(ka) K_n^{(1)}(kr_0)] [1 / K_n(ka)] \quad B_n = 0$

In the problem 7.34 solution, we dealt with the θ EV problem and got quantization of n to integers and we had for our spectrum $\nu = -n^2$. In the current solution, we dealt with the r EV problem and got quantization to $\nu = -\mu_i^2$. I suspect that the only way to prove these two results are the same is to do 7.34 and 7.35 just as we have done. Maybe you could find some general contour integral that you close one way and get one result, and close another way and get the other result, but that is a little hard to imagine!

Another look at the zeros of K

I am wondering about the zeros in variable μ of $K_\mu(ka)$ where k and a are constants > 0. So we might just ask about the zeros in μ of $K_\mu(z)$ when z > 0. If there are zeros exactly on the imaginary axis, then our analysis above would mean that we only pick up half a residue at each one, so we would add a factor of 1/2 to our contour evaluation! Not too bad. Also, this fixes our noted problem and gets the spectrum to be on the positive λ axis where it should be. Watson of course is thinking of μ fixed and finding zeros in z, which I guess is the same thing.

But Watson right at the start says he assumes μ > 0, and he looks then for zeros in z, and this is then not what we want! Wolfram I think quotes this

> The function $K_\nu(z)$ has no zeros in the region $|\mathrm{Arg}(z)| \le \frac{\pi}{2}$ for any real ν.

I am maybe onto something. From GR7 we have from p 384

4. $\int_0^\infty \exp(-\beta \cosh x) \cosh \gamma x \, dx = K_\gamma(\beta) \quad [\mathrm{Re}\,\beta > 0]$ ET I 168(16)a, WA 201(5)

Some guy **Stromberg** claims that

$$K_z(A) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-A \cosh u} e^{zu} du.$$

and so we know he really is talking about Bessel k functions. Notice that he has "exposed" the order which he calls z in a simple form here. Here is his general flow

2. Preliminaries

We need a few facts about K-Bessel and Γ-functions.

Definition 2.1. For $A > 0$ and $z \in \mathbb{C}$, set

$$K_z(A) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-A\cosh u} e^{zu} du.$$

Proposition 2.2. *Let $F(z) = 2K_z(2A)$. Then:*

(i) *$F(z)$ is entire,*
(ii) *$F(-z) = F(z)$,*
(iii) *$F(x) > 0$ for $x \in \mathbb{R}$,*
(iv) *$F(iy) \in \mathbb{R}$ for $y \in \mathbb{R}$,*
(v) *for generic $z \in \mathbb{C}$,*

$$F(z) = \Gamma(z)A^{-z}\left[1 + \sum_{n=1}^{\infty} \frac{A^{2n}}{n!(1-z)_n}\right] + \Gamma(-z)A^{z}\left[1 + \sum_{n=1}^{\infty} \frac{A^{2n}}{n!(1+z)_n}\right],$$

where $(w)_n = w(w+1)\cdots(w+n-1)$.

Proof. (i)-(iv) are obvious; for (v) see [7] or [2]. □

So he is talking my language here, treating the order as the variable. Notice that he says $K_{iy}(x)$ is real!

Then instead of just considering one K function, he considers a linear combination of them:

3. Linear combinations of K-Bessel functions

Proposition 3.1. *Let $0 < A_1 < \cdots < A_N$ and $c_1, c_2, \ldots, c_N \in \mathbb{R}$. Put*

$$f(z) = 2\sum_{j=1}^{N} c_j \frac{K_z(2A_j)}{\Gamma(z)}.$$

Then:

(i) *$f(z)$ is entire,*
(ii) *$f(z) = \sum_{j=1}^{N} c_j A_j^{-z}\left[1 + O(\frac{1}{|z|})\right]$ for $x \geq \frac{1}{2}$, $y \geq 1$,*
(iii) *all zeros of $F(z) = f(z)\Gamma(z)$ are located either along the imaginary axis or symmetrically with respect to it.*

So I can take the special case that there is *only one term* in the series so

$f(z) = K_z(2A)/\Gamma(z)$ so that $f(z)\,\Gamma(z) = K_z(2A)/\Gamma(z) = F(z)$.

Now we know that Γ(z) has poles at z = 0,-1,-2... so 1/ Γ(z) has zeros there. But that is not what he is talking about here. He claims that if F(z) has zeros, they must be on the imaginary axis (as I have been

suspecting) or he allows they might come in pairs symmetric to the imaginary axis. He is talking my language. This paper is

ON THE ZEROS OF LINEAR COMBINATIONS OF K-BESSEL FUNCTIONS

FREDRIK STRÖMBERG

You would think he would first worry about the zeros of individual K functions. But he does not mention this, nor does he give a reference for same.

I brought in the ponderous NSTD site http://dlmf.nist.gov/10.21 and amazingly it gives these references

For information on zeros of Bessel and Hankel functions as functions of the order, see Cochran (1965), Cochran and Hoffspiegel (1970), Hethcote (1970), and Conde and Kalla (1979).

J. A. Cochran (1965)
The zeros of Hankel functions as functions of their order,
Numer. Math. **7** (3), pp. 238–250.

J. A. Cochran and J. N. Hoffspiegel (1970)
Numerical techniques for finding ν-zeros of Hankel functions,
Math. Comp. **24** (110), pp. 413–422.

H. W. Hethcote (1970)
Error bounds for asymptotic approximations of zeros of Hankel functions occurring in diffraction problems,
J. Mathematical Phys. **11** (8), pp. 2501–2504.

S. Conde and S. L. Kalla (1979)
The ν-zeros of $J_{-\nu}(x)$,
Math. Comp. **33** (145), pp. 423–426.

So if you want leads, there they are! But then I searched on

"zeros of Hankel functions as functions of"

and got a paper I think has my answer, by Ferreira and Sesma, Brazil and Spain, a 2008 preprint! Ah but they are doing zeros in z, now order.

The Cochran reference seems the main act, here in more detail:

Mathematical and Computational Physics - AuthorMapper
www.authormapper.com/search.aspx?coll...val...size... - Cached
The zeros of Hankel functions as functions of their order · Numerische Mathematik (1965) 7: 238-250, June 01, 1965. By **Cochran**, James Alan · **Download** PDF ...

I am able to look at this paper here

http://www.digizeitschriften.de/en/dms/img/?PPN=PPN362160546_0007&DMDID=dmdlog30

and I got it as a PDF, amazing! Remember that website! Recall now that

$$K_\nu(z) \equiv (i\pi/2)\, i^\nu\ H_\nu^{(1)}(iz) \ \sim\ e^{i\pi\nu/2}\ H_\nu^{(1)}(iz)$$

Cochran actually studies the zeros of $H_\nu^{(1)}(w)$. Here is one result for large w and also large integer s.

$$\nu_s = \frac{i\pi s}{i\left(\frac{1}{2}\pi - \arg w\right) + \ln\left(\frac{3\pi s}{e\,|w|}\right)}\left[1 + \frac{\ln\ln s}{\ln s}O(1)\right] \qquad (5.5)$$

where s is the labeling integer. If w is real, then if |w| >> s for our integer s, we seem to get

$$\nu_s = i\pi s/(i\pi/2) = s/2$$

But I am interested in w = iz for z real, so arg(w) = π/2 and then we get

$$\nu_s = i\pi s/\ln(3\pi s/e|w|) \qquad \text{for large } z$$

$$\nu_s = i\pi s/\ln(3\pi s/ez) \qquad z >> 0 \qquad s = 1,2,3...$$

so I guess there are my zeros lying on the imaginary axis! But in general the zeros are out somewhere in the complex ν plane, they are not all just on the imaginary axis. It is just as Stromberg says.

I am not sure what the complex zeros to do our spectrum! They seem to make it be complex.

OK, this has been fun, but I will stop. A little view of current research I guess.

My own comments on the zeros in ν of $K_\nu(z)$

If we look at an integral representation [try Bateman p 82 (21)] or series for K, you conclude that $K_\nu(z)$ is real analytic in both ν and z as variables. This means that

$$[K_\nu(z)]^* = K_{\nu^*}(z^*)$$

In the case that z is real, such as the z = ka we are concerned with above, we get

$[K_\nu(z)]^* = K_{\nu^*}(z)$

Suppose ν_i is a zero of $K_\nu(z)$. Then LHS = 0, so RHS = 0, so ν_i^* must also be a zero!

Theorem 1: For real z, the zeros of $K_\nu(z)$ must occur in pairs which are mirrored in the Re(ν) axis of the complex ν plane. If it happens that ν_s is a real zero, then this mirroring does not occur.

This seems to disagree with Stromberg's claim! He considers $K_\nu(z)/\Gamma(\nu)$ it is true. So I would adjust my argument to say, since Γ(ν) is real analytic in ν, and assuming real z,

$[K_\nu(z)/\Gamma(\nu)]^* = K_{\nu^*}(z) / \Gamma(\nu^*)$

Suppose ν_i is a zero of $K_\nu(z)/\Gamma(\nu)$. Then LHS = 0 so RHS = 0, so ν_i^* is a zero of $K_{\nu^*}(z) / \Gamma(\nu^*)$. Such a zero arises either from K or from Γ. If from Γ, then it would occur at ν_i = 0,-1,-2.. If from K, then we would arrive at my conclusion.

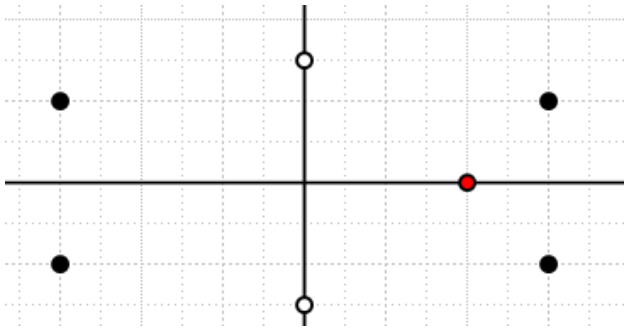
Let's just continue with another theorem. We know that

$K_\nu(z) = K_{-\nu}(z)$

Therefore, if ν_i is a zero, then so is $-\nu_i$.

Theorem 2: For any z, the zeros of $K_\nu(z)$ must occur in pairs reflected in the origin of the ν plane.

Theorem 3: For real z, zeros can in principle only occur in three ways, as illustrated by this picture:



You can in theory have real zeros, shown in red. If you have an imaginary zero, it will occur in a pair as shown. If you have a complex zero as on the upper right, it must occur in a set of 4 as shown by the four black dots.

Now we can quote our claim above

> The function $K_\nu(z)$ has no zeros in the region $|\mathrm{Arg}(z)| \le \frac{\pi}{2}$ for any real ν.

So if z is real, there are no real ν zeros of $K_\nu(z)$, and this then rules out our "red case" above, leaving only the other two cases. This then is in fact in agreement with the claim of Stromberg that the zeros can only occur either on the imaginary axis or in pairs reflected across the imaginary axis. What he does not mention is that all these zeros must be reflected as well across the real axis.

Theorem 4: For real z, $K_\nu(z)$ is real when ν is imaginary.

Proof: From above we had $[K_\nu(z)]^* = K_{\nu^*}(z)$ for real z. thus $[K_{ia}(z)]^* = K_{-ia}(z) = K_{ia}(z)$, QED. We have to use both of our little rules to get this result.

Theorem 5: For z real, $K_\nu(z)$ has no real zeros in ν.

Proof: Look at the integral representation Bateman p 82 (21)

$$K_\nu(z) = \int_0^\infty dt\, e^{-zcht}\, ch(\nu t) \qquad Re(\nu) > 0$$

Both factors in the integrand are positive definite, so this is the integral of a positive integrand and the integral must give a positive result. Thus, $K_\nu(z)$ is positive for real z and $Re(\nu) > 0$. But the reflection rule for K says no zeros for $Re(\nu) < 0$ either. And at $\nu = 0$ the same is true. Thus, if K is always positive, it cannot have any zeros! Here are a few AS graphs of K functions

10.26(i) Real Order and Variable

See Figures 10.26.1–10.26.6.

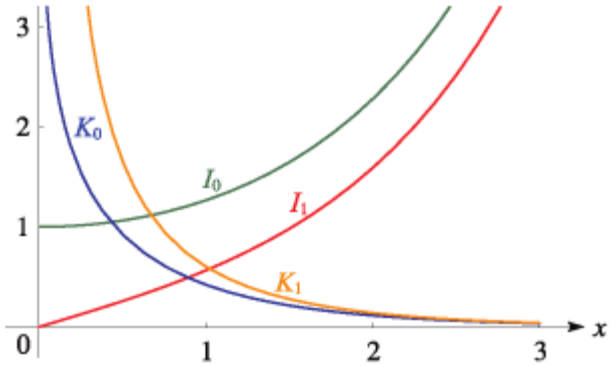
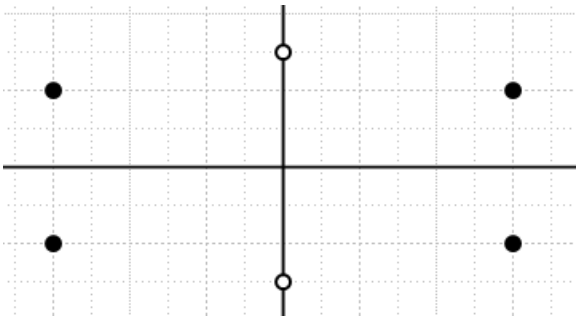


Figure 10.26.1: $I_0(x)$, $I_1(x)$, $K_0(x)$, $K_1(x)$, $0 \le x \le 3$.

Conclusions:

(1) We don't know where the zeros in ν of $K_\nu(z)$ are in fact located for real z, but we know there are no zeros on the real ν axis.

(2) The only *possibilities* for real z are zeros occurring in imaginary axis pairs, or complex quads:



This does not prove that either kind of zero sets actually occur. Just if they occur, they must fall into these categories, white or black.

(3) If we consider the poles of $1/K_\nu(z)$ for z real, they are at the four black dots above. The residues of a vertical pair of such poles must be complex conjugates of each other. If the residues happened to be imaginary, then the sum of the two residues would be 0.

(4) It seems strange that we don't have any kind of table of these zeros.

Reconsider our solution to problem 7.35

$$u(r,\theta) = -\Sigma_i \; K_{\mu i}(kr)\, K_{\mu i}(kr_0)\, I_{\mu i}(ka)\, \mathrm{Res}[1/K_\mu(ka)]|^{\mu=\mu i} \; [\sin(\mu_i \pi)]^{-1} \cos[\mu_i(\pi - |\theta|)]$$

Assume we have a pair of complex zeros in the RHP. We then can add just those two to get

$$K_{\mu i}(kr)\, K_{\mu i}(kr_0)\, I_{\mu i}(ka)\, \mathrm{Res}[1/K_\mu(ka)]|^{\mu=\mu i}\ [\sin(\mu_i\pi)]^{-1} \cos[\mu_i(\pi-|\theta|)]$$

+

$$K_{\mu i*}(kr)\, K_{\mu i*}(kr_0)\, I_{\mu i*}(ka)\, \{\mathrm{Res}[1/K_\mu(ka)]|^{\mu=\mu i}\}^*\ [\sin(\mu_i{}^*\pi)]^{-1} \cos[\mu_i{}^*(\pi-|\theta|)]$$

The second line is the CC of the first so we get a real result. Now consider

$$I_{-\nu}(ka) = I_\nu(ka) + (2/\pi)\sin(\pi\nu)\, K_\nu(ka)$$

$$I_{-\mu i}(ka) = I_{\mu i}(ka) + (2/\pi)\sin(\pi\mu_i)\, K_{\mu i}(ka) = I_{\mu i}(ka)$$

I don't see anyway offhand to show the this sum is 0 for complex zeros. So I am still having trouble getting the $\lambda = -\mu^2$ spectrum to lie on the real axis for complex zeros of K. I know for large ν that there are definitely zeros on the real axis, so these are OK.