Stakgold chap 7 exercise 7_36 PhL 8.8.11

We have here a slab that runs from z=0 to h, but is infinite in the x and y directions. On each of the two faces, we want no heat loss, so to speak, so we want $\partial_z u(x,y,z) = 0$ at both z=0 and z=h. We have a driving Helmholtz source in the form of a ring around the z axis, radius r_0, and at $z = z_0$. I presume he means that this ring lies inside the slab so that $0 < z_0 < h$. Notice that the entire "boundary" consists of the two surfaces and perhaps some surface at infinity. This is a 3D Neumann Helmholtstatics problem with azimuthal symmetry. Here is our PDE in cylindrical coordinates:

$(-\nabla^2-\lambda)u(r,z) = \delta(r-r_0)\delta(z-z_0)/2\pi r$ $\dim(\lambda) = L^{-2}$ $\dim(u) = L^{-1}$ $\dim(RHS)=L^{-3}$

where I assume the total source charge is 1 unit -- if you integrate the above source over $dV = dz\, r dr d\theta$ you get 1.

$-(1/r)\partial_r(r\partial_r u) - \partial_z^2 u - \lambda u(r,z) = \delta(r-r_0)\delta(z-z_0)/2\pi r$

$-\partial_r(r\partial_r u) - r\,\partial_z^2 u - \lambda r u = \delta(r-r_0)\delta(z-z_0)/2\pi$ $\partial_z u(r,0) = \partial_z u(r,h) = 0$ z in (0,h) r in $(0,\infty)$

We are going to do this in "both ways" , r eigenfunctions and z eigenfunctions. r means ρ of course.

1. Atoms which appear in the homo solution

Our usual first order of business is to separate the homo equation so we can know the nature of solutions away from the distributional source. So try u = R(r)Z(z) :

$-\partial_r(r\partial_r u) - r\,\partial_z^2 u - \lambda r u = 0$

$-\partial_r(r\partial_r RZ) - r\,\partial_z^2 RZ - \lambda r RZ = 0$ $\dim(R) = \dim(Z) = L^{-1}$

$Z\,[-\partial_r(r\partial_r R)] - rR\,\partial_z^2 Z - \lambda r RZ = 0$

$[-\partial_r(r\partial_r R)]/R - r\,\partial_z^2 Z/Z - \lambda r = 0$

$(1/r)[-\partial_r(r\partial_r R)]/R - \partial_z^2 Z/Z - \lambda = 0$

$(1/r)[-\partial_r(r\partial_r R)]/R = \partial_z^2 Z/Z + \lambda \equiv \nu$, our separation constant

$(1/r)[-\partial_r(r\partial_r R)]/R = \nu$ r in $(0,\infty)$
$\partial_z^2 Z/Z = \nu-\lambda$ z in (0,h)

$-\partial_r(r\partial_r R)/R = \nu r$ r in $(0,\infty)$
$\partial_z^2 Z/Z = \nu-\lambda$ z in (0,h)

$-(rR')' = \nu rR$ r in $(0,\infty)$ $L^{-2} - L^{-2} = L^{-2}$ so OK

$Z'' = (\nu-\lambda)Z$ z in (0,h)

$-(rR')' - \nu rR = 0$ r in (0,∞)
$Z'' = (\nu-\lambda)Z$ z in (0,h)

where ' and " have the obvious derivative meanings.

We now focus on the R equation only. We now want to refer to ν as λ in this equation, so we write

$-(rR')' - \lambda rR = 0$ r in (0,∞) $\lambda = \nu$
$Z'' = (\lambda-\lambda_H)Z$ z in (0,h)

The R equation is of the Hankel form with Bessel order = 0. If we let $\lambda = \kappa^2$, solutions are $J_0(\kappa r)$. The corresponding transform is the Hankel with order = 0, which is to say

$$R_h(k) = \int_0^\infty dr\, r\, J_0(kr)\, R(r)$$

$$R(r) = \int_0^\infty dk\, k\, J_0(kr)\, R_h(k)$$

The second just says that the $J_0(kr)$ for positive real k form a complete set for functions or r.

Question: if we transform the R equation from r-space to k-space, what do we expect to see in k-space? If we define

$$LR = -(rR')'$$

Then our eigenvalue problem is

$$(L-r\lambda)u_\lambda = 0 \quad \text{or} \quad (L-r\kappa^2)u_\kappa \quad \text{or} \quad Lu_\kappa = r\kappa^2 u_\kappa$$

for example $LJ_0(\kappa r) = r\kappa^2 J_0(\kappa r)$

So we can write our r-equation like this

$$-(rR')' - \lambda rR = 0$$

$$LR - \lambda rR = 0$$

Now let's install a Hankel expansion for R to get

$$L \int_0^\infty dk\, k\, J_0(kr)\, R_h(k) - \lambda r \int_0^\infty dk\, k\, J_0(kr)\, R_h(k) = 0$$

Move in the L and use $LJ_0(kr) = rk^2 J_0(kr)$ and we then have

$$\int_0^\infty dk\, k\, J_0(kr)\, rk^2\, R_h(k) = \int_0^\infty dk\, k\, J_0(kr)\, \lambda r\, R_h(k) = 0$$

Since the $J_0(kr)$ are a complete set, we conclude that

$$rk^2\, R_h(k) = \lambda r\, R_h(k)$$

$$\lambda = k^2$$

I think I just drove off the road. The radial r equation IS the EV equation. All we know is that we can write the solutions of this homo equation in the form $J_0(kr)$ where $k = \sqrt{\lambda}$.

[You are still on the road. The $J_0(\kappa r)$ ARE the eigenfunctions, you have found them. So this is the TYPE of homo R equation solution away from the ring source.]

Back up. Here where we were:

$-(rR')' - \lambda rR = 0$	r in $(0,\infty)$	$\lambda = \nu$
$Z'' = (\lambda - \lambda_H)Z$	z in (0,h)	

We were looking for homo solution forms away from the source. The Z solutions are of the form

$$Z(z) = \exp(\pm\sqrt{\lambda - \lambda_H}\, z)$$

So we know a little know about the "atoms" that will appear in the solution to this problem.

2. Pause for Comment on Approach

At this point all I have done is write down the atomic forms for the homo Helmholtz equation. But I am not sure that is relevant. It would be if I were doing a "Smythian form" solution attempt. In this method, you would then divide the slab region into perhaps 4 regions: inside and outside the ring, above and below the ring. But I don't want to try a Smythian form solution here, I want to exercise the methods taught in this chapter! So time for a **little review**.

• **In Ex 7.31** we had a "similar" problem with a point source involving two variables on the RHS. In THAT problem, what did I do?

(a) In one method, I noted the homo θ eigenfunctions (Legendre) and applied the Legendre projector. After MUCH fiddling, this led to a 1D Green's equation in r with just a single δ driving that equation. The projector trivially removes the other δ from the full RHS.

(b) In a second method, I instead expanded on these same Legendres and ended up with the same 1D Greens.

In both these methods, I essentially had $u(r,z) = \Sigma_{n=0}^{\infty} a_n(r)\, P_n(z)$ and it was for $a_n(r)$ that I had a 1D Green's equation. I found that $a_n(r) = C\, [r_<^{-1/2}\, J_{n+1/2}(\sqrt{\lambda}\, r_<)][r_>^{-1/2} H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_>)]$ and I went on to find the constant C, which was then $C = i\, [\, (2n+1)/8\,]$. I then "put all the pieces together" to get

$$u(r,z) = \Sigma_{n=0}^{\infty} a_n(r)\, P_n(z) = \Sigma_{n=0}^{\infty} i\,[\,(2n+1)/8\,]\,[r_<^{-1/2}\, J_{n+1/2}(\sqrt{\lambda}\, r_<)][r_>^{-1/2} H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_>)]\; P_n(z)$$

So basically this is the old "partial EF method" of the Laplace world.

• Then **in Ex 7.32**, we had a ring source lying on the cone instead of a point charge on the z axis. In this problem, the LHS is *exactly the same* as in Ex 7.31, so I applied that same Legendre projector to both sides and just reused the LHS result and got this for the 1D Green's function problem:

$$-(r^2 a_n'(r))' - [\lambda r^2 - n(n+1)]\, a_n(r) = \delta(r-r_0)\,(2n+1)\,P_n(\cos\theta_0)/4\pi$$

Using the same Legendre expansion, the solution was

$$a_n(r) = i\,[\,(2n+1)\,P_n(\cos\theta_0)/[8\sqrt{r r_0}\,]\; J_{n+1/2}(\sqrt{\lambda}\, r_<)\, H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_>)$$

so

$$u(r,\theta;\lambda) = (i/[8\sqrt{r r_0}\,])\,\Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, P_n(\cos\theta_0)\,(2n+1)\, J_{n+1/2}(\sqrt{\lambda}\, r_<)\, H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r_>)$$

• So how does that relate to the current problem? Well, the problem is this:

$$(-\nabla^2-\lambda)u(r,z) = \delta(r-r_0)\delta(z-z_0)/2\pi r \quad = \text{ring source}$$

where now $z = r\cos\theta$. Since we are now in cylindrical coordinates, only single r in denominator. So this is going to be a different problem in this sense: since we are no longer in spherical coordinates, we can no longer use the Legendre projector. But that is fine, I see that there are two ways to proceed.

Method A. Learn the r-space atoms (they are $J_0(kr)$ with the Hankel projector), then apply the Hankel projector to both sides, and you will end up with a z-space 1D Green's problem which you then have to go and solve.

Method B. Learn the z-space atoms (they are $\exp(\pm\sqrt{\lambda-\lambda_H}\, z)$, but probably now we have to do something like $\lambda = -\kappa^2$ so we have a transform in z) , then apply this transform projector to both sides, and you should end up with an r-space 1D Green's problem.

Stak in fact asks for a solution by both methods!

Method A. I will do expansion instead of projector, either is OK.

Now maybe we are supposed to just realize this fact, and go back to the inhomo equation (this is cylindrical coordinates!!)

$$-\partial_r(r\partial_r u) - r\,\partial_z^2 u - \lambda_H\, ru = \delta(r-r_0)\delta(z-z_0)/2\pi$$

$$Lu - r\,\partial_z^2 u - \lambda_H\, ru = \delta(r-r_0)\delta(z-z_0)/2\pi \qquad Lu = -\partial_r(r\partial_r u)$$

Now expand on the known radial homo atoms,

$$u(r,z) = \int_0^\infty dk\, J_0(kr)\, U(k,z)$$

$$Lu = \int_0^\infty dk\, LJ_0(kr)\, U(k,z) = \int_0^\infty dk\, rk^2\, J_0(kr)\, U(k,z)$$

Then our PDE becomes

$$\int_0^\infty dk\, rk^2\, J_0(kr)\, U(k,z) - r\int_0^\infty dk\, J_0(kr)\, \partial_z^2 U(k,z) - \lambda_H\, r\int_0^\infty dk\, J_0(kr)\, U(k,z)$$

$$= \delta(z-z_0)/2\pi * r\int_0^\infty dk\, k\, J_0(kr)J_0(kr_0) \quad \text{// } \delta(r-r_0) \text{ from "transforms"}$$

We cancel all the r's,

$$\int_0^\infty dk\, k^2\, J_0(kr)\, U(k,z) - \int_0^\infty dk\, J_0(kr)\, \partial_z^2 U(k,z) - \int_0^\infty dk\, J_0(kr)\, \lambda_H\, U(k,z)$$

$$= \int_0^\infty dk\, k\, J_0(kr)J_0(kr_0)\, \delta(z-z_0)/2\pi$$

Now we appeal to completeness and conclude that

$$k^2\, U(k,z) - \partial_z^2 U(k,z) - \lambda_U\, U(k,z) = J_0(kr_0)\, \delta(z-z_0)/2\pi$$

$$-\partial_z^2 U(k,z) + (k^2 - \lambda_H)\, U(k,z) = [J_0(kr_0)/2\pi]\, \delta(z-z_0)$$

So if we start with our PDE and do a Hankel transform from u(r,z) to U(k,z), we end up with a 1D Green's function problem in variable z. This is what happens if you start with the correct "r-eigenfunctions". This is what happens if you transform the PDE with the Hankel transform of order 0.

So our next business is to identify homo solutions of this 1D thing, and then find the Green's function. Suppose we define

$$\kappa^2 = -[k^2 - \lambda_H]$$

Then homo solutions are

$$e^{\pm i\kappa z} \quad \text{or} \quad \cos(\kappa z) \text{ and } \sin(\kappa z)$$

It seems then that our Green's might have this form

$$U(k,z) = [J_0(kr_0)/2\pi]\cos(\kappa z_<)\,[\, A\sin(\kappa z_>) + B\cos(\kappa z_>)\,]$$

I suspect that A and B will be determined by the usual Green's 1D jump condition and by the BC at z = h. We have already met the BC at z = 0 with our form above. So then the final solution will be

$$u(r,z) = \int_0^\infty dk\, J_0(kr)\, U(k,z)$$

$$= \int_0^\infty dk\, J_0(kr)\, [J_0(kr_0)/2\pi]\cos(\kappa z_<)\, [\, A\sin(\kappa z_>) + B\cos(\kappa z_>)\,] \qquad \kappa^2 = -\,[k^2 - \lambda_H]$$

So this is just an outline of how Method A would go.

Method B. Start again with the full PDF

$$-\partial_r(r\partial_r u) - r\,\partial_z^2 u - \lambda_H\, ru = \delta(r-r_0)\delta(z-z_0)/2\pi$$

$$Lu - r\,\partial_z^2 u - \lambda_H\, ru = \delta(r-r_0)\delta(z-z_0)/2\pi \qquad Lu = -\partial_r(r\partial_r u)$$

Now we do a different starting point expansion. But first it seems a good idea to select z atoms that meet the z BC's. How about

$$f_k(z) = \cos(kz) \qquad f'_k(z) = -k\sin(kz)$$

Thus meets the BC at z = 0 for any k. But we also need

$$f'_k(h) = -k\sin(kh) = 0 \qquad \Rightarrow kh = n\pi \qquad k = n\pi/h$$

So we have a discrete spectrum here and our guys are then

$$f_n(z) = \cos(k_n z) \qquad k_n = n\pi/h$$

This sounds like a Fourier Cosine Series Transform situation. Let's define an angle θ which is NOT the spherical polar angle, it is a special thing for this problem. I make it range θ = 0 to π.

$$\theta = \pi\,(z/h) \qquad k_n z = n\pi/h * h\theta/\pi = n\theta$$

I am not saying anything is even here, our only z range is 0 to h. I think all is OK. So here are the EF's of interest

$$f_n(\theta) = \cos(n\theta)$$

Now we call upon transforms.doc which says

$$f(\theta) = a_0/2 + \Sigma_{n=1}^{\infty} a_n \cos(n\theta) = (1/2)\,\Sigma_{n=0}^{\infty}\, \varepsilon_n\, a_n \cos(n\theta) \qquad \text{// expansion}$$

$$a_n = (2/\pi) \int_0^\pi d\theta\, f(\theta)\cos(n\theta) \qquad \text{where } \varepsilon_n = 2-\delta_{n,0} \qquad \text{// projection}$$

$\int_0^{\pi} d\theta \cos(n\theta)\cos(n'\theta) = (\pi/\varepsilon_n)\,\delta_{nn'}$ // orthogonality

$\Sigma_{n=0}^{+\infty}\varepsilon_n \cos[n(\theta-\theta')] = 2\pi\,\delta(\theta-\theta')$ // completeness

Then we have

$\delta(z-z_0) = \delta(\,h\theta/\pi - h\theta_0/\pi) = (\pi/h)\,\delta(\theta-\theta_0)$.

$\partial_z^2 = ?$ $\quad dz = (h/\pi)\,d\theta \quad dz^2 = (h/\pi)^2\,d\theta^2 \quad \partial_z^2 = (\pi/h)^2\partial_\theta^2$

I think it might be best to apply "the projector" in this problem rather than the expansion. But first, here is our equation transformed from z to θ, and u = u(r,θ) here:

$Lu - r\,(\pi/h)^2\partial_\theta^2 u - \lambda_H\, ru = (\pi/h)\,\delta(r-r_0)\delta(\theta-\theta_0)/2\pi = (1/2h)\delta(r-r_0)\delta(\theta-\theta_0)$ $\quad Lu = -\partial_r(r\partial_r u)$

Now apply $(2/\pi)\int_0^{\pi} d\theta \cos(n\theta)$ to all terms. Acting on u this gives $u_n(r)$ we can call it

$LHS = Lu_n - r\,(\pi/h)^2(2/\pi)\int_0^{\pi} d\theta \cos(n\theta)\,\partial_\theta^2 u - \lambda_H\, ru_n$

$RHS = (2/\pi)\int_0^{\pi} d\theta \cos(n\theta)\,(1/2h)\;\delta(r-r_0)\delta(\theta-\theta_0) = (2/\pi)\,(1/2h)\,\delta(r-r_0)\cos(n\theta_0)$

$= (1/\pi h)\,\delta(r-r_0)\cos(n\theta_0)$

Now we have to do the parts twice in the usual manner. First parts

$\int_0^{\pi} d\theta \cos(n\theta)\,\partial_\theta^2 u = -\int_0^{\pi} d\theta\,\partial_\theta \cos(n\theta)\,\partial_\theta u + [\cos(n\theta)\,\partial_\theta u]\,|\,^{\pi}_0$

But $\partial_\theta u \sim \partial_z u = 0$ at both these endpoints from our BC's, so no parts. so

$\int_0^{\pi} d\theta \cos(n\theta)\,\partial_\theta^2 u = -\int_0^{\pi} d\theta\,\partial_\theta \cos(n\theta)\,\partial_\theta u$

Now second parts on the RHS:

$= -\{\; -\int_0^{\pi} d\theta\,\partial^2_\theta \cos(n\theta)\,u + [\partial_\theta \cos(n\theta)\,u]\,|\,^{\pi}_0\;\}$

But this time the parts vanish because sin(nθ) = 0 at both endpoints. So as expected,

$\int_0^{\pi} d\theta \cos(n\theta)\,\partial_\theta^2 u = +\int_0^{\pi} d\theta\,\partial^2_\theta \cos(n\theta)\,u = -n^2\int_0^{\pi} d\theta \cos(n\theta)\,u$

and then

$$(2/\pi)\ \int_0^{\pi} d\theta\ \cos(n\theta)\ \partial_\theta^2 u = \ -n^2\ (2/\pi)\ \int_0^{\pi} d\theta\ \cos(n\theta)\ u\ \ = \ -(2n^2/\pi)\ u_n$$

So here then is our projected equation,

$$Lu_n - r\ (\pi/h)^2[-(2n^2/\pi)]\ \ u_n - \lambda_H\ r u_n\ \ =\ \ (1/\pi h)\ \delta(r-r_0)\ \cos(n\theta_0)$$

$$Lu_n + r\ (2n^2/h^2)\ u_n - \lambda_H\ r u_n\ \ =\ \ (1/\pi h)\ \cos(n\theta_0)\delta(r-r_0)$$

and this then is our Method B 1D Green's equation in r, where r is really cylindrical ρ. So write it that way

$$-\partial_\rho(\rho\partial_\rho u_n) + \rho\ (2n^2/h^2)\ u_n - \lambda_H\ \rho u_n\ \ =\ \ (1/\pi h)\ \cos(n\theta_0)\delta(\rho-\rho_0)$$

We then consult our Bessel doc to find the homo solutions of this thing:

$$-(\rho u_n')' + \ [(2n^2/h^2)\ -\lambda_H]\rho\ u_n\ \ =\ \ (1/\pi h)\ \cos(n\theta_0)\delta(\rho-\rho_0)$$

We should define

$$\kappa^2 = -\ \ [(2n^2/h^2)\ -\lambda_H] \qquad\qquad k_n = n\pi/h$$

so we have

$$-(\rho u_n')' -\kappa^2\ \rho\ u_n\ \ =\ \ (1/\pi h)\ \cos(n\theta_0)\delta(\rho-\rho_0)$$

and our doc says

$$-(xu')' + \nu^2 u/x - k^2 xu = 0 \qquad\qquad Z_\nu(kx)$$

so I conclude that ν = 0 and our solutions are of the form

$$Z_0(\kappa\rho).$$

So in the usual way, I would conclude that

$$u_n = a_n(\rho) =\ C\ (1/\pi h)\ \cos(n\theta_0)\ J_0(\kappa\rho_<)\ H^{(1)}{}_0(\kappa\rho_>)$$

and assume that we trivially find C, skip for now. Then our solution becomes

$$u(z,\rho) =\ (1/2)\ C\ (1/\pi h)\ \ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ \cos(n\theta)\ \cos(n\theta_0)\ \ J_0(\kappa\rho_<)\ H^{(1)}{}_0(\kappa\rho_>)$$

$$=\ (1/2)\ C\ (1/\pi h)\ \ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ \cos(n\pi z/h)\ \cos(n\pi z_0/h)\ \ J_0(\kappa\rho_<)\ H^{(1)}{}_0(\kappa\rho_>)$$

OK, I don't have to do any more detail here. The result has the correct symmetry. The ring is at z_0 and some ρ_0 .

and we will have to include how our BC's which sort of pass through I think.

I need to pause and write down the general theory here, though Stak did that in his "partial eigenfunction method", but I think this deserves its own doc.