Stakgold chap 7 exercise 7_38 PhL 12.28.11

I select this problem (am sick right now) because lots of the results are given so I will be able to verify my work. Page 291. Our "cylindrical" waveguide has some general cross section area R, not necessarily a disk. It is infinite in both +z and - z directions. This is just a generic Helmholtz problem where the BC is the Dirichlet one that u = 0 on the inner surface of the cylinder.

We try to find a solution in the obvious factored form shown. Here is what happens if we stuff this form into the 3D Helmholtz equation

$(\nabla_2^2 + \partial_z^2 + \lambda)u = 0 \qquad u = Z(z)\,\varphi(x,y)$

This is the usual sign relation Stak uses between ∇^2 and parameter λ. So here we go

$(\nabla_2^2 + \partial_z^2 + \lambda)[\,Z(z)\,\varphi(x,y)] = 0$

$[Z(z)\nabla_2^2\,\varphi(x,y) + \varphi(x,y)\partial_z^2\,Z(z) + \lambda\,Z(z)\,\varphi(x,y)] = 0$

$\nabla_2^2\,\varphi(x,y)/\varphi + Z''(z)/Z + \lambda = 0$

$\nabla_2^2\,\varphi(x,y)/\varphi + \lambda = -\,Z''(z)/Z$

LHS is function of x,y only, RHS is function of z only, so set to constant μ to get

$\nabla_2^2\,\varphi(x,y)/\varphi + \lambda = \mu \quad => \quad \nabla_2^2\,\varphi + (\lambda-\mu)\,\varphi = 0 \quad OR \quad \nabla_2^2\,\varphi + \nu\,\varphi = 0$

$-\,Z''(z)/Z = \mu \quad => \quad -Z'' = \mu Z \quad => -\nabla_2^2\,\varphi = \nu\varphi$

At this point, I have derived all results on page 291. Notice that λ is the original H parameter, but we end up now with a 2D H equation with a new parameter $\nu = \lambda-\mu$, same sign relation. Could have used λ' .

What about BC's? The Z one is that things die off far away I presume, while the φ one is that we must have $\varphi = 0$ on the 2D boundary R.

Part I: Solve the transverse problem first and use those φ_n as the partial eigenfunctions.

In our first approach, imagine that we solve this Laplace EV problem:

$-\nabla_2^2\,\varphi_n = \nu_n\varphi_n \qquad$ with $\varphi_n = 0$ on R

We now install a Green's point source inside the cylinder on the z=0 plane at (x_0, y_0). Our task is to compute the Green's function, fair enough.

My first instinct is to use the "partial eigenfunction method". I will then write

$g(x,y,z|\,x_0,y_0,0) = \Sigma_n\; G_n(z)\,\varphi_n(x,y)$

and try to find the $G_n(z)$. Our Greens PDE is this

$$-(\nabla_2^2 + \partial_z^2 + \lambda)\, g(x,y,z|\, x_0,y_0,\, 0) = \delta(x-x_0)\delta(y-y_0)\delta(z)$$

Now insert the above expansion to get

$$\text{LHS} = -(\nabla_2^2 + \partial_z^2 + \lambda)\, g(x,y,z|\, x_0,y_0,\, 0) \;=\; -(\nabla_2^2 + \partial_z^2 + \lambda)\, \Sigma_n \; G_n(z)\, \varphi_n(x,y)$$

$$= \; -\,\Sigma_n\, (\nabla_2^2 + \partial_z^2 + \lambda)\, G_n(z)\, \varphi_n(x,y)$$

$$= \; -\,\Sigma_n\, (\nabla_2^2\, G_n(z)\, \varphi_n(x,y) \;+\; \partial_z^2\, G_n(z)\, \varphi_n(x,y) \;+ \lambda\, G_n(z)\, \varphi_n(x,y)\,)$$

$$= \; -\,\Sigma_n\, (G_n(z)\, \nabla_2^2\, \varphi_n(x,y) \;+ \varphi_n(x,y)\, \partial_z^2\, G_n(z) + \lambda\, G_n(z)\, \varphi_n(x,y)\,)$$

But we know that

$$\nabla_2^2\, \varphi_n(x,y) \;=\; -\nu_n \varphi_n$$

so we then have

$$\text{LHS} = \; -\,\Sigma_n\, (-\, G_n(z)\, \nu_n \;\varphi_n(x,y) \;+ \varphi_n(x,y)\, \partial_z^2\, G_n(z) + \lambda\, G_n(z)\, \varphi_n(x,y)\,)$$

$$= \; -\,\Sigma_n\, [\, -\, G_n(z)\, \nu_n \;+ \partial_z^2\, G_n(z) + \lambda\, G_n(z)\,] \;\varphi_n(x,y)$$

$$= \; -\,\Sigma_n\, [\, \partial_z^2\, G_n(z) + (\lambda - \nu_n)\, G_n(z)\,] \;\varphi_n(x,y)$$

If we now assume that the φ_n are discrete and normalized, we can then apply $\int_R \; dx\, dy\, \varphi_m^*(x,y)$ to both sides of our equation to get

$$\int_R \; dx\, dy\, \varphi_m^*\,(x,y) \;\; \text{LHS} \;=\; \int_R \; dx\, dy\, \varphi_m^*(x,y) \;\; \text{RHS}$$

$$-\,\Sigma_n\, [\, \partial_z^2\, G_n(z) + (\lambda - \nu_n)\, G_n(z)\,]\, \delta_{n,m} \;=\; \varphi_m^*(x_0,y_0)\delta(z)$$

$$-\,[\, \partial_z^2\, G_m(z) + (\lambda - \nu_m)\, G_m(z)\,] = \; \varphi_m^*(x_0,y_0)\delta(z)$$

$$-\,[\, \partial_z^2 + (\lambda - \nu_m)\,]\, G_m(z) = \; \varphi_m^*(x_0,y_0)\delta(z)$$

We can now regard this last equation as a **1D Green's problem in variable z**. I am now inclined to define

$$-\,k^2 \;= (\lambda - \nu_m)$$

so

$$- [\partial_z^2 - k^2] G_m(z) = \varphi_m^*(x_0,y_0)\delta(z)$$

The homo solution for z > 0 will be

$$G_m(z)_{homo+} = \varphi_m^*(x_0,y_0)e^{-kz}$$

since ∂_z^2 on this produces $+k^2$ and then $[\partial_z^2 - k^2] = 0$ as desired. So here are the left and right solutions:

$$G_m(z)_{homo+} = A\, \varphi_m^*(x_0,y_0)e^{-kz}$$
$$G_m(z)_{homo-} = B\, \varphi_m^*(x_0,y_0)e^{+kz}$$

Continuity of the solution at z = 0 requires B = A. We now integrate the ODE from z = -ε to ε (I always do this from scratch to save the trouble of looking up the sign)

$$- \int_{-\varepsilon}^{\varepsilon} dz [\partial_z^2 - k^2] G_m(z) = \varphi_m^*(x_0,y_0) \int_{-\varepsilon}^{\varepsilon} dz\, \delta(z)$$

$$- \int_{-\varepsilon}^{\varepsilon} dz\, d [\partial_z G_m(z)] = \varphi_m^*(x_0,y_0)$$

$$\partial_z G_m(z)^+ - \partial_z G_m(z)^- = - \varphi_m^*(x_0,y_0)$$

Installing our homo± solutions from above gives

$$A\, \varphi_m^*(x_0,y_0)\, \partial_z\, e^{-kz} - A\, \varphi_m^*(x_0,y_0)\, \partial_z\, e^{+kz} = - \varphi_m^*(x_0,y_0)$$

$$-kA\, \varphi_m^*(x_0,y_0)\, e^{-kz} - k\, A\, \varphi_m^*(x_0,y_0)\, e^{+kz} = - \varphi_m^*(x_0,y_0)$$

$$-kA - k\, A = - 1 \qquad \Rightarrow \qquad 2kA = 1 \qquad \Rightarrow \qquad A = 1/2k$$

Therefore our 1D Green's function is this:

$$G_m(z) = (1/2k)\, \varphi_m^*(x_0,y_0)\, e^{-k|z|}$$

Now recall from above that

$$g(x,y,z|\, x_0,y_0,\, 0) = \Sigma_n\, G_n(z)\, \varphi_n(x,y)$$

$$= \Sigma_n\, (1/2k)\, e^{-k|z|} \varphi_n^*(x_0,y_0)\, \varphi_n(x,y) \qquad\qquad k = \sqrt{\nu_n - \lambda}$$

and this agrees with 7.227.

Now suppose $\lambda < \nu_n$ for all n. Then k is real, and we damp in both z directions, and we get no propagation in the waveguide! In particular, this is true for $\lambda = 0$ which is the Laplace Green's problem and we then just have $k = \sqrt{\nu_n}$. So let's write down this problem and its solution:

Consider the Laplace problem $\lambda=0$ in the cylinder. The Green's function is given by

$$g(x,y,z|\, x_0,y_0,0) \;=\; \Sigma_n\, (1/2k)\, e^{-k|z|}\varphi_n^*(x_0,y_0)\, \varphi_n(x,y) \qquad \text{where} \quad k=\sqrt{\nu_n}$$

where φ_n are the transverse eigenfunctions and ν_n the corresponding EV's,

$$-\nabla_2^2\, \varphi_n \;=\; \nu_n \varphi_n$$

Thus we have a solution to an electrostatics problem I don't think we ever did in Chapter 6.

A key fact that Stak does not mention much is that in the general λ case, if $\lambda >$ some ν_i, then you will get propagation of your wave down the waveguide! So only a finite number of modes will be activated for any finite λ. As you increase λ, you get more modes going, carrying power down the guide.

Part II:

Now we are supposed to redo this entire problem using Z_n eigenfunctions instead of φ_n ones. Our equation is $-Z'' = \mu Z$ which has solution $Z = e^{-ikz}$ so that $Z'' = -k^2 Z$ and $k^2 = \mu$ and we have a continuous spectrum of oscillatory eigenfunctions (see later for why I add the 2π)

$$Z_k(z) \;= e^{-ikz}/2\pi \qquad\qquad k = \text{in } (-\infty,\infty)$$

So now we want to expand G on this set of partial eigenfunctions,

$$g(x,y,z|\, x_0,y_0,0) \;=\; \int_{-\infty}^{\infty} dk \;\; Z_k(z)\, G_k(x,y) \;=\; (1/2\pi) \int_{-\infty}^{\infty} dk \;\; e^{-ikz}\, G_k(x,y)$$

and this is the form Stak shows in page 292. But this is precisely a Fourier Transform and he uses α instead of k as the integration variable, and he has a constant $1/2\pi$ added in. Once again, here is how Stak likes to do his FT's: (page 23)

$$f(x) = (1/2\pi) \int_{-\infty}^{\infty} du\; e^{-iux}\, \hat{f}(u) \qquad\qquad \hat{f}(u) \;=\; \int_{-\infty}^{\infty} dx\; f(x)\, e^{iux}$$

$$f(x) = (1/2\pi) \int_{-\infty}^{\infty} dk\; e^{-ikx}\, \hat{f}(k) \qquad\qquad \hat{f}(k) \;=\; \int_{-\infty}^{\infty} dx\; f(x)\, e^{ikx}$$

$$f(z) = (1/2\pi) \int_{-\infty}^{\infty} d\alpha\; e^{-i\alpha z}\, \hat{f}(\alpha) \qquad\qquad \hat{f}(\alpha) \;=\; \int_{-\infty}^{\infty} dz\; f(z)\, e^{i\alpha z}$$

So taking the last line with f = g, we can then identify $G_k(x,y) \;=\; \hat{g}(k; x,y)$ and

$$g(x,y,z|\, x_0,y_0,0) = (1/2\pi) \int_{-\infty}^{\infty} d\alpha \;\; e^{-i\alpha z}\, \hat{g}(\alpha; x,y) \qquad \text{// agrees with p 292 B.}$$

OK fine. Now we have a continuous spectrum $\mu = \alpha^2$ whereas earlier we had a discrete spectrum doing things the other way with n as the discrete label. So as we did before, we want to jam this expansion above into the full Green's PDE which was

$$-(\nabla_2^2 + \partial_z^2 + \lambda)\, g(x,y,z|\, x_0,y_0,\, 0) = \delta(x-x_0)\delta(y-y_0)\delta(z)$$

$$-(\nabla_2^2 + \partial_z^2 + \lambda)\, (1/2\pi) \int_{-\infty}^{\infty} d\alpha\ e^{-i\alpha z}\, g\hat{}(\alpha;\, x,y) = \delta(x-x_0)\delta(y-y_0)\delta(z)$$

Now write $\partial_z^2\, e^{-i\alpha z} = (-i\alpha)^2\, e^{-i\alpha z} = -\alpha^2\, e^{-i\alpha z}$ so that

$$-\,(1/2\pi) \int_{-\infty}^{\infty} d\alpha\ (\nabla_2^2 + \partial_z^2 + \lambda)\, e^{-i\alpha z}\, g\hat{}(\alpha;\, x,y) = \delta(x-x_0)\delta(y-y_0)\delta(z)$$

$$-\,(1/2\pi) \int_{-\infty}^{\infty} d\alpha\ (\nabla_2^2 - \alpha^2 + \lambda)\, e^{-i\alpha z}\, g\hat{}(\alpha;\, x,y) = \delta(x-x_0)\delta(y-y_0)\delta(z)$$

$$-\,(1/2\pi) \int_{-\infty}^{\infty} d\alpha\ \ e^{-i\alpha z}\, (\nabla_2^2 - \alpha^2 + \lambda)\, g\hat{}(\alpha;\, x,y) = \delta(x-x_0)\delta(y-y_0)\delta(z)$$

Now my inclination is to write $\delta(z) = (1/2\pi) \int_{-\infty}^{\infty} d\alpha\ e^{-i\alpha z}$ on the RHS to get

$$(1/2\pi) \int_{-\infty}^{\infty} d\alpha\ \ e^{-i\alpha z}\, \{ -\, (\nabla_2^2 - \alpha^2 + \lambda)\, g\hat{}(\alpha;\, x,y) \}\ \ = (1/2\pi) \int_{-\infty}^{\infty} d\alpha\ e^{-i\alpha z}\ \delta(x-x_0)\delta(y-y_0)$$

and then we use completeness of the $e^{-i\alpha z}$ to conclude that

$$-\ (\nabla_2^2 - \alpha^2 + \lambda)\, g\hat{}(\alpha;\, x,y)\ = \delta(x-x_0)\delta(y-y_0)$$

and this agrees with p 292 C. Now we need some way to express this 2D Green's function $g\hat{}(\alpha;\, x,y)$.

In the previous part of this problem we determined that

$$-\,\nabla_2^2\, \varphi_n\ = \nu_n \varphi_n \qquad\qquad \text{with } \varphi_n = 0 \text{ on R}$$

determined the set of EF's φ_n and EV's ν_n and let's write

$$(\nabla_2^2 + \nu_n)\varphi_n\ = 0$$

So think of $\nu_n = \lambda - \alpha^2$ and then we can compose our Green's function by the "general formula" for a Helmholtstatics Green's Function. But now I have to locate this thing somewhere. Again, Stak was so busy in his Helmholtz Section in this chapter that he neglected to produce this thing. I think I can conjecture the general result in n dimensions.

Digression: Find the general bilinear form for a Helmholtz Green's function:

Assume the problem is this (where I try to use the usual signs)

$$(-\nabla^2 - k^2)g(x|\xi) = \delta(x-\xi)$$

and assume that this is our set of Laplace EF's

$$-\nabla^2 \varphi_n = \lambda_n \varphi_n$$

Then try

$$g(x|\xi) = -\Sigma_n \varphi_n^*(\xi)\, \varphi_n(x) / (k^2-\lambda_n)$$

To verify this conjecture, apply operator to both sides, where ∇ means ∇_x ,

$$-(\nabla^2 + k^2)g(x|\xi) = (\nabla^2 + k^2)\Sigma_n \varphi_n^*(\xi)\, \varphi_n(x) / (k^2-\lambda_n)$$

$$= \Sigma_n (k^2-\lambda_n)^{-1} \varphi_n^*(\xi)\, (\nabla^2 + k^2)\, \varphi_n(x)$$

$$= \Sigma_n (k^2-\lambda_n)^{-1} \varphi_n^*(\xi)\, (-\lambda_n + k^2)\, \varphi_n(x)$$

$$= \Sigma_n \varphi_n^*(\xi)\, \varphi_n(x) = \delta(x-\xi)$$ // completeness with std normalization

so this verifies our little formula.

Now we want to *apply* this digression result to this equation,

$$-(\nabla_2^2 - \alpha^2 + \lambda)\, g\hat{}(\alpha; x,y) = \delta(x-x_0)\delta(y-y_0)$$

where n=2 and we then identify $k^2 = \lambda-\alpha^2$ and also $\lambda_n = \nu_n$ and we then get

$$g\hat{}(\alpha; x,y \mid x_0,y_0) = -\Sigma_n \varphi_n^*(\xi)\, \varphi_n(x) / (k^2-\lambda_n)$$

$$= -\Sigma_n \varphi_n^*(x_0,y_0)\, \varphi_n(x,y) / (\lambda-\alpha^2-\nu_n)$$

$$= \Sigma_n \varphi_n^*(x_0,y_0)\, \varphi_n(x,y) / (-\lambda+\alpha^2+\nu_n)$$

$$= \Sigma_n \varphi_n^*(x_0,y_0)\, \varphi_n(x,y) / (\nu_n+\alpha^2-\lambda)$$

Now insert this into our continuous z expansion:

$$g(x,y,z|\, x_0,y_0,\, 0) = (1/2\pi) \int_{-\infty}^{\infty} d\alpha\ e^{-i\alpha z}\, g\hat{}(\alpha;\, x,y)$$

$$= (1/2\pi) \int_{-\infty}^{\infty} d\alpha\ e^{-i\alpha z} \Sigma_n\, \varphi_n{}^*(x_0,y_0)\, \varphi_n(x,y) / (\nu_n+\alpha^2-\lambda)$$

$$= (1/2\pi) \Sigma_n \int_{-\infty}^{\infty} d\alpha\ e^{-i\alpha z}\, \varphi_n{}^*(x_0,y_0)\, \varphi_n(x,y) / (\nu_n+\alpha^2-\lambda)$$

and indeed this agrees with Stak's presented result on p 292 D.

We can now compare the two solutions to this problem. The first is a single sum on n, while the second is a "double expansion" as he calls it. Now to make them look the same, we need to show that

$$\int_{-\infty}^{\infty} d\alpha\ e^{-i\alpha z} / (\nu_n+\alpha^2-\lambda) = \pi \exp(-\sqrt{\nu_n-\lambda}\, |z|) / \sqrt{\nu_n-\lambda}$$

So I will do this little "extra credit" integral now. Let $\nu_n - \lambda = a^2$. Then I need to show that

$$\int_{-\infty}^{\infty} d\alpha\ e^{-i\alpha z} / (a^2+\alpha^2) = \pi \exp(-a|z|) / a \qquad (*)$$

This is just an inverse FT one can look up, but I can do it directly faster this way

$$\text{LHS} = \int_{-\infty}^{\infty} d\alpha\ e^{-i\alpha z} / [(\alpha+ia)(\alpha-ia)]$$

If $z > 0$ we close the contour down since then $e^{-i\alpha z} = e^{-i(-100i)z} = e^{-100z}$. We then pick up the residue of the pole at $\sigma = -ia$ so (wrong way makes -)

$$(1/2\pi i)\ \text{LHS} = (1/2\pi i) \int_{-\infty}^{\infty} d\alpha\ e^{-i\alpha z} / [(\alpha+ia)(\alpha-ia)]$$

$$= -\,\text{Res}\{e^{-i\alpha z} / [(\alpha+ia)(\alpha-ia)]\}^{\alpha=-ia} = -\,e^{-i(-ia)z} / (-ia-ia) = e^{-az}/2ai$$

Therefore

$$\text{LHS} = 2\pi i\, e^{-az}/2ai = \pi\, e^{-az}/a \qquad \text{// agrees for } z > 0$$

If $z < 0$, close contour up and then get (right way makes +)

$$(1/2\pi i)\ \text{LHS} = (1/2\pi i) \int_{-\infty}^{\infty} d\alpha\ e^{-i\alpha z} / [(\alpha+ia)(\alpha-ia)]$$

$$= +\,\text{Res}\{e^{-i\alpha z} / [(\alpha+ia)(\alpha-ia)]\}^{\alpha=+ia} = e^{-i(ia)z} / (ia+ia) = e^{az}/2ai$$

Therefore

$$\text{LHS} = 2\pi i\, e^{az}/2ai = \pi\, e^{-a|z|}/a$$

and then the two results can be combined and we have shown (*) above.

It certainly seems better to solve this problem the Part I way.