Stakgold chap 7 exercise 7_39 (p 292) PhL 12.29.11

Part 1. Derive p 292 E. First, recall what 7.217 (p 285) is all about. It is the solution (a H Green's Function) to the H problem (λ) in 2D with the H source near the endpoint of a line on which g = 0. So what happens in this formula if we put the source at $\varphi_0 = \pi$? Then the point H source is off the left end of the half line.

$R^2 = r^2+r_0^2 - 2rr_0\cos(\varphi-\pi) = r^2+r_0^2 - 2rr_0\cos(\pi-\varphi) = r^2+r_0^2 + 2rr_0\cos(\varphi)$

R^{*2}= the same and R=R*

This makes the two integrands identical. Now the integral upper endpoints are

$2\sqrt{rr_0}\cos[(\varphi-\pi)/2] = 2\sqrt{rr_0}\cos[\pi/2-\varphi/2] = 2\sqrt{rr_0}\sin(\varphi/2) \equiv \alpha$ // first integral

$2\sqrt{rr_0}\cos[(\varphi+\pi)/2] = 2\sqrt{rr_0}\cos[\pi/2+\varphi/2] = -2\sqrt{rr_0}\sin(\varphi/2) = -\alpha$// 2nd integral

Since the integrands are the same, the two -∞ lower endpoints cancel and we end up with

$$g = (1/4\pi)\int_{-\alpha}^{\alpha} dv\, \exp(i\sqrt{\lambda}\sqrt{R^2+v^2})/\sqrt{R^2+v^2}$$

Since R does not depend on v, the integrand is even in v, so it is then

$$g = (1/2\pi)\int_{0}^{\alpha} dv\, \exp(i\sqrt{\lambda}\sqrt{R^2+v^2})/\sqrt{R^2+v^2}$$

and this gives p 292 E, so we are off and running!

Part 2. Now I need to **clarify the meaning of I**. At this point in the text, Stak really has not mentioned the notion of I, so I need to move ahead to page 300-1 and review a bit. At this point I derived the equations p 300A, p 301A,B,C and E and put these derivations into the raw3 notes. I conclude there that -I is the induced source which creates u_s, that is what p 301B says.

Part 3. Derive p 292 F. Now back to our problem. Think of the half line sitting there. On the upper surface what is the outpointing normal? Well we really want dg/dy and we have $y = r\sin\theta \approx r\theta$ so dy = rdθ and that produces the first term in p 292. On the bottom side we really want dg/(-dy) since the normal points down. IN terms of magnitude we still have dy = rdθ and so we get dg/(-dy) = - $(1/r)\partial_\theta g$ for the second term, AND we have to set θ = 2π inside g. So we are now happy with p 292 F.

Part 4. Derive 7.228. Now comes the task of actually computing the ∂_φ derivatives of g. Notice that φ appears lots of places!

Consider this more general problem

$$g = \int_0^{\alpha(\varphi)} dv\, f(v,\varphi)$$

There are two contributions in $\partial_\varphi g$. We really want

$$[g(\varphi+d\varphi)-g(\varphi)] / d\varphi = \{ \int_0^{\alpha(\varphi+d\varphi)} dv\, f(v,\varphi+d\varphi) - \int_0^{\alpha(\varphi)} dv\, f(v,\varphi) \} / d\varphi$$

$$= \{ \int_0^{\alpha(\varphi)} dv\, f(v,\varphi+d\varphi) + \int_{\alpha(\varphi)}^{\alpha(\varphi+d\varphi)} dv\, f(v,\varphi+d\varphi) - \int_0^{\alpha(\varphi)} dv\, f(v,\varphi) \} / d\varphi$$

The first and third terms give this

$$\int_0^{\alpha(\varphi)} dv\, \partial_\varphi f(v,\varphi)$$

whereas the second term gives this

$$\approx f(v=\alpha(\varphi), \varphi)\, v|^{up}_{dn} = f(v=\alpha(\varphi), \varphi)\, \{ \alpha(\varphi+d\varphi)-\alpha(\varphi)\} = f(v=\alpha(\varphi), \varphi)\, \partial_\varphi \alpha\, d\varphi$$

and then this divided by $d\varphi$, so this second term gives

$$\partial_\varphi \alpha\ f(v=\alpha(\varphi),\varphi)$$

where we have dropped a second order term. Thus

$$g = \int_0^{\alpha(\varphi)} dv\, f(v,\varphi)$$

$$\partial_\varphi g = \partial_\varphi \alpha\ f(\alpha(\varphi),\varphi) + \int_0^{\alpha(\varphi)} dv\, \partial_\varphi f(v,\varphi)$$

Now I see some work ahead! Start with the second term:

$$f(v,\varphi) = (1/2\pi)\exp(i\sqrt{\lambda}\ Q)/Q \qquad Q^2 = (R^2+v^2) \qquad R^2(\varphi) = r^2+r_0^2 + 2rr_0\cos(\varphi)$$

$$2RdR = -\,2rr_0\sin(\varphi)d\varphi$$

$$\partial_\varphi R = -\,2rr_0\sin(\varphi)/(2R) = rr_0\sin(\varphi)/R$$

$$2QdQ = 2RdR = 2R(\partial_\varphi R)d\varphi \qquad => \qquad QdQ = R(\partial_\varphi R)d\varphi$$

$$\partial_\varphi Q = R(\partial_\varphi R)/Q = (R/Q)\, rr_0\sin(\varphi)/R = (1/Q)\, rr_0\sin(\varphi)$$

Then next we have

$$\partial_\varphi f = (1/2\pi)\ \{ Q(\varphi)\, i\sqrt{\lambda}\, (\partial_\varphi Q) \exp(i\sqrt{\lambda}\, Q(\varphi) - \exp(i\sqrt{\lambda}\, Q(\varphi)\, (\partial_\varphi Q)\} / Q^2$$

$$= (1/2\pi) \exp(i\sqrt{\lambda}\, Q(\varphi))\, (\partial_\varphi Q)\ \{ Q(\varphi)\, i\sqrt{\lambda} - 1\} / Q^2$$

$$= (1/2\pi) \exp(i\sqrt{\lambda}\, Q(\varphi))\, (1/Q)\, rr_0\sin(\varphi)\ \{ Q(\varphi)\, i\sqrt{\lambda} - 1\} / Q^2$$

$$= (1/2\pi) \exp(i\sqrt{\lambda}\, Q(\varphi))\, rr_0\sin(\varphi)\ \{ Q(\varphi)\, i\sqrt{\lambda} - 1\} / Q^3$$

Now one contribution to our desired result is this

$$\int_0^{\alpha(\varphi)} dv\ \partial_\varphi f(v,\varphi) = (1/2\pi)\, rr_0\sin(\varphi) \int_0^{\alpha(\varphi)} dv \exp(i\sqrt{\lambda}\, Q(\varphi))\ \{ Q(\varphi)\, i\sqrt{\lambda} - 1\} / Q^3$$

Barring weirdness, the integral is some finite thing, and then as we take $\varphi = 0$ or $\varphi = 2\pi$ in our $\partial_\varphi g$ this term will vanish since in both cases sinφ 0. Therefore we will get not contribution from this term!!

So our total result will be just the first term, for either of the two terms of interest,

$$\partial_\varphi g = \partial_\varphi \alpha\ f(\alpha(\varphi),\varphi) \qquad \alpha = 2\sqrt{rr_0}\ \sin(\varphi/2)$$

Now go compute

$$\partial_\varphi \alpha = 2\sqrt{rr_0}\cos(\varphi/2)\,(1/2) = \sqrt{rr_0}\cos(\varphi/2)$$

Then our total answer will be this:

$$I(r) = (1/r)\,[\partial_\varphi g(0) - \partial_\varphi g(2\pi)]$$

$$= (1/r)\,[\partial_\varphi \alpha(0)\ f(\alpha(0),0) - \partial_\varphi \alpha(2\pi)\ f(\alpha(2\pi),2\pi)]$$

Now what are these four pieces?

$$\partial_\varphi \alpha(0) = \sqrt{rr_0} \qquad \partial_\varphi \alpha(2\pi) = -\sqrt{rr_0}$$

Recall now that

$$f(v,\varphi) = (1/2\pi)\exp(i\sqrt{\lambda}\ Q)/Q \qquad Q^2 = (R^2+v^2) \qquad R^2(\varphi) = r^2+r_0^2 + 2rr_0\cos(\varphi)$$

Then we know that

$f(v,0) = (1/2\pi)\exp(i\sqrt{\lambda}\ Q)/Q \qquad Q^2 = (R^2+v^2) \qquad R^2(0) = (r+r_0)^2$

$\alpha(0) = 0$ so then

$f(\alpha(0),0) = f(0,0) = (1/2\pi)\exp(i\sqrt{\lambda}\ Q)/Q \qquad Q^2 = R^2 \qquad R^2(0) = (r+r_0)^2$

$= (1/2\pi)\exp(i\sqrt{\lambda}\ R)/R \qquad R = (r+r_0)$

Next,

$f(v,2\pi) = (1/2\pi)\exp(i\sqrt{\lambda}\ Q)/Q \qquad Q^2 = (R^2+v^2) \qquad R^2(0) = (r+r_0)^2$

$\alpha(2\pi) = 0$ so then

$f(\alpha(2\pi),2\pi) = f(0,2\pi) = (1/2\pi)\exp(i\sqrt{\lambda}\ Q)/Q \qquad Q^2 = R^2 \qquad R^2(0) = (r+r_0)^2$

$= (1/2\pi)\exp(i\sqrt{\lambda}\ R)/R \qquad R = (r+r_0)$ // same thing!

So

$I(r) = (1/r)\ [\partial_\varphi\alpha(0)\ \ f(\alpha(0),0) - \partial_\varphi\alpha(2\pi)\ \ f(\alpha(2\pi),2\pi)]$

$= (1/r)\ [\sqrt{rr_0}\ \ f(\alpha(0),0) + \sqrt{rr_0}\ \ f(\alpha(2\pi),2\pi)]$

$= (\sqrt{rr_0}/r)\ [\ f(\alpha(0),0) + \ f(\alpha(2\pi),2\pi)]$

$= 2(\sqrt{rr_0}/r)\ [\ f(\alpha(0),0)]$

$= 2(\sqrt{rr_0}/r)\ (1/2\pi)\exp(i\sqrt{\lambda}\ R)/R \qquad R = (r+r_0)$

$= (1/\pi)(\sqrt{r_0/r}\ )\ \exp(i\sqrt{\lambda}\ [r+r_0])/[r+r_0]$ // agrees with 7.228 p 293

This is really the negative of the induced source density on the half line. In this Green's problem the Helmholtz source is + 1, and as in electrostatics we expect to have a negative induced density. Certainly that would be the case in the electrostatics limit $\lambda = 0$.

Part 5. I would now like to understand better what this problem is really about.

Stak presents this source + half-line problem inside his self-contained Helmholtz World, so we don't get a clear picture of the time-domain starting point, so I will try to make that up now.

Assume a source doing $e^{-i\omega t}$ time dependence as on page 259. Imagine the true wave equation with unit velocity

$(\partial_t^2 - \nabla^2)\ [\ e^{-i\omega t}\ u(r)\ ] = e^{-i\omega t}\delta(r-r_0)$

$((-i\omega)^2 - \nabla^2)\,[\, e^{-i\omega t}\, u(r)\,] = e^{-i\omega t}\delta(r-r_0)$

$(-\omega^2 - \nabla^2)\, u(r) = \delta(r-r_0)$

$(-\lambda - \nabla^2)\, u(r) = \delta(r-r_0)$

So here we arrive at our H equation driven by a source and $\lambda = \omega^2$.

So the basic idea here is to say that

$u(r,t) = e^{-i\omega t}u(r)$ // the monochromatic assumption

Then everything is going to have this time dependence. If we look at 7.250, we can stick $e^{-i\omega t}$ onto u_i on the LHS and onto I on the right. Therefore, I think this is what our induced source looks like

induced source $= -Ie^{-i\omega t} = -\,(1/\pi)(\sqrt{r_0/r}\,)\, e^{-i\omega t} \exp(i\omega\,[r+r_0])/[r+r_0]$

$= -\,(1/\pi)(\sqrt{r_0/r}\,) \exp(i\omega\,[r+r_0-t])/[r+r_0]$

The range of r is r in $(0,\infty)$ and things blow up in the usual quadratic way at r=0. Otherwise we see that the current has phrase fronts running down the wire at the expected velocity of 1.

I am hard pressed to come up with the electromagnetic physical situation to model this problem. What would the field u be? If it is E_x then do we really have $E_x = 0$ on the wire? If that were so, we would have no "current" in the wire since $J = \sigma E$. Maybe u = V, the electric potential, and maybe it does solve a wave equation (I think it does), and then V=0 means the wire is grounded. It must have infinite conductivity so it can remain at V = 0 and yet support the above surface current.

I think Stak would say it is not really his job to apply all his stuff to E&M theory or any other wave theory. He is just providing a set of tools, and there certainly are a lot of tools. Solving a generic Helmholtz equation with BC is one of those tools and that is what have been doing in the above problem.

Perhaps if we think of low frequency E&M, and then the source is a sort of pulsating charge monopole and then σ on the half-wire does vary in time and if it varies then it must move around on the wire and thus there must be current in the wire. If $\sigma(r,t) = e^{-i\omega t}\,\sigma(r)$, then $\partial_t\sigma * C$ = current $= -i\omega\, e^{-i\omega t}\,\sigma(r)$ and then it looks like a "current" proportional to ω on the wire.