Exercise 7.46 Redo Scattering Theory for Neumann BC PhL 12.30.11

Scattering theory starts Section 7.15 p 299. Luckily I just recently worked through details of the first part of this section and I can maybe copy paste and edit what I did there.

Comment: I get rolling on this problem, but give up after 4 pages only because Stak gives me no waypoints so I can check my progress. Since in OTHER problems he DOES give waypoints, I would do better doing THOSE problems. (And maybe I can find more typos.)

Derive Neumann p 300A and 7.247. The first step is to state a modified 7.245. Neumann says $\partial_n u_i = -\partial_n u_s$ on the surface, so we write

$\nabla^2 u_s + \lambda u_s = 0$ x in R, the exterior region $u_s = u_s(x)$ $\partial_n u_i = -\partial_n u_s$ on σ

As I read my notes and do the same processing steps, everything goes along exactly as before for quite a few steps since the BC's are not used yet. We end up then with this same result for p 300A,

$$u_s(\xi) = \int_\sigma dS\, [u_s(\partial E/\partial n) - E(\partial u_s/\partial n)]$$

As before, we now set $u_s = u - u_i$ so that the above reads

$$u_s(\xi) = \int_\sigma dS\, [(u-u_i)(\partial E/\partial n) - E\{(\partial u/\partial n) - (\partial u_i/\partial n)\}]$$

$$= \int_\sigma dS\, [(u-u_i)(\partial E/\partial n) + E(\partial u_i/\partial n)\}]$$ // since $\partial_n u = 0$ on σ, different!

$$= \int_\sigma dS\, [-u_i(\partial E/\partial n) + E(\partial u_i/\partial n) + u(\partial E/\partial n)]$$

$$= \int_\sigma dS\, [E(\partial u_i/\partial n) - u_i(\partial E/\partial n) + u(\partial E/\partial n)]$$

$$= \int_\sigma dS\, [E(\partial u_i/\partial n) - u_i(\partial E/\partial n)] + \int_\sigma dS\, u(\partial E/\partial n)$$ // the Neumann 7.247

The first term is the same as it was for Dirichlet, but now the last term is different!

Derive Neumann p 301A. This result is unchanged because no step in the derivation uses the BC!

Derive Neumann p 301B . Well, combine 301A with 7.247 Neumann and get this new result

$u_s(\xi) = + \int_\sigma dS\, u\,(\partial E/\partial n)$ for ξ inside R // this is Neumann p 301B

$$u_s(\xi) = \quad + \int_\sigma dS_x\, u(x)\, \partial_{nx} E(x|\xi)$$

Swapping x and ξ as he does, this can be rewritten as

$$u_s(x) = \quad + \int_\sigma dS_\xi\, u(\xi)\, \partial_{n\xi} E(\xi|x)$$

But maybe need to write it this way

$$u(x) = u_i(x) + \int_\sigma dS_\xi\, u(\xi)\, \partial_{n\xi} E(\xi|x) \qquad \text{// this is Neumann 7.249}$$

It seems better now to take u as the unknown function rather than $u_{s\geq}$

If we now take x and approach s on the surface, I am not sure what happens. As I look at my potential theory notes, I think this is the dipole layer situation, but where the dipole layer density is u ! Also, E has the expo in its numerator since we are now Helmholtz, but I don't think that will affect much. As I look at page 19 of my Chap 6 meta notes, I see this:

$$v(\mathbf{x}) = \int_\sigma dS_\xi\, b(\xi)\, k(x,\xi) = \int_\sigma dS_\xi\, b(\xi)\, \partial_{n\xi} E(x|\xi)$$

where $k(x,\xi) = \cos(x-\xi,n)/[4\pi|\mathbf{x}-\xi|^2] = \partial_{n\xi} E(x|\xi)$ // see Noted added below

$$\lim_{x\to s} v \quad = \pm\, b(s)/2 + \int_\sigma dS_\xi\, b(\xi)\, k(s,\xi) \qquad \text{// 6.44 becomes 6.46}$$

so applying that here and assuming the phase is OK we would get

$$u(s) = u_i(x) + u(s)/2 + \int_\sigma dS_\xi\, u(\xi)\, \partial_{n\xi} E(s|\xi)$$

or

$$u(s)/2 = u_i(x) + \int_\sigma dS_\xi\, u(\xi)\, \partial_{n\xi} E(s|\xi) \qquad \text{// this is Neumann 7.250}$$

I have picked the plus sign because I think we approach from the outside of the surface in the usual manner. This is the Fred 2 form on page 195 of volume 1 with μ = 1/2 and f = $-u_i$. So the result is that in the Neumann version of things, we get a Fred 2 integral equation directly for our unknown total amplitude u . This is a very different result, but Stak on page 327 in 7.309 gives a Fred 2 version of the Dirichlet world equation for I which at least has some similarity to the above! Since u_i is known, we do in theory have full knowledge of the inhomo driving function f = f = $-u_i$ in the above and of the kernel.

Plane Wave Excitation

We continue along now to see what other changes there are for Neumann! Well, change here is pretty serious. We must now start with our Neumann version of 7.249 which from above is

$$u_s(x) = \quad + \int_\sigma dS_\xi \, u(\xi) \, \partial_{n\xi} E(\xi|x)$$

Now we need a large-x limit of $\partial_{n\xi} E(\xi|x)$. I do this for Laplace on page 19 of Ch 6 meta notes, but now we have the extra phase to deal with which adds an extra term. Using my picture in those Ch 6 notes, we find that if $R = |x-\xi|$, then $\partial_n R = \cos\theta = \cos(x-\xi, n)$ which for very large x will be $\cos(x,n)$. So if we have

$$E = \exp(i\omega |x-\xi|)/ \ 4\pi|x-\xi| \ = e^{i\omega R}/4\pi R$$

then

$$\partial_n E \ = [4\pi R \, e^{i\omega R} \, i\omega \partial_n R - e^{i\omega R} \, 4\pi \partial_n R \ ] \ / \ (4\pi R)^2$$

$$= 4\pi \, e^{i\omega R} \ [R \, i\omega \partial_n R - \partial_n R \ ] \ / \ (4\pi R)^2$$

$$= 4\pi \, e^{i\omega R} \, (\partial_n R) \ [R \, i\omega - 1 \ ] \ / \ (4\pi R)^2$$

$$= e^{i\omega R} \, (\partial_n R) \ [R \, i\omega - 1 \ ] \ / \ (4\pi R^2)$$

$$= e^{i\omega R} \, (\partial_n R) \ [i\omega - 1/R \ ] \ / \ (4\pi R)$$

$$\approx \ e^{i\omega R} \cos(x,n) \ [i\omega - 1/|x| \ ] \ / \ (4\pi |x|)$$

and now we have to assume that $\omega >> 1/|x|$ or $|x| >> 1/\omega$ which is fine for large ω or fixed ε, and then

$$\partial_n E \ \approx e^{i\omega |x|} \cos(x,n) \ [i\omega] \ / \ (4\pi |x|)$$

$$\approx \ e^{i\omega |x|} \ / \ (4\pi |x|) \quad * \quad i\omega \cos(x,n)$$

Now we need to install this into Neumann 7.249 which says from above,

$$u(x) = \ u_i(x) + \int_\sigma dS_\xi \, u(\xi) \, \partial_{n\xi} E(\xi|x)$$

$$= u_i(x) + \int_\sigma dS_\xi \, u(\xi) \, e^{i\omega |x|} \ / \ (4\pi |x|) \quad * \quad i\omega \cos(x,n)$$

$$= \ u_i(x) \quad + \ e^{i\omega |x|} \ / \ (4\pi |x|) \ \{ \ i\omega \int_\sigma dS_\xi \, u(\xi) \cos(x,n) \ \}$$

Therefore our conclusion would be

$$u_s(x) \ \approx \ e^{i\omega |x|} \ / \ (4\pi |x|) \ \{ \ i\omega \int_\sigma dS_\xi \, u(\xi) \cos(x,n) \ \}$$

and we then have a scattering amplitude of the form

$$A = \{ i\omega \int_{\sigma} dS_{\xi}\, u(\xi) \cos(x,n) \}/4\pi$$

But somehow I am now missing information about the incoming plane wave! Maybe now we have to make the assumption that $u_i >> u_s$ and then we could say

$$A(\alpha,\beta) = \{ i\omega \int_{\sigma} dS_{\xi}\, u_i(\xi) \cos(x,n) \}/4\pi$$ // this is Neumann 7.253

Stopping. OK, I think it is now time to stop on this problem! I am hanging way out on many thin limbs and Stak is giving me no waypoints whatsoever. I will have to find this problem treated elsewhere to know how it all comes out. I could finish the problem, but I feel it is a waste of time without a waypoint for the above expression for amplitude A. I gave the problem due diligence!