Stakgold Exercise 7.49 (p 327) PhL 12.31.11

This is the 3D problem of a point charge off the edge of a half plane. Charge is at x = -a as usual. Helmholtz of course is this, where right off the bat he uses $\lambda = -k^2$,

$$(-\partial_x^2-\partial_y^2-\partial_z^2+k^2)v = \delta(x+a)\delta(y)\delta(z) \qquad (*)$$

We are Dirichlet BC so u = 0 on the half plane. This half plane emerges from the plane of paper with z pointing to the viewer, so the two sides of the half plane are at $y = \pm\varepsilon$. That is to say, y is up in our little 2D projected picture with the half plane appearing as a half line on the right $x \geq 0$. I therefore agree with all parts of p 327C for the problem setup.

We are instructed to do a z FT with α conjugate to z. I will use the Stak p 23 FT form

$$\hat{v}(x,y;\alpha) = \int_{-\infty}^{\infty} dz\, e^{i\alpha z}\, v(x,y,z) \qquad v(x,y,z) = (1/2\pi) \int_{-\infty}^{\infty} d\alpha\, e^{-i\alpha z}\, \hat{v}(x,y;\alpha)$$

So apply the projection operator $\int_{-\infty}^{\infty} dz\, e^{i\alpha z}$ to both sides of (*) to get

$$\int_{-\infty}^{\infty} dz\, e^{i\alpha z}\, (-\partial_x^2-\partial_y^2-\partial_z^2+k^2)\, v(x,y,z) = \int_{-\infty}^{\infty} dz\, e^{i\alpha z}\, \delta(x+a)\delta(y)\delta(z)$$

On the LHS we do double parts with the ∂_z^2 term so that $-\partial_z^2 \rightarrow -(i\alpha)^2 = \alpha^2$ and we through the actual parts away at $z = \pm\infty$. We then have

$$(-\partial_x^2-\partial_y^2+\alpha^2+k^2) \int_{-\infty}^{\infty} dz\, e^{i\alpha z}\, v(x,y,z) = \delta(x+a)\delta(y)$$

$$(-\partial_x^2-\partial_y^2+\alpha^2+k^2)\, \hat{v}(x,y;\alpha) = \delta(x+a)\delta(y)$$

$$(-\partial_x^2-\partial_y^2+\kappa^2)\, \hat{v}(x,y;\alpha) = \delta(x+a)\delta(y) \qquad \kappa^2 = \alpha^2+k^2$$

This is a 2D Green's function problem, but what are the BC's? Well

$$\hat{v}(x,\pm\varepsilon;\alpha) = \int_{-\infty}^{\infty} dz\, e^{i\alpha z}\, v(x,\pm\varepsilon,z) = \int_{-\infty}^{\infty} dz\, e^{i\alpha z}\, 0 = 0$$

where I don't think there are technical problems with the integral. So our 2D Green's problem is:

$$(-\partial_x^2-\partial_y^2+\kappa^2)\, \hat{v}(x,y;\alpha) = \delta(x+a)\delta(y) \qquad \kappa^2 = \alpha^2+k^2$$

$$\hat{v}(x,\pm\varepsilon;\alpha) = 0$$

As Stak points out, this is the same as 7.295 (321) but with κ for k and with v^ for w. This was the setup for the 2D half line problem, or the 3D problem with the full line source. This is in the Weiner-Hopf section of the text I suddenly realize! This in fact is WH example 2, and in ω-space we get to (7.301)

which is the ω-space equation we need to "segregate". This required lots of work. Remember we are doing the "integral equation method" with WH. When the dust all settles, we find the current on the wire is 7.308 on p 326. WE can use instead 7.307R if we want the result in k instead of λ.

Although Stak did not do the last step, that step would be to insert this current I(ξ) into 7.249 (301) to get u_s and we would call this $v\hat{}_s(x,y;\alpha)$. We would then do inverse FT for $v_s(x,y,z)$ and then since v_i = E_3(source at point x = -a) we would know the solution v. That outlines the method.

Stak is only asking us here about the current, so we can forget the rest of the outline above. We therefore know this fact: (this is the total current on both sides of half-plane

$$I\hat{}(x,0;\alpha) = a^{1/2} \exp(-(k^2+\alpha^2)^{1/2}(x+a) / [\ \pi x^{1/2}(x+a)\ ]$$

But in our current problem, this is not the current, it is the z FT of the current, so we now have to do this:

$$I(x,0,z) = (1/2\pi) \int_{-\infty}^{\infty} d\alpha\, e^{-i\alpha z}\, I\hat{}(x,0;\alpha)$$

$$= (1/2\pi) \int_{-\infty}^{\infty} d\alpha\, e^{-i\alpha z}\, a^{1/2} \exp(-(k^2+\alpha^2)^{1/2}(x+a)) / [\ \pi x^{1/2}(x+a)\ ]$$

$$= a^{1/2}\,(1/2\pi^2)\,[\ x^{1/2}(x+a)\ ]^{-1} \int_{-\infty}^{\infty} d\alpha\, e^{-i\alpha z} \exp[-(k^2+\alpha^2)^{1/2}(x+a)]$$

We now need this integral

$$I = \int_{-\infty}^{\infty} d\alpha\, e^{-i\alpha z} \exp[-(k^2+\alpha^2)^{1/2}(x+a)]$$

Let x+a = c so this says $I = \int_{-\infty}^{\infty} d\alpha\, e^{-i\alpha z} \exp[-c(\alpha^2+k^2)^{1/2}]$ with c > 0. The integrand is even in α, so this integral is twice the Fourier Cosine Transform of the function $\exp[-c(\alpha^2+k^2)^{1/2}]$. This does not appear in my simple Schaum table. Let's try Bateman IT 1 from which I now quote:

Exponential functions (cont'd)

	$f(x)$	$g(y) = \int_0^\infty f(x)\cos(xy)\,dx \qquad y > 0$
(26)	$\exp[-\beta(x^2+\alpha^2)^{1/2}]$ Re $\alpha > 0$, Re $\beta > 0$	$\alpha\beta(y^2+\beta^2)^{-1/2} K_1[\alpha(y^2+\beta^2)^{1/2}]$

So first I rewrite my integral this way

$$I = \int_{-\infty}^{\infty} dx\, e^{-ixz} \exp[-c(x^2+k^2)^{1/2}] = 2 \int_{0}^{\infty} dx \cos(xz) \exp[-c(x^2+k^2)^{1/2}] = 2\, g(z)$$

So set $\beta = c$ and $\alpha = k$ and then we seem to have

$$I = 2\, kc\, (z^2+c^2)^{-1/2}\, K_1[\, k\, (z^2+c^2)^{1/2}]$$

Now I can replace $c = x+a$ to get

$$I = 2\, k(x+a)\, (z^2+(x+a)^2)^{-1/2}\, K_1[\, k\, (z^2+(x+a)^2)^{1/2}]$$

And finally we look at K properties from AS2010 p 251

10.29.3 $I_0'(z) = I_1(z), \quad K_0'(z) = -K_1(z).$

so that I then have

$$I = -\,2\, k(x+a)\, (z^2+(x+a)^2)^{-1/2}\, K_0'[\, k\, (z^2+(x+a)^2)^{1/2}]$$

MY answer is then

$$I(x,0,z) = a^{1/2}\, (1/2\pi^2)\, [\; x^{1/2}(x+a)\,]^{-1}\; I$$

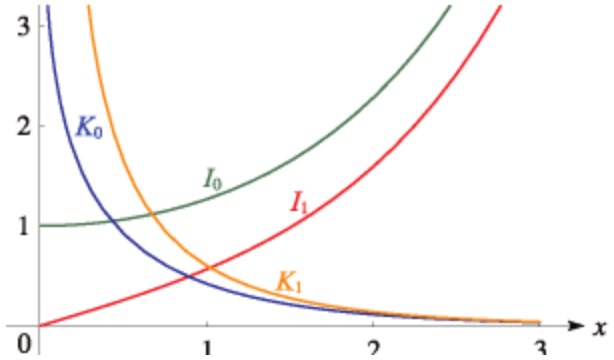
$$= -\,a^{1/2}\, (1/2\pi^2)\, [\; x^{1/2}(x+a)\,]^{-1}\; 2\, k(x+a)\, (z^2+(x+a)^2)^{-1/2}\, K_0'[\, k\, (z^2+(x+a)^2)^{1/2}]$$

$$= -\,(\sqrt{a}\, k/\pi^2)\, (1/\sqrt{x}\,)\, (z^2+(x+a)^2)^{-1/2}\, K_0'[\, k\, (z^2+(x+a)^2)^{1/2}]$$

and this agrees exactly with p 328A, hurray!

What is this result saying? as you approach the edge, you get the usual $1/\sqrt{x}$ for any z! But surely the other factors show decay in the z direction. Here is K_1

See Figures 10.26.1–10.26.6.



so as |z| increases, the K_0' function decays very fast and is helped by the other z factor in this decay.

Now what is the DC limit? k = 0, so need small arg behavior of K_1

10.30.2 $K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)(\frac{1}{2}z)^{-\nu}, \qquad \Re\nu > 0,$

Therefore

$K_1(z) \sim (1/2)\,(z/2)^{-1}$

$-K_0'[\,k\,(z^2+(x+a)^2)^{1/2}] \;\sim\; (1/2)\,[(k/2)\,(z^2+(x+a)^2)^{1/2}]^{-1} = (1/2)\,(2/k)\,(z^2+(x+a)^2)^{-1/2}$

$I(x,0,z) = -\,(\sqrt{a}\,k/\pi^2)\,(1/\sqrt{x}\,)\,(z^2+(x+a)^2)^{-1/2}\,K_0'[\,k\,(z^2+(x+a)^2)^{1/2}]$

$= (\sqrt{a}\,k/\pi^2)\,(1/\sqrt{x}\,)\,(z^2+(x+a)^2)^{-1/2}\;\;(1/k)\,(z^2+(x+a)^2)^{-1/2}$

$= (\sqrt{a}\,/\pi^2)\,(1/\sqrt{x}\,)\;(z^2+(x+a)^2)^{-1}$

Now this is I and Stak says that σ = -I. I will verify this momentarily against my own notes, but first let's assume it is true, then my result says

$\sigma = -\;(\sqrt{a}\,/\pi^2)\,(1/\sqrt{x}\,)\;(z^2+(x+a)^2)^{-1}$

and this then is the charge induced on a half plane due to a point charge of +1 sitting off at x = -a. So I have exposed another Stak typo!! He has k in the numerator but of course k = 0 in our limit so that should not be there. Was this corrected in Stak's later edition? No it was not!

I just updated my errata doc with the last two errors I found (p 312 and 328, both not fixed in 2000) and added a comment about our email exchange.

Regarding I, this is introduced by Stak on page 301. From my memory of Ch6 and more directly from the minus sign in 7.249 if is clear that $I \rightarrow -\sigma_{induced}$ in the DC limit!

So we have this result:

$\sigma(x,z) = -\;(\sqrt{a}\,/\pi^2)\,(1/\sqrt{x}\,)\;(z^2+(x+a)^2)^{-1}$

The following we know must be true

$$\int_{-\infty}^{\infty} dz \int_{0}^{\infty} dx \;\; (\sqrt{a}\,/\pi^2)\,(1/\sqrt{x}\,)\;(z^2+(x+a)^2)^{-1} \;=\; +1$$

Stak asks us to do the integral and verify this fact. So we need to show that

$$K \equiv \int_{0}^{\infty} dx \;(1/\sqrt{x}\,) \;\int_{-\infty}^{\infty} dz \;(z^2+(x+a)^2)^{-1} \;=\; \pi^2/\sqrt{a}$$

where I will try to do the z integral first. Let $(x+a)^2 = b^2$ so we write

$$K = \int_0^\infty dx \ (1/\sqrt{x}\,)\, I$$

$$I = \int_{-\infty}^{\infty} dz \ 1/(z^2+b^2) = \int_{-\infty}^{\infty} dz \ 1/[(z+ib)(z-ib)]$$ and I will close up and assume $b > 0$ to get

$$= 2\pi i \ \mathrm{Res}\{1/[(z+ib)(z-ib)\}^{z=+ib} = 2\pi i \ \{1/[(ib+ib)\} = 2\pi i/2ib = \pi/b$$

which agrees with Maple. So we are then faced with

$$K = \int_0^\infty dx \ (1/\sqrt{x}\,) \ \pi/(x+a) = \pi \int_0^\infty dx \ x^{-1/2}/(x+a)$$

For $a > 0$, Maple says this integral is $\pi/\sqrt{a}$ (we do not run through the pole), so then

$$K = \pi \ \ \pi/\sqrt{a} = \pi^2/\sqrt{a}$$

and that is what we wanted to show!

Summary: This is probably a fairly hard 3D problem. In the above method, we take advantage of our Weiner-Hopf solution of the line-charge problem (really a 2D problem) and with a z Fourier Transform, we are able to come up with the solution current on the half plane for the full 3D problem.

This problem got me back into serveral things of Olde:

- Bateman ET lookup
- Maple integrals
- AS2010 lookups on function properties
- residue integrals
- use of the FT