Stakgold Exercise 7.50 (p 328) PhL 12.31.11

Plane wave from the left is $u_i(z) = e^{i\omega z}$ where I write ω in place of $\sqrt{\lambda}$. Thus $u_i(0) = 1$. In the z=0 plane we have a "metal" screen and it has the usual total current as shown p 328 C. Our integral equation 7.250 p 301 is therefore

$$1 = \int_{\text{screen}} dS_\xi \, E(s|\xi) I(\xi) \qquad E = E_3 \text{ since this is a 3D problem}$$

where s is any point on the screen ("screen" means plate with hole in it, and screen = σ).

What would happen here it there were no aperture? Well, we have u=0 on the screen, so $u_s = -u_i = -1$ there and whatever the current is on the screen, it is then the same at every point on the screen, so this integral is not going to converge. You know this since $E \sim e^{i\omega R}/R$ and then

$$\text{tail of } \int_{\text{screen}} dS_\xi \, E(s|\xi) I(\xi) \sim \int_\Lambda^\infty d\rho \, \rho \, e^{i\omega\rho}/\rho \sim \int_\Lambda^\infty d\rho \, e^{i\omega\rho} = \infty \text{ if } \omega = 0 \text{ for example}$$

You might be able to rescue things here with some fancy distribution theory, subtracting off δ(ω) or some such, but Stak has a simpler plan.

1. The situation is modeled in this simple manner

$$u(z) = e^{i\omega z} - e^{-i\omega z} + v(z) \qquad z < 0$$
$$u(z) = v(z) \qquad z > 0$$

I think the idea is that $|v| << 1$ in this model. You get the same diffracted component on both sides, but on the left you also get the incoming and reflected waves. In the case v = 0 and no hole, you get u = 0 on the plate and that explains the phase of the reflected wave shown above as $-e^{-i\omega z}$.

2. It is an easy matter to compute the current on the plate when there is no hole and v = 0. We get

$$I_0 \equiv \partial_z u^+ - \partial_z u^- = \partial_z 0 - \partial_z[\, e^{i\omega z} - e^{-i\omega z}] = -2i\omega e^{i\omega z} = -2i\omega$$

and all this current flows on the left side of the plate. It generates the reflected beam and cancels the incoming beam.

3. In the model with a "small hole" and $v \neq 0$, you can imagine that the current on the "screen" σ differs by a small amount from the no-hole current and you then have some current

$$J = I - I_0 = \text{small}$$

It is trivial to show that $J = I - I_0$ is consistent with J being generated by v as follows:

$$J = \partial_z v^+ - \partial_z v^-$$

Proof: Define J by this difference, so you then have

$$J = \partial_z v^+ - \partial_z v^-$$
$$I = \partial_z u^+ - \partial_z u^-$$

Now we have

$$u^+ = v^+$$
$$u^- = e^{i\omega z} - e^{-i\omega z} + v_-$$

$$\partial_z u^+ = \partial_z v^+$$
$$\partial_z u^- = \partial_z v^- + \partial_z [e^{i\omega z} - e^{-i\omega z}] = \partial_z v^- + 2i\omega$$

Therefore, we find that

$$I = \partial_z u^+ - \partial_z u^- = \partial_z v^+ - [\partial_z v^- + 2i\omega] = \partial_z v^+ - \partial_z v^- - 2i\omega = J - 2i\omega = J + I_0$$

Therefore we have just shown that

$$J = I - I_0$$

So this takes us down through equation C, D and E on page 328.

4. Superposition Idea. Here is the part I had trouble with. You imagine three situations:

A: plate with no hole, vector current I_0 everywhere on the plate, it just reflects. We know from above that this current is $I_0 = -2i\omega$.

B: a disk with current $-I_0$ on it. This is an artifact of the mind. This thing will radiate and we know that the radiated field will be

$$w(x) = -\int_{disk} dS_\xi \, E(x|\xi) [-I_0]$$

The reason for the external minus sign is that this is the same minus sign you see in 7.249 p 301. The actual radiating source is the negative of the current in Stak's convention. Here we have $I(\xi) = -I_0$ it just happens.

C: plate with hole. We know that the radiated field here is going to be

$$v(x) = -\int_\sigma dS_\xi \, E(x|\xi) J(\xi)$$

The external minus sign is for the same reason just discussed in B. The actual source is minus the "current" in Stak's convention. So this equation is just superposition with the radiating sources. I think

this is what Stak is looking for when he says to "express v in terms of J". The above equation is perhaps best motivated on the right side where that excess current $J = I - I_0$ is what makes the diffraction field v,

Now we make this superposition claim:

situation A + situation B = situation C

When we superpose the disk with current $-I_0$ of situation B onto situation A, the currents cancel in the hole and we get situation C. We can therefore superpose the fields of situations A and B on the right side (which is to say, those fields are 0 and w(x)) and obtain the field of situation C, which is v(x). Therefore we claim that v(x) = w(x). Therefore we write

$v(x) = w(x)$ for any x

$$\int_\sigma dS_\xi \, E(x|\xi)[-J(\xi)] = -\int_{disk} dS_\xi \, E(x|\xi) [\, -I_0 \,]$$

$$\int_\sigma dS_\xi \, E(x|\xi)J(\xi) = -I_0 \int_{disk} dS_\xi \, E(x|\xi)$$

$$\int_\sigma dS_\xi \, E(x|\xi)J(\xi) = 2i\omega \int_{disk} dS_\xi \, E(x|\xi) = -v(x)$$

My derivation says this equation is true for any x. If course ξ is a point on the screen ξ,η,0 in Stak notation. Stak chose to put his point x = x,y,0 at some point in the z = 0 plane AND on the screen and then this equation becomes = 328 F.

In theory, you could pick any point x,y,0 and solve the above equation for J(ξ). Then from C above

$$v(x) = -\int_\sigma dS_\xi \, E(x|\xi)J(\xi) = v(x) \text{ at any point x}$$

5. Plan A (ignore, see Plan B). Now the last part is to express v(x) for general x in terms of v in the hole. I presume this means that J would not appear in such an equation. The hint is to make some kind of odd Green's function which leaves me blank. We could say

$$v(x) = -\int_\sigma dS_\xi \, E(x|\xi)J(\xi) = x \text{ is an arbitrary point}$$

$$v(x_h) = -\int_\sigma dS_\xi \, E(x_h|\xi)J(\xi) = x_h \text{ is a point in the hole}$$

Now in general, if we have an integral equation like these for function J(ξ), we should be able to "invert" the integral equation to get a differential equation. The Laplace/Poisson prototype here is

$$-\nabla^2 u(x) = \rho(x) \quad \Rightarrow \quad u(x) = \int dx \, g(x|\xi) \, \rho(\xi) \quad \text{where} \quad -\nabla^2 g(x|\xi) = \delta(x-\xi)$$

In terms of a surface charge distribution (instead of volume) we might say

$$-\nabla^2 u(x) = \delta(z)\sigma(x) \Rightarrow \quad u(x) = \int dS_\xi \, g(x|\xi)\, \sigma(\xi) \qquad \text{where} \quad -\nabla^2 g(x|\xi) = \delta(x-\xi)$$

and if we convert this to Helmholtz world I think is says

$$(-\nabla^2 - \omega^2)\, u(x) = \delta(z)J(x) \Rightarrow \quad u(x) = \int dS_\xi \, g(x|\xi)\, J(\xi) \qquad \text{where} \quad (-\nabla^2 - \omega^2) g(x|\xi) = \delta(x-\xi)$$

Now negate J and replace u with v

$$(-\nabla^2 - \omega^2)\, v(x) = -\,\delta(z)J(x) \Rightarrow \quad v(x) = -\int dS_\xi \, g(x|\xi)\, J(\xi) \qquad \text{where} \quad (-\nabla^2 - \omega^2) g(x|\xi) = \delta(x-\xi)$$

We recognize here our integral equation from above where g = E, so we should say

$$(-\nabla^2 - \omega^2)\, v(x) = -\,\delta(z)J(x) \Rightarrow \quad v(x) = -\int dS_\xi \, E(x|\xi)\, J(\xi) \qquad \text{where} \quad (-\nabla^2 - \omega^2) E(x|\xi) = \delta(x-\xi)$$

Now we need some notation. Let x,y on the screen be called x',y'. Then we can say

$$(-\nabla'^2 - \omega^2)\, v(x',y',z') = -\,\delta(z')J(x',y') \qquad = -\delta(z')\, J(\xi)$$

Now take the first integral equation above and mult by δ(z')

$$\delta(z')v(x,y,z) = -\int_\sigma dS_\xi \, E(x,y,z|x',y',z')\, \delta(z')J(x',y') \qquad = \text{x is an arbitrary point}$$

$$= \int_\sigma dS_\xi \, E(x|\xi)\, (-\partial_{x'}^2 - \partial_{y'}^2 - \partial_{z'}^2 - \omega^2)\, v(x',y')$$

and I have sort of "eliminated J". Now integrate -∞ ∞ on z' maybe

$$v(x,y,z) = \int_{-\infty}^{\infty} dz' \int_\sigma dS_\xi \, E(x|\xi)\, (-\partial_{x'}^2 - \partial_{y'}^2 - \partial_{z'}^2 - \omega^2)\, v(x',y')$$

This ain't worth much, but at least I have sort of expressed v(x) anywhere in terms of v on the screen! But Stak wants it in terms of v in the hole. STOP.

5. Plan B. We know these facts:

$$v(x) = -\,2i\omega \int_{disk} dS_\xi \, E(x|\xi) \qquad x = \text{anywhere}$$

$$v(x) = -\int_{\sigma} dS_{\xi}\, E(x|\xi) J(\xi) \qquad x = \text{anywhere}$$

The first can be applied to a general point x and to a hole point x_h

$$v(x) = -\ 2i\omega \int_{disk} dS_{\xi}\, E(x|\xi) \qquad x = \text{anywhere}$$

$$v(x_h) = -\ 2i\omega \int_{disk} dS_{\xi}\, E(x_h|\xi) \qquad x = \text{anywhere}$$

Then

$$v(x) = v(x_h) \ -\ 2i\omega \int_{disk} dS_{\xi}\, \{\, E(x|\xi) \ - E(x_h|\xi)\, \}$$

This seems to express the field anywhere in terms of the field in the hole and an integral which is fully determined. I give up because I don't understand what he is asking for (this seems to answer his question) . In fact just look at

$$v(x) = -\ 2i\omega \int_{disk} dS_{\xi}\, E(x|\xi)$$

This gives the field anywhere period in terms of known E. So does he want this in terms of ONLY the field in the hole and no E appearing anywhere? Stak has dropped the ball here IMHO. By not even mentioning superposition, he also dropped the ball for me. Enough already!

I will perhaps run into this same problem in some optics reading someday.