Exercise 7.54 PhL 12.31.11

Do a certain integral

$$F(a) = \int_k^\infty dx\, e^{-a(x+p)} / [\,(x+p)(x-k)^{1/2}\,]$$

$$dF/da = \int_k^\infty dx\, (-1)(x+p) e^{-a(x+p)} / [\,(x+p)(x-k)^{1/2}\,]$$

$$= -\int_k^\infty dx\, e^{-a(x+p)} / (x-k)^{1/2}$$

$$= -e^{-ap} \int_k^\infty dx\, e^{-ax} / (x-k)^{1/2}$$

Change to y = x-k so dy = dx to get

$$= -e^{-ap} \int_0^\infty dy\, e^{-a(y+k)} / y^{1/2}$$

$$= -e^{-ap} e^{-ak} \int_0^\infty dy\, e^{-ay} / y^{1/2}$$

Change to z = ay so dz = ady

$$= -e^{-a(p+k)} (1/a) \int_0^\infty dz\, e^{-z} / (z/a)^{1/2}$$

$$= -e^{-a(p+k)} a^{-1} a^{1/2} \int_0^\infty dz\, e^{-z} / z^{1/2}$$

The integral is $\pi^{1/2}$ says Maple, so

$$dF/da = -\pi^{1/2} a^{-1/2} e^{-a(p+k)}$$

$$= -\pi^{1/2} (e^{-a(p+k)} / \sqrt{a}\,)$$

$$dF = -\pi^{1/2} (e^{-a(p+k)} / \sqrt{a}\,)\, da$$

Now what? Integrate both sides to get

$$F(a) = -\pi^{1/2} \int^a da' (e^{-a'(p+k)} / \sqrt{a'}\,) + \text{constant}$$

Let p+k = c so this says

$$F(a) = -\pi^{1/2} \int^{a} dx\, (e^{-cx}/\sqrt{x}) + \text{constant}$$

Maple says the integral is

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int(exp(-c*x)/sqrt(x),x);
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$$\frac{\sqrt{\pi}\ \mathrm{erf}(\sqrt{c}\ \sqrt{x})}{\sqrt{c}}$$

so we get

$$F(a) = -\pi^{1/2}\ \pi^{1/2}\ \mathrm{erf}(c^{1/2}\ a^{1/2}) / c^{1/2} + \text{constant}$$

Now if a = 0 we have erf(0) which is 0 so we get

$$F(0) = 0 + \text{constant} \quad \Rightarrow$$

$$F(a) = -\pi\ \mathrm{erf}(c^{1/2}\ a^{1/2}) / c^{1/2} + F(0)$$

We also know that erf(∞) = 1 so

$$F(\infty) = -\pi/c^{1/2} + F(0)$$

Now look at our original integral

$$F(a) = \int_{k}^{\infty} dx\, e^{-a(x+p)} / [\,(x+p)(x-k)^{1/2}\,]$$

It sure seems that F(∞) = 0 here, so then

$$F(0) = \pi/c^{1/2}$$

and then

$$F(a) = -\pi\ \mathrm{erf}(c^{1/2}\ a^{1/2}) / c^{1/2} + \pi/c^{1/2}$$

$$= \pi/c^{1/2}\,[\,1 - \mathrm{erf}(c^{1/2}\ a^{1/2})] = \pi/c^{1/2}\ \mathrm{erfc}(c^{1/2}\ a^{1/2})$$

$$= (\,\pi/\sqrt{p+k}\,)\ \mathrm{erfc}(\sqrt{p+k}\ \sqrt{a}\,)$$

and we then get Stak's result ! There must be an easier way to do this integral but OK.