Exercise 7.56 PhL 12.31.11 - 1.2.12.

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Overview. This is a Wiener-Hopf problem. The physical setup is the usual 2D half-wire with u=0. A plane wave of unit amplitude comes down from above. We have our usual Fred 1 integral equation 7.250 which we would like to solve for wire current I, in which E = E_2 which is a K_0 function. Due to the half wire, we have our $\int_0^\infty$ Wiener-Hopf situation and the WH technique can then be applied.

In the Opener section. I quickly derive the first few results p 330 E, F and G, the last of which is the diagonalized equation in ω space which needs to be "segregated". I then did a quotient split on $1/\sqrt{\omega^2+k^2}$ and got the diagonalized equation into this form

$$(1/2)[1/[\sqrt{\omega+ik}]^{+}\ I_+^{+}(\omega) = \sqrt{\omega-ik}^{-}\ i/\omega + \sqrt{\omega-ik}^{0}\ g_-^{-}(\omega)$$

I then made a bad conceptual mistake by thinking of the i/ω factor as being either a plus or a minus function. Since it is the projection of f+(x) = 1, it should be a + function only. But if you assume i/ω is a - function, then the solution to the above seems to be $I_+^{+}(\omega) = 0$ which is wrong. This led me into the section called **Plan A Incorrect**. Looking for something wrong, I at least studied the large x behavior of the various functions involved to see if we were meeting the WH assumptions OK. This is shown at the very end of Plan A. I found for example that u(x) ~ $e^{d''x}$ where d'' < k which meets conditions. In the very end I found that d'' = 0.

I then started into **Plan B** fixing my misconception about i/ω. I was then here:

$$(1/2)[1/[\sqrt{\omega+ik}]^{+}\ I_+^{+}(\omega) = \sqrt{\omega-ik}^{-}\ i/\omega + \sqrt{\omega-ik}^{+}\ g_-^{-}(\omega)$$

I had then to deal with the - + term, and for some reason I thought for a long time I had to do a quotient split there, when in fact what is required is a sum split. I think I flailed for half a day on that.

Then in **Plan C** I studied Stak's example 2. This had a mixed term similar to mine, but his term had a slightly different problem than mine, but still he had a mixed term. So I finally realized I needed to do a sum split.

Then in **Phil's Sum Split Problem** I did this required sum split. This led finally to this fully segregated equation,

$$(1/2)[1/[\sqrt{\omega+ik}]^{+}\, I_+^{+}(\omega) - i\, e^{-i\pi/4}\sqrt{k}/\omega = \sqrt{\omega-ik}^{-}\, g_-^{-}(\omega) + i\,[\sqrt{\omega-ik}^{-} - e^{-i\pi/4}\sqrt{k}]/\omega$$

and since both sides have the required → 0 property in their directions, both sides = 0, and thus I had an expression for $I_+^+(\omega)$ which matched p 330 H. It took 2 days to get this result due to my flailings.

The next job was to compute I(x) by doing the Fourier Transform expansion formula from $I_+^+(\omega)$. The FT integral is a contour integral which picks up a pole at ω=0 and the discontinuity of a cut. I was quite concerned because Stak's answer had only the cut piece! I stuck by my guns, however. I first drew the cut to the left, but then realized why it is better to pull the cut downwards. I then got My result for I(x) and it different from Stak's result in 2 ways, a factor of 2, and then the pole contribution term 2k.

Next, I luckily got Wolfram to do the integral and I got I(x) in closed form "my way". I then used Laplace transforms to verify that My result was correct because I could regenerate H doing the Fourier Projection (via Laplace with s = -iω). So I think now my result is correct, and thus Stak had another erratum. In fact this one problem had two errata. I did the Laplace check twice in fact and then added these errata to my doc.

At the very end I ask two questions. **Question 1** is why I = 0 when k=0 for the electrostatics limit. **Question 2** was just to make a plot of the current.

This Exercise got me back into many things of old: using Maple, looking things up in AS2010 and in Bateman and in Schaum and in Wolfram Alpha, doing Visio graphics, etc. It was a fitting way to end my long Stakgold saga! The last problem of The Final Exam. It was a fine course!

Opener. Recall the integral equation 7.250

$$u_i(s) = \int_\sigma dS_\xi\, E(s|\xi)\, I(\xi)$$

Our application here is a plane wave incident from the positive y direction so it is going down. He writes this as e^{ky} but I think it should say something else. A plane wave from the -z direction was exp(-ikx) in problem 7.50 where $\lambda = -k^2$. So a way from the -y would be exp(-iky) and then from the +y should be e^{+iky} so he gas omitted the i, else OK.

$$u_i(x,y) = e^{iky}$$

So this plane wave comes down from above. This is a 2D problem. We assume Dirichlet since that is what 7.295 uses, so we need u = 0 on the half-line. If s is a point on the half line, then $u_i(s) = 1$ since y = 0 everywhere on both sides of the half line. Our integral equation is then

$$1 = \int_0^\infty dx\, E(x_0,0|\, x,0)\, I(x) \qquad \text{where we take } s = (x_0,0)$$

Now this is E_2 which is given by 7.155 p 266 as

$E_2 = (1/2\pi)\, K_0(k\,|x-x_0|)$ for our positioning of the two points

and so our integral equation is then

$$1 = (1/2\pi) \int_0^\infty dx\, K_0(k\,|x-x_0|)\, I(x)$$

Now change to integration variable ξ and rename x_0 to be some $x > 0$, so then

$$1 = \int_0^\infty d\xi\, (1/2\pi)\, K_0(k\,|\xi-x|)\, I(\xi) \qquad \text{for } x > 0 \qquad \text{ie, x is on the half-wire}$$

At least we are now off and running, this agrees with p 330 E.

We now want to think of this integral equation in terms of Weiner -Hopf! Looking at 7.284 we set

$u = I$ the unknown function to be solved for

$\mu = 0$ no external I term!

$f = 1$

$k(x-\xi) = (1/2\pi)\, K_0(k\,|\xi-x|)$

$k(x) = (1/2\pi)\, K_0(k|x|)$

So let's rewrite in terms of these more familiar function names

$$\int_0^\infty d\xi\, k(x-\xi)\, u(\xi) = 1 \qquad \text{for } x > 0$$

I will then define

$$g(x) = \int_0^\infty d\xi\, k(x-\xi)\, u(\xi) \qquad \text{when } x < 0 \qquad g(x) = \text{unknown!}$$

Then $g_-(x)$ would be exactly as he shows it in p 330 F.

Looking now at 7.285 and 7.286, the "transformed equation" should be

$$k\hat{}(\omega)\, u_+\hat{}(\omega) = f_+\hat{}(\omega) + g_-\hat{}(\omega)$$

Now exactly as in Example 1 p 318 we have f_+ as in p 318 F and so $f_+\hat{}(\omega) = i/\omega$ as in p 318 G. So going back from u to I, we then have

$k^{\wedge}(\omega)\, I_+^{\wedge}(\omega) = i/\omega + g_-^{\wedge}(\omega)$

Now p 322 7.298 shows that

$k^{\wedge}(\omega) = 1/[2\sqrt{\omega^2+k^2}\,]$ analytic in $-k < \mathrm{Im}(\omega) < k$, this can be our strip

so we arrive quickly at

$1/[2\sqrt{\omega^2+k^2}\,]\, I_+^{\wedge}(\omega) = i/\omega + g_-^{\wedge}(\omega)$ $-k < \mathrm{Im}(\omega) < k$

This certainly would also be true on the upper half of this strip $0 < \mathrm{Im}(\omega) < k$, so we now have p 330 G.

Our task now is to do segregation and find the problem solution with the W-H method! So today I got a solid start on this baby!

Well, one more step. On page 323 we do a quotient split to show that

$1/\sqrt{\omega^2+k^2} = 1/[\sqrt{\omega+ik}\,\sqrt{\omega-ik}\,]$

where page 323 also showed (and I confirmed) that

$\sqrt{\omega+ik}$ analytic for $\mathrm{Im}(\omega) > -k$ so this is a + function
$\sqrt{\omega-ik}$ analytic for $\mathrm{Im}(\omega) < k$ and this is a - function

I now have to ponder this more. Done for 2011 !

Plan A Incorrect. My error here was thinking that i/ω was "either + or -" when it is only +. I leave all these notes here for the record. Please skip to Plan B!

// Resume 1.1.12. Our equation then becomes:

$$(1/2)[1/[\overset{+}{\sqrt{\omega+ik}}\,\overset{-}{\sqrt{\omega-ik}}\,]\, \overset{+}{I_+^{\wedge}(\omega)} = \overset{\text{either}}{i/\omega} + \overset{-}{g_-^{\wedge}(\omega)}$$

Let's mult through by $\sqrt{\omega-ik}$ to get

$$(1/2)[1/[\overset{+}{\sqrt{\omega+ik}}\,]\, \overset{+}{I_+^{\wedge}(\omega)} = \overset{-}{\sqrt{\omega-ik}}\, \overset{0}{i/\omega} + \overset{-}{\sqrt{\omega-ik}}\, \overset{-}{g_-^{\wedge}(\omega)}$$

As shown above, this thing IS segregated. Without doing the work, suppose we could show that the LHS for ω going up went to 0 and the same for RHS when going down. Then we could conclude that $I_+^{\wedge}(\omega) = 0$ which is a no-go! We need to have two terms on each side! It must be that as shown above, the large ω behavior of the two sides is not right. The only term that is obvious is $\sqrt{\omega-ik}\, i/\omega \rightarrow 1/\sqrt{\omega} \rightarrow 0$ going down.

So here we have the issue that was always hazy in Stak's presentation. How do we get some inkling of how I and g behave in x space? Here are all the facts we have:

$$\int_0^\infty d\xi\, k(x-\xi)\, u(\xi) = 1 \qquad x > 0 \qquad (1)$$

$$g(x) = \int_0^\infty d\xi\, k(x-\xi)\, u(\xi) \qquad x < 0$$

In both equations ξ takes only positive values. We know that (note overloaded symbol k)

$$k(x) = (1/2\pi)\, K_0(k|x|) \;\rightarrow\; (1/2\pi)\,(\pi/2)^{1/2} e^{-k|x|} / (k|x|)^{1/2}$$

$$k(x-\xi) = (1/2\pi)\, K_0(k\,|\xi-x|) \;\rightarrow\; (1/2\pi)\,(\pi/2)^{1/2} e^{-k|x-\xi|} / (k|x-\xi|)^{1/2}$$

since AS2010 tell us that

10.40.2

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \sum_{k=0}^{\infty} \frac{a_k(\nu)}{z^k}, \qquad |\operatorname{ph} z| \le \tfrac{3}{2}\pi - \delta,$$

Now in equation (1), let's assume that $u(\xi) \rightarrow e^{d''\xi}$ (to use Stak's symbol d''). Then we have

$$\int_0^\infty d\xi\, k(x-\xi)\, u(\xi) = 1$$

$$\int_0^\Lambda d\xi\, k(x-\xi)\, u(\xi) + \int_\Lambda^\infty d\xi\, k(x-\xi)\, u(\xi) = 1$$

$$\int_0^\Lambda d\xi\, k(x-\xi)\, u(\xi) + c \int_\Lambda^\infty d\xi\, e^{-k|x-\xi|}\, e^{d''\xi} \sim 1$$

The second term on the LHS for ξ >> x has this tail integral

$$\int_\Lambda^\infty d\xi\, e^{-k(\xi-x)}\, e^{d''\xi} = e^{kx} \int_\Lambda^\infty d\xi\, e^{-k\xi}\, e^{d''\xi}$$

In order that this integral converge, we need k > d'' . So we can conclude that d'' < k, so that the worst that u can possibly do is $u(x) \sim e^{kx}$.

Since Stak does not give us the final answer for u apart from an integral, it is hard to tell if this is correct. Look at his result p 330 I. For large x, it seems that the integral is controlled by the LOW endpoint! We can apply the theory of Appendix B to this integral! I think (B.3) is the correct formula in our case. Then we have from p 330 I

$$I(x) = (\sqrt{k}/\pi) \int_{\mathbf{k}}^{\infty} dR\ e^{-xR} \sqrt{R-k}/R$$

$$I(t) = (\sqrt{k}/\pi) \int_{\mathbf{k}}^{\infty} dR\ e^{-tR} \sqrt{R-k}/R$$

$$I(t) = (\sqrt{k}/\pi) \int_{\mathbf{k}}^{\infty} dx\ e^{-tx} \sqrt{x-k}/x \qquad \text{// now in the form of B.1}$$

This tells us that $h(x) = x$ and $f(x) = \sqrt{x-k}/x$ and $a = k$ and $b = \infty$. Clearly it is the a endpoint that controls things. I will then use formula (B.3)

$$I(t) \sim e^{-ta} f(a)/[\,t\ 1] = (e^{-ta}/t)\sqrt{a-k}/a$$

But $f(a) = 0$ so we fail! So let's try our manual method

$$I(t) \approx (\sqrt{k}/\pi) \int_{\mathbf{k}}^{\mathbf{k+\varepsilon}} dx\ e^{-tx} \sqrt{x-k}/x$$

$$\approx (\sqrt{k}/\pi)\, e^{-kt}\ (1/k) \int_{\mathbf{k}}^{\mathbf{k+\varepsilon}} dx \sqrt{x-k}$$

$$\approx (\sqrt{k}/\pi)\, e^{-kt}\ (1/k)\ (2/3)\ \varepsilon^{3/2}$$

So this manual method suggests pretty clearly that $I(t) \sim e^{-kt}$. So this says d" = -k and indeed, this turns out to be d" < k.

Now suppose we cheat and make an ansatz that $I(x) \sim e^{-kx}$. Then what can we say about g?

$$g(x) = \int_{\mathbf{0}}^{\infty} d\xi\ k(x-\xi)\ u(\xi) \qquad x < 0$$

$$g(-x) = \int_{\mathbf{0}}^{\infty} d\xi\ k(-x-\xi)\ u(\xi) \qquad x > 0$$

Again the integral tail just converge for finite x and that tail is this

$$\int_{\mathbf{\Lambda}}^{\infty} d\xi\ e^{-k|-x-\xi|}\ e^{d"\xi}$$

In the above formula, $x > 0$ and $\xi > 0$ and therefore $-x-\xi < 0$ so that $|-x-\xi| = x+\xi$. Thus

$$\sim \int_{\mathbf{\Lambda}}^{\infty} d\xi\ e^{-k(x+\xi)}\ e^{k\xi} \sim e^{-kx} \int_{\mathbf{\Lambda}}^{\infty} d\xi\ 1$$

So as long as d" < k, this converges. Assuming d" = -k adds nothing to our data here. So I think we can conclude that

$g(-x) \sim e^{-kx}$ $\quad x \to +\infty$

$g(x) \sim e^{+kx} = e^{-k|x|}$ $\quad x \to -\infty$

so that c = k. Is this problematical? I think not. Look at the four assumptions on page 314

(1) $k(x) \sim e^{-c|x|}$ need $c > 0$ in our problem, we have $c = k > 0$ so OK

(2) $f(x) \sim e^{d'x}$ $x \to +\infty$ $d' < c$ in our problem f = 1 so d' = 0 and 0 < k so OK

(3) $u(x) \sim e^{d"x}$ $x \to +\infty$ $d" < c$ in our problem d" < k not knowing the answer so OK

(4) $g(x) \sim e^{-c|x|}$ $x \to -\infty$ in our problem c = k so all OK

Now when we make the + and - functions, where are each of these four analytic?

(1) $k(x) \sim e^{-c|x|}$ strip -c,c in our problem, the strip is $-k < Im(\omega) < k$ for $k^(\omega)$

(2) $f_+(x) \sim e^{d'x}$ $x \to +\infty$ $d' < c$ in our problem d' = 0 so $Im(\omega) > 0$ for $f+^(\omega)$

(3) $u(x) \sim e^{d"x}$ $x \to +\infty$ $d" < c$ in our problem $d" = k - \varepsilon$ not knowing the answer so OK
so therefore get $Im(\omega) > k-\omega$ for $u+^(\omega)$

But with ansatz that d" = -k, we get instead that
$Im(\omega) > -k$ for $u+^(\omega)$. [this ansatz is wrong! d"=0 is right.]

(4) $g(x) \sim e^{-c|x|}$ $x \to -\infty$ in our problem c = k so all OK $Im(\omega) < k$ for $g-^(\omega)$

I really have to make the u ansatz to even have a chance.

Plan B My error above was thinking that i/ω was "either + or -" when it is only +. So here I correct this misconception: // Resume 1.1.12. Our equation then becomes:

$\sqrt{\omega+ik}$ analytic for $Im(\omega) > -k$ so this is a + function
$\sqrt{\omega-ik}$ analytic for $Im(\omega) < k$ and this is a - function

$$\begin{matrix} & + & - & + & + & - \\ (1/2)[1/[\sqrt{\omega+ik}\sqrt{\omega-ik}\,]\, I_+^(\omega) & = & i/\omega & + & g_-^(\omega) & \end{matrix}$$

Let's mult through by $\sqrt{\omega-ik}$ to get

$$\begin{matrix} + & + & - & + & - & - \end{matrix}$$

$(1/2)[1/[\sqrt{\omega+ik}\ ]\ I_+\hat{}(\omega) = \sqrt{\omega-ik}\ i/\omega + \sqrt{\omega-ik}\ g_-\hat{}(\omega)$

So here we see the problem: we are not fully segregated because of the first term on the right! Can we do a quotient split here?

$$\sqrt{\omega-ik}\ \ (i/\omega)$$

We have a product of two functions here. The first $\sqrt{\omega-ik}$ is analytic $\mathrm{Im}(\omega) < k$, while the second is analytic $\mathrm{Im}(\omega) > 0$, so the product is analytic in the limited strip $0 < \mathrm{Im}(\omega) < k$. Our goal is to break this into two terms each of which is analytic in a full half plane such that the half planes overlap.

We now seek two overlapping half-plane analytic functions $b_\pm$ such that this is true:

$$\sqrt{\omega-ik}\ \ (i/\omega) \ = \ b_+(\omega)/\ b_-(\omega)$$

But this is already of the proper form! So there is nothing I can do here!

Plan C Go back to an earlier point which is p 330G for which we have a waypoint. Here I show analytic ranges for each term, and I make the "ansatz" for I+^ as noted at the end of Plan A:

	strip	+	+	-
	$0 < \mathrm{Im}(\omega) < k$	$\mathrm{Im}(\omega) > -k$	$\mathrm{Im}(\omega) > 0$	$\mathrm{Im}(\omega) < k$
$1/[2\sqrt{\omega^2+k^2}\ ]$	* $I_+\hat{}(\omega)$	= i/ω	+	$g_-\hat{}(\omega)$

Our candidate overlap strip is $0 < \mathrm{Im}(\omega) < k$.

The question now: What the hell do we do next here? I tried to mimic the text and I took this next step:

$$(1/2)[1/[\sqrt{\omega+ik}^{\,+}\sqrt{\omega-ik}^{\,-}\ ]\ I_+\hat{}(\omega) = i/\omega + g_-\hat{}(\omega)$$

Commentary: Even though I read and studied WH pretty hard, here when we get down to a knowledge test, I flunk completely! If I multiply through by $\sqrt{\omega-ik}$, we do NOT get segregation. So why did this work for Stak in his Example 2, but it does not work for me here? When I mult through I get

$$(1/2)[1/[\sqrt{\omega+ik}^{\,+}\ ]\ I_+\hat{}(\omega) \ - \ \sqrt{\omega-ik}^{\,+}\ i/\omega \ = \ \sqrt{\omega-ik}^{\,-}\ \ g_-\hat{}(\omega)$$

Now compare this to Stak 7.304. My result is similar to his. He had a "problem child" second term on the left and so do I, but our problems are a little different.

What was his problem? He was stuck with this term for his second on right

$$\mathrm{Im}(\omega)^{+} > -k$$

$$- e^{-i\omega a} / \sqrt{\omega+ik}$$

This expo in the upper half plane goes as

$$e^{-i\omega a} = \exp(-i\,[\,\mathrm{Re}(\omega) + i\,\mathrm{Im}(\omega)\,]\,a) = e^{-i\mathrm{Re}(\omega)a}\; e^{+\mathrm{Im}(\omega)a}$$

I guess the problem is that $e^{-i\omega a}$ does not → 0 as you go up, it blows up! Stak did the "procedure" that I worked hard on and which I do understand, and that procedure resulted in this sum-split:

$$a(\omega) = e^{-i\omega a} / \sqrt{\omega+ik} \;\; = \; a+(\omega) + a-(\omega)$$

where he found that (recall that erf+erfc = 1)

$$a+(\omega) = (e^{-i\omega a} / \sqrt{\omega+ik}\,)\; \mathrm{erfc}[\,\sqrt{a}\; e^{-i\pi/4}\; \sqrt{\omega+ik}\;\,]$$
$$a-(\omega) = (e^{-i\omega a} / \sqrt{\omega+ik}\,)\; \mathrm{erf}\,[\,\sqrt{a}\; e^{-i\pi/4}\; \sqrt{\omega+ik}\;\,]$$

and add these up to get

$$a(\omega) = (e^{-i\omega a} / \sqrt{\omega+ik}\,)$$

Now what can I say about analyticity of a+(ω) ? Well, the real question is what happens as ω → up?

As $z \to \infty$

7.12.1

$$\mathrm{erfc}\, z \sim \frac{e^{-z^2}}{\sqrt{\pi} z} \sum_{m=0}^{\infty} (-1)^m \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{(2z^2)^m},$$

In our case we have

$$z = \sqrt{a}\; e^{-i\pi/4}\; \sqrt{\omega+ik}$$

$$z^2 = a\; e^{-i\pi/2}(\omega+ik) \;\; = -ia(\omega+ik)$$

For large ω this becomes

$$z^2 = -ia\omega$$

Therefore in our case

$$\mathrm{erfc}[\,\sqrt{a}\; e^{-i\pi/4}\; \sqrt{\omega+ik}\;\,] \;\sim\; e^{+ia\omega} / \sqrt{a\omega}$$

This $e^{+ia\omega}$ factor then repairs our problem, and we then have $a^+ \sim 1/\omega$ and all is well.

What about a-(ω) ? Well we can say

$$\text{erf}\,[\,\sqrt{a}\,e^{-i\pi/4}\sqrt{\omega+ik}\,\,] \;\sim\; (1 - e^{+ia\omega}/\sqrt{a\omega}\,)$$

Then

$$a_- = (e^{-i\omega a}/\sqrt{\omega+ik}\,)\,(1 - e^{+ia\omega}/\sqrt{a\omega}\,) \;=\; (e^{-i\omega a}/\sqrt{\omega+ik}\,) \;- 1/\omega\;(1/\sqrt{a}\,)$$

As we go to Im(ω) → -∞, both terms → 0 so we are OK.

Therefore the sum split results in two terms each of which → 0 in its half plane, and that is the goal.

So that was HIS problem and we see how he resolved his problem.

MY problem is this problem child term shown in red.

$$(1/2)[1/[\,\sqrt{\omega+ik}\,\,]^{+}\, I_+\hat{}(\omega) \;-\; \sqrt{\omega-ik}^{-}\, i/\omega \;=\; \sqrt{\omega-ik}^{+}\, g_-\hat{}(\omega)$$

What exactly is the problem with this term? For large ω either up or down it → 0, so THAT is not the problem. The problem is that the term is analytic only in the strip 0 < Im(ω) < k so it is neither a + nor a - function. My problem is then to find a sum split!

Phil's Sum Split Problem

Write

$$a(\omega) = \sqrt{\omega-ik}\,/\omega \;= a_+(\omega) + a_-(\omega)$$

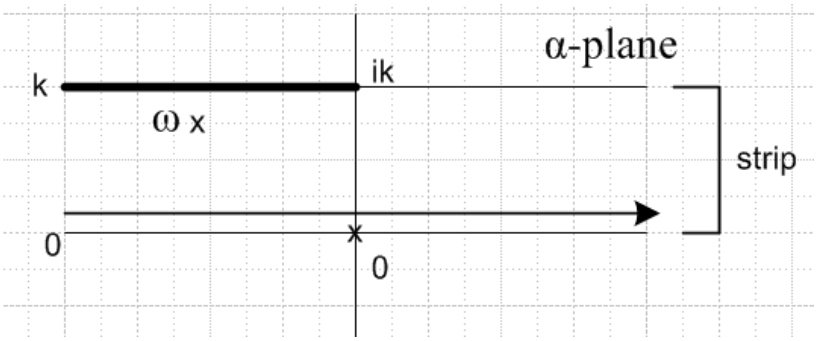
My problem is to DO this sum split! It has taken me about 4 hours to realize that this is my problem!

So my strip is 0 ≤ Im(ω) ≤ k and I have teh page 317 top picture then in mind. I can write

$$2\pi i\, a(\omega) = [\,\underbrace{\int_{C1} d\alpha\; a(\alpha)/(\alpha-\omega)}_{2\pi i\, a_+(\omega)}] \;+\; [\,\underbrace{-\int_{C2} d\alpha\; a(\alpha)/(\alpha-\omega)}_{2\pi i\, a_-(\omega)}]$$

I need only compute the first term just as Stak did. So explicitly α-plane

$$a_+(\omega) = (1/2\pi i)\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} d\alpha\; a(\alpha)/(\alpha-\omega) \;=\; (1/2\pi i)\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} d\alpha\,\sqrt{\alpha-ik}\,/\,[\,\alpha\,(\alpha-\omega)]$$



The contour is the arrow. The obvious thing is to close down. Integrand goes as $\alpha^{1/2-1-1} = \alpha^{-3/2}$ which is sufficient to kill the GC going down there, so we pick up a negative residue at $\alpha = 0$, so

$$a_+(\omega) = (1/2\pi i) \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} d\alpha \sqrt{\alpha - ik} / [\alpha(\alpha-\omega)]$$

$$= -\text{Res}\{\sqrt{\alpha-ik} / [\alpha(\alpha-\omega)]\}^{\alpha=0} = -\sqrt{-ik}/(-\omega) = \sqrt{-ik}/\omega = e^{-i\pi/4}\sqrt{k}/\omega$$

Well, this seems awfully simple, but let's assume it is correct. Then

$$a_-(\omega) = \sqrt{\omega-ik}/\omega - e^{-i\pi/4}\sqrt{k}/\omega = [\sqrt{\omega-ik} - e^{-i\pi/4}\sqrt{k}]/\omega$$

The claim is that this is then a - function. Let's hold on confirming that and see if this will give us the right answer:

$$\begin{array}{ccccc} a(\omega) & = & a_+(\omega) & + & a_-(\omega) \\ \sqrt{\omega-ik}/\omega & = & e^{-i\pi/4}\sqrt{k}/\omega & + & [\sqrt{\omega-ik} - e^{-i\pi/4}\sqrt{k}]/\omega \end{array}$$

Go back to our projected equation

$$(1/2)[1/[\sqrt{\omega+ik}]^{+} I_+^{+}(\omega) - \sqrt{\omega-ik}\, i/\omega = \sqrt{\omega-ik}^{-}\, g_-^{-}(\omega)$$

Therefore with our sum-splitting we get

$$(1/2)[1/[\sqrt{\omega+ik}]^{+} I_+^{+}(\omega) - i\, e^{-i\pi/4}\sqrt{k}/\omega - i\,[\sqrt{\omega-ik} - e^{-i\pi/4}\sqrt{k}]/\omega = \sqrt{\omega-ik}^{-}\, g_-^{-}(\omega)$$

Now we segregate to get

$$(1/2)[1/[\sqrt{\omega+ik}]^{+} I_+^{+}(\omega) - i\, e^{-i\pi/4}\sqrt{k}/\omega = \sqrt{\omega-ik}^{-}\, g_-^{-}(\omega) + i\,[\sqrt{\omega-ik}^{-} - e^{-i\pi/4}\sqrt{k}]/\omega$$

If both sides → 0 for Im(ω) in the apropo direction, then our solution for I should be

$I_+\hat{}(\omega) = [\ i\, e^{-i\pi/4}\sqrt{k}\,/\,\omega\ ]\ 2\sqrt{\omega+ik}$

$= 2i\, e^{-i\pi/4}\sqrt{k}\,\sqrt{\omega+ik}\,/\omega$

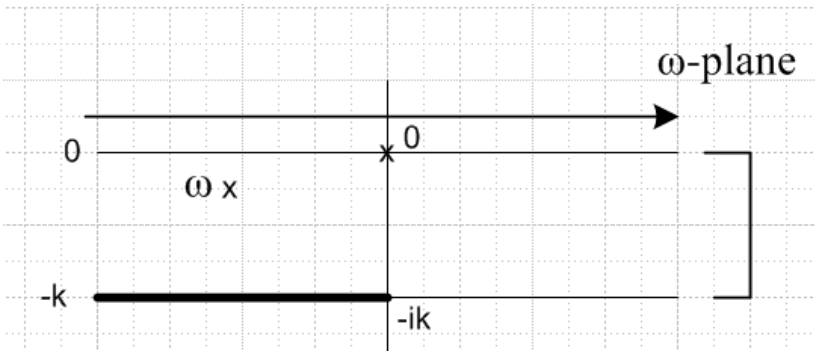
and indeed this is p 330 H. (*That* only took 2 days).

Now transform this back to get I(x). See raw3 notes on this (step 6), and we have

$I_+(x) = (1/2\pi) \int_{-\infty+iv}^{\infty+iv} d\omega\ I_+\hat{}(\omega)\, e^{-i\omega x}$ where $e^{-vx}\, I_+(x)$ converges x>0

$= (1/2\pi)\, 2i\, e^{-i\pi/4}\sqrt{k} \int_{-\infty+iv}^{\infty+iv} d\omega\, e^{-i\omega x}\, \sqrt{\omega+ik}\,/\omega$

I am pretty sure any v>0 is OK here, so the contour goes above the pole at ω = 0,. Here is the picture



Consider $e^{-i\omega x} = e^{-i[Re(\omega) + iIm(\omega)]x} \sim e^{xIm(\omega)}$

If x < 0, this decays going up in the ω plane, so we close up and get 0 as required for + function.

But if x > 0, we have to close down. We are going to pick up a pole and a discontinuity ! But I suspect we should only get the discontinuity, so something is wrong. Maybe this is another Stak typo!

Now let's first do the pole reside. We get

$\int_{-\infty+iv}^{\infty+iv}{}_{pole} d\omega\, e^{-i\omega x}\, \sqrt{\omega+ik}\,/\omega = -(2\pi i) * (1/2\pi i) \oint_{pole} d\omega\, e^{-i\omega x}\, \sqrt{\omega+ik}\,/\omega$

$= -(2\pi i)\, \mathrm{Res}\{\, e^{-i\omega x}\, \sqrt{\omega+ik}\,/\omega\,\}^{\omega=0} = -(2\pi i)\sqrt{ik}$

So this pole gives a constant term in $I_+(x)$ which is equal to

$(1/2\pi)\, 2i\, e^{-i\pi/4}\sqrt{k} * \{ -(2\pi i)\sqrt{ik}\,\} = (1/2\pi)\, 2i\, e^{-i\pi/4}\sqrt{k} * \{ -(2\pi i)\sqrt{k}\ e^{+i\pi/4}\}$

$= 2i\, e^{-i\pi/4}\sqrt{k} * \{ -(i)\sqrt{k}\ e^{+i\pi/4}\} = 2\, e^{-i\pi/4}\sqrt{k} * \{ \sqrt{k}\ e^{+i\pi/4}\}$

$= 2\sqrt{k} * \{ \sqrt{k}\ \} = 2k$

I think Stak has omitted this constant term in his p 330 I. the 2000 edition shows nothing different so if he has erred, he has done so in both editions.

Now let's go after the discontinuity integral. It is this: (leave off same overall factor $(1/2\pi)\ 2i\ e^{-i\pi/4}\sqrt{k}$)

$$\int_{-\infty-ik}^{-ik} d\omega \quad \text{Disc}\left[e^{-i\omega x}\ \sqrt{\omega+ik}\ /\omega\ \right] = \int_{-\infty-ik}^{-ik} d\omega\ e^{-i\omega x}\ \omega^{-1}\ \text{Disc}\left[\sqrt{\omega+ik}\ \right]$$

At this point, it is good to change complex integration variable to

$z = \omega + ik$

so endpoints are then $(-\infty, 0)$ and we have

$$= \int_{-\infty}^{0} dz\ e^{-ix(z-ik)}\ (z-ik)^{-1}\ \text{Disc}[\ z^{1/2}\ ]$$

But in the usual scratch paper way you show that

$$\text{Disc}[\ z^{1/2}\ ] = 2i\ (-z)^{1/2}$$

so we then get

$$= 2i \int_{-\infty}^{0} dz\ e^{-ix(z-ik)}\ (z-ik)^{-1}\ (-z)^{1/2}$$

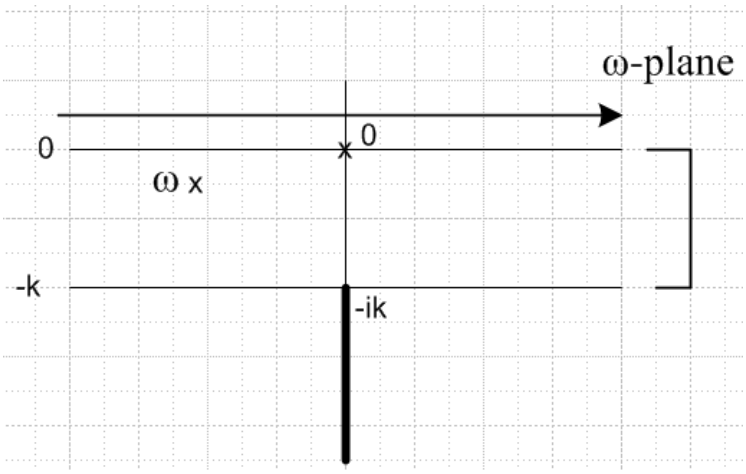
Now finally let $s = -z$ and we have

$$= 2i \int_{0}^{\infty} ds\ e^{-ix(-s-ik)}\ (-s-ik)^{-1}\ (s)^{1/2}$$

$$= 2i \int_{0}^{\infty} ds\ s^{1/2}\ e^{ixs}\ e^{-kx}\ (-s-ik)^{-1}$$

$$= 2i\ e^{-kx} \int_{0}^{\infty} ds\ s^{1/2}\ e^{ixs}\ (-s-ik)^{-1}$$

Well I think this is right, but it is not the form Stak used. The result has to be the same no matter what direction you pull off the cut, so let's do it again with this different picture,



Now let Disc = right minus left and we get

$$\int_{-ik}^{-i\infty} d\omega \quad \text{Disc}\left[e^{-i\omega x}\sqrt{\omega+ik}\,/\omega\right] = \int_{-ik}^{-i\infty} d\omega\, e^{-i\omega x}\,\omega^{-1}\quad \text{Disc}\left[\sqrt{\omega+ik}\right]$$

This is all like his presentation on page 325. So lets set

$$\omega = -iR \qquad R = i\omega$$

Then endpoints are

$$\omega = -ik \quad \Rightarrow \quad R = i(-ik) = k$$

$$\omega = -i\infty \quad \Rightarrow \quad R = i(-i\infty) = +\infty$$

and we then have

$$= \int_{k}^{\infty} (-i)dR \quad e^{-xR}\,(-iR)^{-1}\,\text{Disc}\left[\ (-iR+ik)^{1/2}\right]$$

A scratch calculation says that

$$\text{Disc}\left[\sqrt{\omega+ik}\right] = f_+ - f_- = |\omega+ik|^{1/2}\,2\,e^{-i\pi/4}$$

But then

$$|\omega+ik|^{1/2} = |-iR+ik|^{1/2} = |k-R|^{1/2} = (R-k)^{1/2} \qquad \text{since } R > k$$

so that

$$\text{Disc}[] = 2\,e^{-i\pi/4}\,(R-k)^{1/2}$$

and then we have

$$= \int_k^\infty (-i)dR \quad e^{-xR} (-iR)^{-1} \{ 2\, e^{-i\pi/4} (R-k)^{1/2} \}$$

$$= 2\, e^{-i\pi/4} \int_k^\infty dR\, e^{-xR} R^{-1} (R-k)^{1/2}$$

and then inserting the leading term,

cut contribution to I+(x) =

$$(1/2\pi)\, 2i\, e^{-i\pi/4}\sqrt{k} \quad 2\, e^{-i\pi/4} \int_k^\infty dR\, e^{-xR} R^{-1} (R-k)^{1/2}$$

$$= (1/\pi)\, 2i\, e^{-i\pi/2}\sqrt{k} \quad \int_k^\infty dR\, e^{-xR} R^{-1} (R-k)^{1/2}$$

$$= (1/\pi)\, 2i\, (-i)\sqrt{k} \quad \int_k^\infty dR\, e^{-xR} R^{-1} (R-k)^{1/2}$$

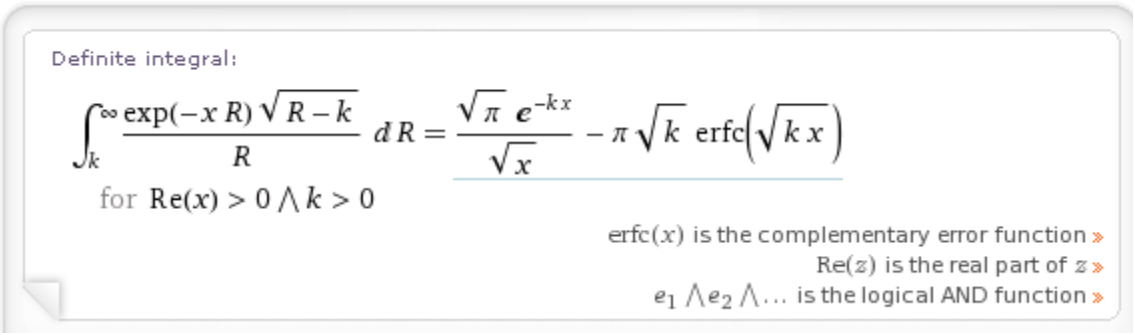
$$= (2/\pi)\sqrt{k} \quad \int_k^\infty dR\, e^{-xR} R^{-1} (R-k)^{1/2}$$

I am off only by a factor of 2. I think I am right. So here is my complete answer:

$$I+(x) = 2k + (2/\pi)\sqrt{k} \quad \int_k^\infty dR\, e^{-xR} R^{-1} (R-k)^{1/2} \qquad \text{my result}$$

$$I+(x) = (1/\pi)\sqrt{k} \quad \int_k^\infty dR\, e^{-xR} R^{-1} (R-k)^{1/2} \qquad \text{his result p 330 I}$$

Regardless of who is right, I really should do the integral to finish off the problem. Since the radical is upstairs, this is not a special case of 7.54 as I first thought. Let's try Maple just for fun: Maple cannot do it. Let's try WolframAlpha. It took "extra time" but they could do it:



So the conclusion is

$$\int_{\mathbf{k}}^{\infty} dR\, e^{-xR} R^{-1} (R-k)^{1/2} = \sqrt{\pi}\, e^{-kx}/\sqrt{x} - \pi \sqrt{k}\, \mathrm{erfc}(\sqrt{kx})$$

Then My answer becomes

$$I_+(x) = 2k + (2/\pi)\sqrt{k}\, [\sqrt{\pi}\, e^{-kx}/\sqrt{x} - \pi \sqrt{k}\, \mathrm{erfc}(\sqrt{kx})]$$

$$= 2k + [(2/\pi)\sqrt{k}\sqrt{\pi}\, e^{-kx}/\sqrt{x} - (2/\pi)\sqrt{k}\,\pi\sqrt{k}\, \mathrm{erfc}(\sqrt{kx})]$$

$$= 2k + (2\sqrt{k}/\sqrt{\pi})\, e^{-kx}/\sqrt{x} - 2k\, \mathrm{erfc}(\sqrt{kx})$$

$$= 2k\,[1 - \mathrm{erfc}(\sqrt{kx})] + 2\sqrt{k/\pi}\, e^{-kx}/\sqrt{x}$$

$$= 2k\, \mathrm{erf}(\sqrt{kx}) + 2\sqrt{k/\pi}\, e^{-kx}/\sqrt{x}$$

My pole part seems to fit in somewhat. But now what happens for large x in my answer? The only asymptotic we have is for erfc which I requote

$$\operatorname{erfc} z \sim \frac{e^{-z^2}}{\sqrt{\pi} z} \sum_{m=0}^{\infty} (-1)^m \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{(2z^2)^m},$$

So My answer would have

$$\mathrm{erfc}(\sqrt{kx}) \sim e^{-kx}/\sqrt{\pi} * 1/\sqrt{kx}$$

so I would then have

$$I_+(x) \sim 2k(1 - e^{-kx}/\sqrt{\pi} * 1/\sqrt{kx}) + 2\sqrt{k/\pi}\, e^{-kx}/\sqrt{x}$$

$$= 2k - 2\sqrt{k/\pi}\, e^{-kx}/\sqrt{x} + 2\sqrt{k/\pi}\, e^{-kx}/\sqrt{x} = 2k + e^{-kx}\, \mathrm{order}(1/kx)$$

so in my answer, the leading expos cancel and I have the constant left over. My constant term must be wrong because it would stop the FT from converging! No not right. My constant term is like f+(x) = 1 which is a constant term and would give then 2k i/ω for the FT.

Idea: Assume My answer is right and go the other direction! Then I should get

$$\int_{\mathbf{0}}^{\infty} dx\, e^{i\omega x} I_+(x) = I_+\hat{}(\omega)$$

which for My solution would say that

$$\int_0^\infty dx\, e^{i\omega x}\{2k + (2\sqrt{k}/\sqrt{\pi})\, e^{-kx}/\sqrt{x} - 2k\,\mathrm{erfc}(\sqrt{kx})\} = 2i\, e^{-i\pi/4}\sqrt{k}\,\sqrt{\omega+ik}/\omega$$

Maybe I can show this is or is not true. There are three terms on the left. Write

$$\text{LHS} = 2k\int_0^\infty dx\, e^{i\omega x} + (2\sqrt{k}/\sqrt{\pi})\int_0^\infty dx\, e^{i\omega x}(e^{-kx}/\sqrt{x}) - 2k\int_0^\infty dx\, e^{i\omega x}\,\mathrm{erfc}(\sqrt{kx})$$

$$= 2k\, I_1 + (2\sqrt{k}/\sqrt{\pi})\, I_2 - 2k\, I_3$$

I ought to be able to find these 3 integrals. Do we have a table of such things? Well suppose we set

$i\omega = -s$

Then we have

$$I_1 = \int_0^\infty dx\, e^{-sx} = 1/s$$ Schaum p 164

```
laplace(1,t,s);
```

$$\frac{1}{s}$$ verified!

$$I_2 = \int_0^\infty dx\, e^{-sx}(e^{-kx}/\sqrt{x})$$

Why can't I find this even in Bateman? Well there is a rule for this one

$$I_2 = \int_0^\infty dx\, e^{-sx}[e^{-kx}/\sqrt{x}]$$ think -k = a use Schaum p 162 32.5 so $I_2 = J_2(s+k)$

$$J_2(s) \equiv \int_0^\infty dx\, e^{-sx}\, 1/\sqrt{x}$$ think $\nu = -1/2$ so $= \Gamma(1/2)s^{-1/2}$ since

(1)	t^ν	$\mathrm{Re}\ \nu > -1$	$\Gamma(\nu+1)p^{-\nu-1}$	$\mathrm{Re}\ p > 0$

Bateman p 137

but $\Gamma(1/2) = \sqrt{\pi}$ so then

$$I_2 = J_2(s+k) = \sqrt{\pi}\,(s+k)^{-1/2}$$

```
laplace(exp(-k*t)/sqrt(t),t,s);
```

$$\frac{\sqrt{\pi}}{\sqrt{s+k}}$$ verified!

Finally,

$I_3 = \int_0^\infty dx\, e^{-sx}\, \mathrm{erfc}(\sqrt{kx}) = (1/s)\,(s+k)^{-1/2}[\,(s+k)^{1/2} - k^{1/2}]$

$= (1/s) - k^{1/2}\,(s+k)^{-1/2}(1/s)$

```
laplace(erfc(sqrt(k*t)),t,s);
```

$$\frac{1}{s} - \frac{\sqrt{k}}{s\sqrt{s+k}}$$ verified.

since

(9)	$\mathrm{Erfc}(a^{1/2} t^{1/2})$	$p^{-1}(p+a)^{-1/2}[(p+a)^{1/2} - a^{1/2}]$ $\mathrm{Re}\, p > -\mathrm{Re}\, a$

So let's try to assemble all the pieces (surely lots of errors have been made...)

LHS = $2k\, I_1 + (2\sqrt{k}/\sqrt{\pi})\, I_2 - 2k\, I_3$

$= 2k(1/s) + (2\sqrt{k}/\sqrt{\pi})\sqrt{\pi}\,(s+k)^{-1/2} - 2k\,\{(1/s) - k^{1/2}\,(s+k)^{-1/2}(1/s)\}$

$= (2\sqrt{k}/\sqrt{\pi})\sqrt{\pi}\,(s+k)^{-1/2} - 2k\,\{ - k^{1/2}\,(s+k)^{-1/2}(1/s)\}$ // two terms cancelled!

$= (2\sqrt{k}/\sqrt{\pi})\sqrt{\pi}\,(s+k)^{-1/2} + 2k^{3/2}\,(s+k)^{-1/2}(1/s)$

$= (s+k)^{-1/2}\,\{(2\sqrt{k}/\sqrt{\pi})\sqrt{\pi} + 2k^{3/2}\,(1/s)\}$

$= (s+k)^{-1/2}\,\{(2\sqrt{k}) + 2k^{3/2}\,(1/s)\}$

$= 2\sqrt{k}\,(s+k)^{-1/2}\,\{1 + k/s)\}$

$= 2\sqrt{k}\,(s+k)^{-1/2}\,\{1 + k/s)\}s/s$

$= 2\sqrt{k}\,(s+k)^{-1/2}\,\{s + k)\}(1/s)$

$= 2\sqrt{k}\ (s+k)^{+1/2}/s$

$= 2\sqrt{k}\ (-i\omega+k)^{+1/2}/(-i\omega)$

$= 2i\sqrt{k}\ (-i\omega+k)^{+1/2}/\omega$

$= 2i\sqrt{k}\ (-i\omega-i[ik])^{+1/2}/\omega$

$= 2i\sqrt{k}\ (-i)^{1/2}(\omega+[ik])^{+1/2}/\omega$

$= 2i\sqrt{k}\ e^{-i\pi/4}(\omega+ik)^{+1/2}/\omega$

and amazingly this is the correct result! This seems to verify that My answer is correct.

Let's just do this one more time with my simpler form answer:

$I+(x) = 2k\ \mathrm{erf}(\sqrt{kx}\ ) + 2\sqrt{k/\pi}\ e^{-kx}/\sqrt{x}$ // this is "my answer"

Then going to ω space with iω = -s I get

$$\int_0^\infty dx\ e^{i\omega x}\ I+(x) = \int_0^\infty dx\ e^{i\omega x}\ \{\ 2k\ \mathrm{erf}(\sqrt{kx}\ ) + 2\sqrt{k/\pi}\ e^{-kx}/\sqrt{x}\ \}$$

$$= 2k \int_0^\infty dx\ e^{-sx}\ \mathrm{erf}(\sqrt{kx}\ ) + 2\sqrt{k/\pi} \int_0^\infty dx\ e^{-sx}\ (e^{-kx}/\sqrt{x}\ )$$

Maple does these two Laplace transforms:

```
laplace(exp(-k*t)/sqrt(t),t,s);
```

$$\frac{\sqrt{\pi}}{\sqrt{s+k}}$$

```
laplace(erf(sqrt(k*t)),t,s);
```

$$\frac{\sqrt{k}}{s\sqrt{s+k}}$$

to give

$= 2k\ k^{1/2}\ (1/s)(1/\sqrt{s+k}\ ) + 2\sqrt{k/\pi}\ \pi^{1/2}(1/\sqrt{s+k}\ )$

$= 2k\ k^{1/2}\ (1/s)(1/\sqrt{s+k}\ ) + 2\ k^{1/2}\ (1/\sqrt{s+k}\ )$

$= 2k^{1/2}\ [(k/s)(1/\sqrt{s+k}\,)+\ (1/\sqrt{s+k}\,)]$

$= 2k^{1/2}\ [(k/s) +\ 1]\ /\ \sqrt{s+k}$

$= 2k^{1/2}\ [k+s]\ \ (1/\sqrt{s+k}\,)(1/s)$

$= 2k^{1/2}\sqrt{s+k}\ \ (1/s)$

and I won't repeat the rest of the steps. So here is my best answer for p 330 I:

$$I+(x) = 2k + (2/\pi)\sqrt{k}\ \int_{\mathbf{k}}^{\infty} dR\ e^{-xR}\ R^{-1}\ (R-k)^{1/2}$$ my result

and when I do that integral in Wolfram I get

$$I+(x) = \ 2k\ \mathrm{erf}(\sqrt{kx}\,)\ + 2\sqrt{k/\pi}\ \ e^{-kx}/\sqrt{x}$$

and when I do Laplace transform on this with $s = -i\omega$ I get

$$I_{+}\hat{}(\omega) \ = \ 2i\ e^{-i\pi/4}\sqrt{k}\ \sqrt{\omega+ik}\ /\omega$$

which agrees with p 330 H.

Now notice that $I+(x) \rightarrow \ 2k \ = e^{0}$ so in fact d" = 0, and this does not conflict with my d" < k earlier conclusion.

So I will now make another entry in the **errata**! Done.

Question 1: Why does I(x) = 0 when k=0 ? In this limit, the plane wave coming down from above is just a constant α which lies in the range (-1,1) depending on the time t I suppose. So $u_i = \alpha$. In this case, we must have $u_s = -\alpha$ on the half wire so that u = 0 there. I think then the solution is $u_s = -\alpha$ in all space, and then u=0 everywhere!! Then of course we expect σ = 0. So here we have a constant negative scattered wave, so to speak, that cancels the constant incoming wave everywhere.

Question 2: Plot the current versus x for some finite value of k: Go back to

$$I+(x) = \ 2k\ \mathrm{erf}(\sqrt{kx}\,)\ + 2k\sqrt{1/\pi}\ \ e^{-kx}/\sqrt{kx}$$

$$= 2k[\mathrm{erf}(\sqrt{kx}\,)\ + \sqrt{1/\pi}\ \ e^{-kx}/\sqrt{kx}\ \ ]$$

The current is really a function of variable kx, with an overall scaling by k. So the curves have a constant shape, so just pick some k and do a 2D plot. Here with k = 1 we get

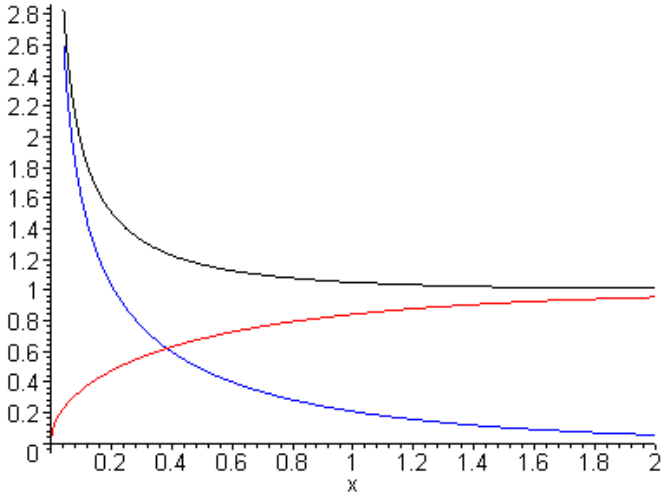
```
restart;
f1 := erf(sqrt(x));
```

$$f1 := \mathrm{erf}(\sqrt{x})$$

```
f2 := (1/sqrt(Pi))*exp(-x)/sqrt(x);
```

$$f2 := \frac{e^{(-x)}}{\sqrt{\pi}\sqrt{x}}$$

```
plot([f1,f2,f1+f2],x=0..2, color=[red,blue,black]);
```



The main feature of the current is the $1/\sqrt{x}$ blowup at the left end of the half-wire. The current then goes to a constant value for large x and this comes entirely from the erf term which is the red plot.

If you had a long wire segment, you might expect this to happen at both ends. Think of this as my short wave antenna if it were grounded. You would need a current sensor at the end (instead of the usual voltage sensor) to pick up the signal.