Sum Splitting Confusion PhL 7.28.11

These are early notes on this confusion, don't read them because most of it is probably wrong. The matter is finally cleared up in my "**Analytic functions defined by integral.doc**". It turns out that each of the two integrals being added together in 7.291 has two different "branches" (ie, completely different functions!) and you have to select the correct branch for each of the functions. It turns out that these "branches" are analytic in full half-planes. The selection of branches is made such that the first integral is analytic above the bottom edge of the strip, and the second is analytic below the top edge of the strip, so that both are then analytic in the strip. It was not obvious to me that there were multiple branches of each integral in the first place, nor was it obvious why each branch should have simple analyticity properties (half plane), but now it is all in the bag (I think). Below I state my confusion and review my web search to get clarification. In the end, clarification had to come from me, I cannot imagine anyone I know coming to the same conclusion.

A similar subject is the analyticity of the Fourier Transforms of one-side functions like $f_+(x)$. That is discussed in D:\Work\MyInterests\Math\Transforms\Fourier Transforms\ **Fourier Transform Notes.doc**. For example we find that $f^_+(\omega)$ is analytic in an upper half plane. You can decompose f(x) and $f^(\omega)$ into a sum of terms which looks a little like the "sum splitting" of Stak p 316. However, on page 316 the + and - labels do not imply one sided functions, nor are the terms even regards as Fourier Transforms.

This relates to Stak page 316. He assumes $a(\omega)$ is analytic only in the strip shown. He assumes ω lies inside the strip and he uses Cauchy's theorem to write $a(\omega)$ as the contour integral shown. He assumes that $a(\omega)$ decays to the left and right such that the side contours vanish, so we just have the two linear contours shown. Yes, C_1 runs below ω, and C_2 runs above ω, also as shown.

He then claims that the first integral in 7.291 is analytic in ω in the entire half plane $\text{Im}(\omega) > d$ where d is the bottom of the strip. I do not understand why that conclusion may be reached. Why should that be true?? I am "missing something", as usual. For example, if ω lies above the upper edge of the strip, then the Cauchy theorem we started with is not justified and so our conclusion is not justified.

I am drawing a complete blank here! I have a Full Mental Block to something that would be obvious to another person. I did a web search and looked at other books, but they just state the same conclusion without giving a reason for the conclusion. As usual, I have nobody to ask about this, I am on my own. So this will be perhaps a 1 week shutdown of progress in Stakgold while I search for the missing fact.

Question 1: Do I believe that the first integral is analytic even for ω in the strip? I know that $a(\omega)$ is analytic in the strip, so I know that the sum of the two integrals is analytic in the strip. In order to SHOW that the first term is analytic at some ω in the strip, I have to be able to adjust contour C_1 so that it is a closed contour enclosing point ω. But I see no way to do that. You could add the side pieces for free, but you cannot add the C_2 top piece for free.

What about adding an upper great circle half to contour C_1? If we knew that $a(\omega)$ decayed at $\text{Im}\omega \to +\infty$, then maybe we could just add that piece. But even so, such a contour might enclose poles of $a(\alpha)$ above the strip, so we don't get our conclusion.

Stak's other book does not have anything but does at least give two new references:

Wiener-Hopf method (see Noble; Carrier, Krook, and Pearson; and Stakgold).

Noble, B., *The Wiener-Hopf technique*, Pergamon Press, London, 1958.

Carrier, G. F., Krook, M., and Pearson, C. E., *Functions of a complex variable*, McGraw-Hill, New York, 1966.

Krook. I found the critical spot in the 2nd reference above, and they do the same as Stak -- they don't say WHY the claim they make is true. Here it is:

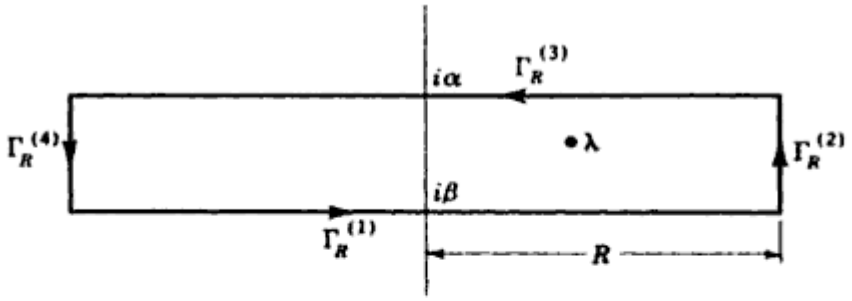


Fig. 8-2 Contour for Eq. (8-31).

Accordingly, we choose Γ_R to be the contour of Fig. 8-2 and identify segments $\Gamma_R^{(j)}$ of the contour as shown in that figure. We consider only functions $G(z)$ for which G is analytic at each point inside Γ_R for all R, and we limit our attention to those $G(z)$ for which the integrals on $\Gamma_\infty^{(1)}$ and $\Gamma_\infty^{(3)}$ exist and for which $\Gamma_\infty^{(2)} + \Gamma_\infty^{(4)} = 0$.

The integral over $\Gamma_\infty^{(1)}$ defines a function which is analytic everywhere above $\Gamma_\infty^{(1)}$. Correspondingly, the integral along $\Gamma_\infty^{(3)}$ defines a function which is analytic everywhere below $\Gamma_\infty^{(3)}$. Thus we have

The reader is supposed to think it is completely obvious that the Γ^1 integral is analytic above that contour. So this source is now used up.

As usual, the problem is finding the right search handle. I tried "sum-splitting" but get no free hits. I do notice that the term "a plus function" means a function analytic I think in the upper half plane, and "a minus function" is the reverse, so that is another possible search phrase. [but this meanings may be wrong]

I found a **Hojuwen** theses that also talks about the "ratio" decomposition idea. But both it and the sum decomposition discussion use the same "missing fact" that is obvious to them but not me. I have to keep putting -elsevier in searches to remove the pay-only flak.

I am getting lots of "plus function" sources now. Often people put decompositions into their Appendix. Here is an example from ***

6 Appendix

The Wiener-Hopf technique requires a multiplicative factorization of the kernel function $K(\lambda)$, defined by (25), into the product of two functions,

$$K(\lambda) = K^+(\lambda)K^-(\lambda)$$

where $K^+(\lambda)$ and $K^-(\lambda)$ are analytic in the upper and lower halves of the complex plane, respectively. This multiplicative factorization is achieved by using the Cauchy integral formula

to obtain an additive decomposition of the function

$$L(\lambda) = \ln(K(\lambda)).$$

The Cauchy integral formula states

$$L(\lambda) = \frac{1}{2\pi i}\oint \frac{L(\zeta)}{\zeta - \lambda}d\zeta,$$

As an aside, we always have two distinct problems, the ratio and the sum. Here he is going to treat the ratio problem by doing the sum of the logs. So far so good, we continue:

where the integral is evaluated around a closed contour that surrounds the pole at $\zeta = \lambda$ and contains no other singularities of $L(\zeta)$. We choose the point $\zeta = \lambda$ to lie in the strip of overlap between the lower and upper half-planes for the Wiener-Hopf analysis, and the contour to be a 'generalized rectangle' within this same region consisting of two long horizontal sides that pass above and below the pole, and two short connecting vertical segments. The form of the function $K(\lambda)$ was chosen so that it approaches one as $\lambda \to \pm\infty$ in the strip. Thus, when we deform the integration contour so that the two long horizontal segments extend from $-\infty$ to ∞, the contributions from the two vertical segments vanish. We can then write

$$L(\lambda) = L^+(\lambda) + L^-(\lambda)$$

I anticipate trouble because he is already assuming analytic half planes exist. We continue nevertheless:

where the plus and minus functions correspond to the contributions from the two horizontal segments of the contour,

$$L^+(\lambda) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{L(\zeta)}{\zeta - \lambda} d\zeta \qquad \text{and} \qquad L^-(\lambda) = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{L(\zeta)}{\zeta - \lambda} d\zeta$$

and the contours of integration for $L^+(\lambda)$ and $L^-(\lambda)$ pass below and above the pole at $\zeta = \lambda$, respectively. The multiplicative factorization of $K(\lambda)$ is then given by

$$K^+(\lambda) = \exp(L^+(\lambda)), \qquad K^-(\lambda) = \exp(L^-(\lambda)).$$

OK, fine, but no claims yet that L^+ is analytic in a half plane. The integral notation is a bit sloppy, but I understand that one goes above the pole and the other below. We continue:

The procedure for calculation of the additive split of $H(\lambda)$ follows directly from the Cauchy integral formula,

$$H^+(\lambda) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{H(\zeta)}{\zeta - \lambda} d\zeta \qquad \text{and} \qquad H^-(\lambda) = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{H(\zeta)}{\zeta - \lambda} d\zeta,$$

where the contours of integration for $H^+(\lambda)$ and $H^-(\lambda)$ pass below and above the pole at $\zeta = \lambda$, respectively.

and then the appendix just ends!!! So he just assumes the thing I don't understand. This is what seems to happen in every source I have found!

I need the reason spelled out in baby steps.

Search: "plus function" -elsevier analytic -springerlink

latorebondeng90245.tripod.com/HMTWSI/c5.pdf

This is Chapter 5 "The Weiner-Hopf and Related Techniques" , but book it is from is not stated.
This source does not have what I want.

www.ms.unimelb.edu.au/publications/HoJuwen.pdf

Already mentioned above, does not explain what I need to know.

gen.lib.rus.ec/get?nametype=orig&md5...

_____ to slow to carefully doc each hit,

A hit I find says this:

0. Then its Fourier integral

$$\hat{f}(\xi) = (2\pi)^{-1} \int_{\mathbb{R}} f(t) \exp(-it\xi) dt$$

makes sense not only for *real* ξ, but also for any $\xi \in \mathbb{C}_-$, the open lower-half-plane, and f is *analytic there* (the condition $f \in L^1(\mathbb{R})$ is not essential, we might speak of a charge, an

The point here is that if f(t) is 0 for t < 0, then the integration has only t > 0. If f^(ξ) exists for real ξ, then if must also exist for complex ξ of the form ξ = x+iy where $e^{-it(x+iy)} = e^{-itx}\, e^{ty}$ converges the integral. Clearly y < 0 will do this, so f^(ξ) "exists" for all ξ in the lower half plane. Does this mean that the function f^(ξ) is *analytic* in the entire lower half plane? Yes, that is what they are saying above. Notice the distinction between upper and lower half plane here! I guess $e^{-it\xi}$ is analytic in ξ, so that makes the integral also analytic. So we have a theorem:

Now it happens that Stak uses the opposite phase in his Fourier projection, so he would conclude that if f(t) is a "plus function", then its Fourier Transform function is analytic in the *upper* half plane! Stak confirms this on page 28.

I see that Stak has more to say on this page. He claims that if $f \sim e^{\alpha x}$ and f is a "plus function" in the sense that f(x<0) = 0, then you can modify the Fourier Transform theorem by shifting the inversion contour up so it lies above the line Im(ω) = α.

Reinterpretation of Stak p 316-317. [probably not correct]

Imagine we have a strip where a(α) is analytic, and above and below that strip we are peppered with various poles. Consider the C_1 integral along the bottom of the strip, as shown p 317. If ω lies above this linear C_1 contour, we shall choose to "close down", add a lower half great circle, which we are allowed to do since a(α) decays in all directions at ∞. In this case, we pick up residues at all the lower poles which are at locations α_i . This means these residues will contain factors $1/(\omega-\alpha_i)$ which arise from the $1/(\alpha-\omega)$ factor of the integrand. Since all the α_i of these residues are *below* C_1, while ω was assumed *above* C_1, the factors $1/(\omega-\alpha_i)$ are just finite numbers. No matter where we put complex ω above the C_1 line, we just get a number for our C_1 integral. If there were a pole in $a_+(\omega)$ above the C_1 line, we would have the integral blow up at that value of ω, but we just said the integral is always just a finite number. Therefore, $a_+(\omega)$ is analytic above the C_1 contour.

By the same argument, $a_+(\omega)$ is also analytic everywhere below the C_1 contour! This is the point Stak and the others do not make clear. If ω lies below the contour, we close "up" and pick up all the northern residues and we get factors $1/(\omega-\alpha_i)$ where now the α_i are located above the C_1 contour. Since they are *above* the contour, while ω is *below* the contour, $a_+(\omega)$ is just a number. So $a_+(\omega)$ has no poles below the contour, and we shall take that to mean it is analytic there.

Furthermore: consider that set of residues we get closing on the opposite side of the selected ω location. We will get as sum of residue terms of the form $\Sigma_i\, C_i/(\omega-\alpha_i)$. As $|\omega| \to \infty$, each of these residues vanishes, so barring some strange situation, the sum of all these residues vanish, and so $a_+(\omega) \to 0$.

Apply this argument to the example. Assume the strip lies in (-1/2,1/2).

OK, if ω lies *above* the strip, we close the contour down and get $a_+(\omega) = k/(\omega+i)$. We find that $a_+(\omega)$ is therefore analytic above the strip. Conversely, if ω lies below the strip, we close the contour up and we then find that $a_+(\omega) = k'/(\omega-i)$ and we conclude that $a_+(\omega)$ is analytic below the strip. But how can the same function $a_+(\omega)$ be these two different functions $k/(\omega+i)$ and $k'/(\omega-i)$? If we are given $a(\omega) = 1/(\omega^2+1)$, how can our $a_+(\omega)$ have two different values?

So let's look more closely at our $a_+(\omega)$ integral which is this

$$a_+(\omega) \equiv \int_{-\infty-i/2}^{+\infty-i/2} d\alpha\, a(\alpha)/(\alpha-\omega) = \int_{-\infty-i/2}^{+\infty-i/2} d\alpha\, (\alpha+i)^{-1}(\alpha-i)^{-1}(\alpha-\omega)^{-1}$$

If we go out to the great circle at some angle θ, we find that along a piece of arc dθ distance is $d\alpha = |\alpha|\, d\theta$ and a finite angle integral there would be $\int d\theta\, |\alpha|\, \alpha^{-3} \sim \int d\theta\, |\alpha|^{-2} \rightarrow 0$ for sure. This says we can always add any piece of great circle we want and not change anything. We have not placed ω any particular place yet. If we agree not to put ω in the lower half of our strip, we can simplify things by raising the contour to the axis, and we then have

$$a_+(\omega) = (1/2\pi i) \int_{-\infty}^{+\infty} d\alpha\, (\alpha+i)^{-1}(\alpha-i)^{-1}(\alpha-\omega)^{-1}$$

We should get the same answer here whether we close the contour up or down! But now we have to select a place for ω. So put ω in the upper half plane, but away from the pole at ω = +i of a(ω).

Close down: In this case, we get (minus sign because wrong contour direction)

$$a_+(\omega) = -\text{ Res } [(\alpha+i)^{-1}(\alpha-i)^{-1}(\alpha-\omega)^{-1}, \alpha = -i\,] = -[(\alpha-i)^{-1}(\alpha-\omega)^{-1}]^{\alpha=-i} = -[(-i-i)^{-1}(-i-\omega)^{-1}]$$

$$= -[(2i)^{-1}(i+\omega)^{-1}] = -(1/2i)(\omega+i)^{-1} = (i/2)\,(\omega+i)^{-1}$$

in agreement with Stak p 317 F.

Close up: In this case we get

$$a_+(\omega) = +\text{ Res } [(\alpha+i)^{-1}(\alpha-i)^{-1}(\alpha-\omega)^{-1}, \alpha = +i\,] + \text{Res } [(\alpha+i)^{-1}(\alpha-i)^{-1}(\alpha-\omega)^{-1}, \alpha = \omega\,]$$

$$= [(\alpha+i)^{-1}\,(\alpha-\omega)^{-1}]^{\alpha=i} + [(\alpha+i)^{-1}(\alpha-i)^{-1}]^{\alpha=\omega}$$

$$= [(i+i)^{-1}\,(i-\omega)^{-1}] + [(\omega+i)^{-1}(\omega-i)^{-1}] = (1/2i)\,(i-\omega)^{-1} + 1/(\omega^2+1)$$

$$= (i/2)/(\omega-i) + 1/(\omega-i)(\omega+i) = [1/(\omega-i)(\omega+i)\,]\,\{\,(i/2)(\omega+i) + 1\,\}$$

$$= (i/2)\,[1/(\omega-i)(\omega+i)\,]\,\{\omega + i + 2/i\,\} = (i/2)\,[1/(\omega-i)(\omega+i)\,]\,\{\omega + i - 2i\,\}$$

$$= (i/2)\,[1/(\omega-i)(\omega+i)\,]\,\{\omega - i\,\} = (i/2)\,[1/\,(\omega+i)\,] = (i/2)\,(\omega+i)^{-1}$$

and we get the same answer as if we had closed down, which is as expected.

Now what happens if we *put* ω in the lower half plane? We are going to get a different answer, and I will compute it the simpler of the two ways.

Close up: We get

$a_+(\omega) = +\text{ Res } [(\alpha+i)^{-1}(\alpha-i)^{-1}(\alpha-\omega)^{-1}, \alpha = +i\,]$ = the first term of our case above

$= (i/2)/(\omega-i)$

But let's do the other way just because it is causing an error elsewhere.

Close down: We get (the first term I steal from the above close-down calculation)

$a_+(\omega) = -\text{ Res } [(\alpha+i)^{-1}(\alpha-i)^{-1}(\alpha-\omega)^{-1}, \alpha = -i\,] - \text{Res } [(\alpha+i)^{-1}(\alpha-i)^{-1}(\alpha-\omega)^{-1}, \alpha = \omega\,]$

$= (i/2)\,(\omega+i)^{-1} - [(\alpha+i)^{-1}(\alpha-i)^{-1}]^{\alpha=\omega}$

$= (i/2)\,(\omega+i)^{-1} - (\omega+i)^{-1}(\omega-i)^{-1}$

$= (i/2)\,[(\omega+i)^{-1}(\omega-i)^{-1}]\,\{(\omega-i) - (2/i)\} = (i/2)\,[(\omega+i)^{-1}(\omega-i)^{-1}]\,\{\omega-i+2i\}$

$= (i/2)\,[(\omega+i)^{-1}(\omega-i)^{-1}]\,\{\omega+i\} = (i/2)\,(\omega-i)^{-1}$

which agrees with the close up result.

How then do we explain the fact that $a_+(\omega)$ is two different functions depending on where we place ω ?

So I am now going to rewrite Stak's section:

Now that we have 7.291, let's refer to the first integral as $a_+(\omega)$ and the second as $a_-(\omega)$ so that we then have obtained $a(\omega) = a_+(\omega) + a_-(\omega)$. We know from our exercise above that the following is true:

$a_+(\omega)$ is analytic for ω above $\text{Im}(\omega) = d$ *and* for ω below $\text{Im}(\omega) = d$
$a_-(\omega)$ is analytic for ω above $\text{Im}(\omega) = c$ *and* for ω below $\text{Im}(\omega) = c$

Aside: This follows intuitively because the integrand is analytic in ω away from the contour and we are just adding these analytic integrands to get a sum that is analytic.

Basically, each of these functions is $a_{\pm}(\omega)$ analytic everywhere, except for a value of ω on its contour because then the contour runs right through a pole and we have the usual issues there. So assume that ω does not lie on either contour. In particular, both $a_{\pm}(\omega)$ are analytic on the open strip.

As a *subset* of the above four true facts, we can *select* these two facts:

$a_+(\omega)$ is analytic for ω above $\mathrm{Im}(\omega) = d$	C_1
$a_-(\omega)$ is analytic for ω below $\mathrm{Im}(\omega) = c$	C_2

And as $|\omega| \to \infty$ in any direction, $a_+(\omega) \to 0$.

We can now examine Stak's example in this context. We see that when ω lies above C_1, we close down and we find that $a_+(\omega) = (i/2)(\omega+i)$. This is just a number when ω lies above C_1. But if ω were below C_1, we would close up and get something that is also just a finite number when ω lies below C_1 , but Stak does not show this in his example.

So I still claim that $a_+(\omega)$ is analytic above *and* below its contour, wherever that line might be. But his restricted claims are also true.

What about the unambiguousness of the decomposition? First of all, it seems that d and c are a bit arbitrary. You could use d/2 and c/2 just as well and get different C_1 and C_2 contours. And you can have the contours just above or below the d,c you chose. So I suspect that the decomposition is only unique if you (1) pick a specific c and d; (2) pick specific τ and σ along which to run the contours.

So for the moment, I guess that resolves my big mystery of the day.

Next Day: Logic test question: Each of our two terms I claim is analytic for all ω away from its contour. Therefore, the sum of the two terms is analytic for all ω away from both contours. Therefore, the LHS should be analytic away from both contours. But we know it is not, just look at the example with strip say (-1/2,+1/2) in y: there are poles in $a(\omega)$ outside the strip above and below it.

Resolution: the identification of $a(\omega)$ with the sum $a_+(\omega) + a_-(\omega)$ is only valid when ω lies in the strip. When ω lies outside the strip, the equality no longer holds, and it is OK for the RHS to be analytic outside the strip while the LHS is not. Thus, our "sum splitting" formula is really *only valid inside the strip*. Or in a that strip enlarged until the two edges hit singularities. This is because it is based on Cauchy's theorem where we assumed no singularities in the strip.

Now can we regard $a(\omega) = a_+(\omega) + a_-(\omega)$ as an "analytic extension" of $a(\omega)$ beyond the strip?