Delta Function Rules PhL 4.28.11

See summary of rules at the end.

Review of Distribution Theory

Back on page 35 Stak derives four rules for distributions:

Translation Rule: $\langle t(x-a),\varphi(x)\rangle = \langle t(x),\varphi(x+a)\rangle$

For example, applied to $t = \delta(x)$ this would say

$$\langle \delta_a,\varphi(x)\rangle = \langle \delta_0,\varphi(x+a)\rangle = \varphi(0+a) = \varphi(a)$$

which is consistent with what we know. Then we have

Scaling Rule:

$$\langle t(ax),\varphi(x)\rangle = 1/|a| \langle t(x),\varphi(x/a)\rangle$$

and we can apply this to a delta function

$$\langle \delta(ax),\varphi(x)\rangle = 1/|a| \langle \delta(x), \varphi(x/a)\rangle = 1/|a| \varphi(0) = 1/|a| \langle \delta, \varphi(x)\rangle$$

Comparing both sides we would say in terms of symbolic functions

$$\delta(ax) = \delta(x)/|a|$$

which we also know to be true. Finally, we have the

Derivative Rule:

$$\langle \partial_x t, \varphi\rangle = - \langle t,\partial_x\varphi\rangle$$

which we can apply to $t = \delta_b$ to get

$$\langle \delta'_b, \varphi\rangle = - \langle \delta_b,\varphi'\rangle = - \varphi'(b)$$

and there is not much more we can say here. The idea is just parts integration with a φ test function having compact support so the parts go away. This does not give us a result δ'(x-b) = something in terms of symbolic functions. More generally

$$\langle \delta^{(n)}_b, \varphi\rangle = (-1)^n \langle \delta_b,\varphi^{(n)}\rangle = (-1)^n \varphi^{(n)}(b)$$

Function Rule:

$\langle f\, t, \varphi \rangle = \langle t, f\varphi \rangle$

This is a sort of nothing rule, but applied to $t = \delta_{\mathbf{b}}$ it says

$$\langle f\, \delta_{\mathbf{b}}, \varphi \rangle = \langle \delta_{\mathbf{b}}, f\varphi \rangle = f(b)\varphi(b) = f(b) \langle \delta_{\mathbf{b}}, \varphi \rangle = \langle f(b)\delta_{\mathbf{b}}, \varphi \rangle$$

from which we conclude in terms of symbolic functions that $f(x)\delta(x-b) = f(b)\delta(x-b)$ which is not too earth shattering.

Combination of the Function Rule with a First Derivative Rule:

$$\langle f\, \partial_{\mathbf{x}} t, \varphi \rangle = \langle \partial_{\mathbf{x}} t, f\, \varphi \rangle = - \langle t, \partial_{\mathbf{x}}(f\varphi) \rangle$$

which we can apply to $t = \delta_{\mathbf{b}}$ to get

$$\langle f\, \delta'_{\mathbf{b}}, \varphi \rangle = \langle \delta'_{\mathbf{b}}, f\, \varphi \rangle = - \langle \delta_{\mathbf{b}}, \partial_{\mathbf{x}}(f\varphi) \rangle = - \partial_{\mathbf{x}}(f\varphi)\big|^{\mathbf{x=b}} = - [\, f(b)\varphi'(b) + f'(b)\varphi(b)]$$

$$= - f(b)\varphi'(b) - f'(b)\varphi(b) = - f(b) \langle \delta_{\mathbf{b}}, \varphi' \rangle - f'(b) \langle \delta_{\mathbf{b}}, \varphi \rangle = + f(b) \langle \delta'_{\mathbf{b}}, \varphi \rangle - f'(b) \langle \delta_{\mathbf{b}}, \varphi \rangle$$

$$= \langle f(b)\, \delta'_{\mathbf{b}} - f'(b)\, \delta_{\mathbf{b}}, \varphi \rangle$$

and the symbolic function result is then

$$f(x)\, \delta'(x-b) = f(b)\, \delta'(x-b) - f'(b)\, \delta(x-b)$$

a result which Maple verifies

```
f(x)*Dirac(1,x-b);
                    f(x) Dirac(1, x - b)
simplify(%);
  f(b) Dirac(1, x - b) - D(f)(b) Dirac(x - b)
```

In the case that b = 0 we get

$$f(x)\, \delta'(x) = f(0)\, \delta'(x) - f'(0)\, \delta(x)$$

and if we set $f(x) = x^{\alpha}$ we get,

$$x^{\alpha}\, \delta'(x) = - \alpha(x^{\alpha-1})^{\mathbf{x=0}}\ \delta(x)$$

Therefore we only get well-defined results for two ranges of α:

$x^{\alpha}\, \delta'(x) = 0$ $\quad$ $\mathrm{Re}(\alpha) > 1$

$x^{\alpha}\, \delta'(x) = -\delta(x)$ $\quad$ $\alpha = 1$

For example

$x\,\delta'(x) = -\delta(x)$
$x^2\delta'(x) = x^3\delta'(x) = \ldots = 0$

Combination of the Function Rule with a Second Derivative Rule:

$\langle f\,\partial_x^2 t, \varphi\rangle = \langle \partial_x^2 t, f\varphi\rangle = +\langle t, \partial_x^2(f\varphi)\rangle$

Applied to $t = \delta_b$ we get

$\langle f\,\delta''_b, \varphi\rangle = \langle \delta_b, \partial_x^2(f\varphi)\rangle = [\partial_x^2(f(x)\varphi(x)]^{x=b}$

$= [\, f''(b)\varphi(b) + 2\, f'(b)\varphi'(b) + f(b)\varphi''(b)$

$= f''(b)\langle\delta_b,\varphi\rangle - 2\, f'(b)\langle\delta'_b,\varphi\rangle + f(b)\,\langle\delta''_b,\varphi\rangle$

and we conclude that

$f(x)\,\delta''(x-b) = f''(b)\delta(x-b) - 2f'(b)\delta'(x-b) + f(b)\,\delta''(x-b)$

And if $b = 0$ and $f(x) = x^n$ this says

$x^n\,\delta''(x) = n(n-1)0^{n-2}\delta(x) - 2n0^{n-1}\delta'(x) + 0^n\,\delta''(x)$

and now we need $n \geq 2$ to get a reasonable result, such as

$x^2\,\delta''(x) = 2\,\delta(x)$
$x^3\,\delta''(x) = 0$ and same for higher powers

Here are some general rules on this:

$x^n\delta^{(n)}(x) = (-1)^n\, n!\,\delta(x)$

$x^{n+1}\delta^{(n)}(x) = 0$ and of course for higher powers of x as well

Combination of the Function Rule with an nth Derivative Rule:

$\langle f\,\partial_x^n t, \varphi\rangle = \langle \partial_x^n t, f\varphi\rangle = +\langle t, \partial_x^n(f\varphi)\rangle$

We use the result

$$\partial_x^n(f\varphi) = \Sigma_{k=0}^n\, \partial_x^k f\ \partial_x^{n-k}\varphi \binom{n}{k}$$

to find in the end that

$$f(x)\,\delta^{(n)}(x-b) = \Sigma_{k=0}^{n} \binom{n}{k} (-1)^k\, f^{(n-k)}(b)\,\delta^{(k)}(x-b)$$

Summary of Delta Function Rules

$$\delta(ax) = \delta(x)/|a|$$

$$f(x)\delta(x-b) = f(b)$$

$$f(x)\,\delta'(x-b) = f(b)\,\delta'(x-b) - f'(b)\,\delta(x-b)$$

$$f(x)\,\delta''(x-b) = f''(b)\delta(x-b) - 2f'(b)\delta'(x-b) + f(b)\,\delta''(x-b)$$

$$f(x)\,\delta^{(n)}(x-b) = \Sigma_{k=0}^{n}(-1)^k \binom{n}{k} f^{(n-k)}(b)\,\delta^{(k)}(x-b)$$

$x^n\delta^{(n)}(x) = (-1)^n\, n!\, \delta(x)$

$x\,\delta'(x) = -\delta(x)$	n=1
$x^2\,\delta''(x) = 2\delta(x)$	n=2
etc	

$x^{n+1}\delta^{(n)}(x) = 0$ eg

$x\,\delta(x) = 0$	n=1
$x^2\,\delta'(x) = 0$	n=2
etc	