Collection of Delta Function Sequences PhL 4.12.11

1. $\mathbf{f_t(x) = \exp(-x^2/(4t))/ (\sqrt{\pi t}\,)}$ $\qquad$ $\mathbf{\lim_{t\to 0} f_t(x) = 2\,\delta(x)}$

We start with

$$f_t(x) = \exp(-x^2/(4t))/ (\sqrt{\pi t}\,)$$

$$\int_{-a}^{a} f_t(x)dx = \int_{-a}^{a} \exp(-x^2/(4t))/ (\sqrt{\pi t}\,)dx = (1/ (\sqrt{\pi t}\,)) \int_{-a}^{a} \exp(-x^2/(4t))\, dx$$

$$= (2/ (\sqrt{\pi t}\,)) \int_{0}^{a} \exp(-x^2/(4t))\, dx \qquad \text{since integrand is even in x}$$

$$= (2/ (\sqrt{\pi t}\,))\sqrt{\pi t}\, \text{erf}(a/(2\sqrt{t}\,) \qquad \text{// Maple}$$

$$= 2\, \text{erf}(a/(2\sqrt{t}\,)$$

We have thus shown that

$$\int_{-a}^{a} f_t(x)dx = 2\, \text{erf}(a/(2\sqrt{t}\,)$$

Now we take the limit t→0 on both sides

$$\lim_{t\to 0} \{ \int_{-a}^{a} f_t(x)dx \} = 2\, \text{erf}(0) = 2$$

Since this result is independent of a, it must be true no matter how small a is. Therefore we conclude this symbolic function fact

$$\lim_{t\to 0} f_t(x) = 2\, \delta(x)$$

2. Corollary. Now suppose we consider instead $g_t(x) = f_t(x)\theta(x)$:

$$g_t(x) = \theta(x)\, \exp(-x^2/(4t))/ (\sqrt{\pi t}\,)$$

Then if we carry out the steps above, we will find that

$$\int_{-a}^{a} g_t(x)dx = 1\, \text{erf}(a/(2\sqrt{t}\,)$$

$$\lim_{t\to 0} g_t(x) = \delta(x)$$

Really we are just claiming here that

$$\delta(x)\theta(x) = \delta(x)\theta(0) = \delta(x)/2$$