About Distributions as discussed in Stakgold Chapter 1. PhL 4.2.09

I am having a lot of trouble grokking this subject, so here is a sandbox to play in for a while with the endless contradictions I keep coming up with.

Contradiction #1. [resolved during consideration]

On page 136 we have Riesz which says:

"Every continuous linear functional T[φ] on HS H can be written as <φ ,f> where f ∈H. "

The inner product of course is the one which defines H. Now, when we talk distributions we are talking functionals. What is the relevant HS H? We are going to talk about things like $T_\delta[\varphi] = \varphi(0)$. My feeling is that the HS is the space he calls D on page 30 and which he says is a real linear space. The HS is the space of test functions, which is a very small subset of L_2.

Now consider this linear functional

$$T_\delta[\varphi] = \varphi(0)$$

Is this linear?

$$T_\delta[a_1\varphi_1 + a_2\varphi_2] = T_\delta[\psi] = \psi(0) = a_1\varphi_1(0) + a_2\varphi_2(0) = a_1 T_\delta[\varphi_1] + a_2 T_\delta[\varphi_2]$$

Sure looks linear to me!

Is this continuous? Let δφ be some test function variation.

$$T_\delta[\varphi+\delta\varphi] = T_\delta[\psi] = \psi(0) = \varphi(0) + \delta\varphi(0) = (\varphi+\delta\varphi)(0) \quad ???$$

Aside: How do you show that a *function* f(x) is "continuous". What is the requirement?

$$f(x) \rightarrow f(a) \text{ as } x \rightarrow a$$

$$| f(x) - f(a)| \rightarrow 0 \text{ as } | x - a | \rightarrow 0$$

For any ε you give me, I can find a δ such that $| x - a | < \delta \Rightarrow | f(x) - f(a)| < \varepsilon$. In other words, I can jam f(x) as close as I want to f(a) by jamming x close to a.

Clarification: What is a "linear function" in the sense we are talking here? If we apply the usual definition of "linear operators" then we have $f(\alpha x) = \alpha f(x)$ as one of our two conditions, which goes along with the fact that f(0) = 0. Thus, a "linear function" must have the form f(x) = Ax. We normally think of a linear function as having the form Ax + B, but in the sense of "linear" here, you cannot have B ! Notice that the other linearity property f(x + y) = f(x) + f(y) is also not met by f(x) = Ax + B.

Show for linear that bounded => continuous

Now if f(x) is a *linear* function in either sense above, we know that:

$| f(x) - f(a)| = |f(x-a)|$

The bound on function f would be $||f|| = \max |f(x)| / | x |$ so we know that $|f(x)| \leq ||f|| \, | x |$.
Assume f is in fact bounded on our interval. Then:

$| f(x) - f(a)| = |f(x-a)| \leq ||f|| \, | x - a |$

So if you require that $| f(x) - f(a)| < \varepsilon$, I can make it so by setting $| x - a | < \varepsilon / || f || = \delta$. Therefore I have just shown that if a linear function is bounded, it must be continuous.

Show for linear that continuous => bounded

This direction is harder to do, at least for me. Stak gives a little proof on page 136 middle where he is proving this just at the point x = 0 in the space, but I don't really like his proof much. Here is a wiki proof:

- Conversely, it follows from the continuity at the zero vector that there exists a $\delta > 0$ such that $\|L(h)\| = \|L(h) - L(0)\| \leq 1$ for all vectors h in X with $\|h\| \leq \delta$. Thus, for all non-zero v in X, one has

$$\|Lv\| = \left\| \frac{\|v\|}{\delta} L\left(\delta \frac{v}{\|v\|}\right) \right\| = \frac{\|v\|}{\delta} \left\| L\left(\delta \frac{v}{\|v\|}\right) \right\| \leq \frac{\|v\|}{\delta} \cdot 1 = \frac{1}{\delta}\|v\|.$$

This proves that L is bounded.

And here is my translation of this wiki proof for functions and away from the origin. If f(x) is linear and continuous, then we can find δ such that $| f(x) - f(a) | = | f(x-a) | \leq 1$ for all $| x - a | \leq \delta$. Here we are just choosing $\varepsilon = 1$. Restate this as $| f(y) | \leq 1$ for all $| y | \leq \delta$. Then

$| f(x-a) | = | \alpha \, f([x-a]/\alpha) |$ // for any α, since f is linear in the operator sense!

$= | \, | x - a |/\delta \; f(\delta[x-a]/ | x - a |) |$ // just pick $\alpha = | x - a |/\delta$

$= | x - a |/\delta \; * | f(\delta[x-a]/ | x - a |) |$

Now the argument of the function f is $y = \delta[x-a]/ | x - a |$ which has magnitude δ and so this argument surely is $|y| \leq \delta$, and therefore we know that $| f(\delta[x-a]/ | x - a |) | \leq 1$ for all vectors x-a . Thus,

$| f(x-a) | \leq | x - a |/\delta$ for all vectors x-a

Thus we have

$| f(x-a) | / | x - a | \leq 1/\delta$ for all vectors x-a

$| f(y) | / |y| \leq 1/\delta$ for all vectors y

$$\max (|\, f(y)\,| \,/\, |y\,|) \le 1/\delta$$

Thus, $M = \max (|\, f(y)\,| \,/\, |y\,|)$ has the property $M \le 1/\delta$ so M is finite. But M is the bound of f. Thus, if f(x) is continuous, it is bounded.

This continuous $\Leftrightarrow$ bounded concept works the same way for functions, functionals and operators.

Now, back to our $T_\delta[\varphi] = \varphi(0)$ which I know is a linear functional. Is it bounded?

$$\text{bound} = \max |\, T_\delta[\varphi]\,| \,/\, \|\, \varphi \,\| \qquad \varphi \in D$$

$$= \max |\varphi(0)\| \,/\, \|\, \varphi \,\|$$

Hmmm.... it is not obvious to me that a bound exists or does not exist. On page 41 top Stak constructs a sequence of test functions φ_n with area 1 which approach a δ function in the limit $n \to \infty$. It is this:

$\varphi_n(x) = n\varphi(nx)$ where $\varphi(x)$ is our normalized standard test function top of page 41.

We have $\varphi_n(0) = n\varphi(0)$ and we know $\varphi(0) =$ some $K > 0$. We can take n as large as we like and we still have a normalized test function, so this tells me that

$$\|\varphi_n(0)\| \,/\, \|\, \varphi_n \,\| \;\; = \|\varphi_n(0)\| \;\; = n\, \varphi(0)$$

and therefore

$$\max \;\; [\|\varphi_n(0)\| \,/\, \|\, \varphi_n \,\|]_n \;\; = \infty$$

Conclusion: The linear functional $T_\delta[\varphi] = \varphi(0)$ which is defined on the HS of test functions D is linear, but it is **not bounded**! It is therefore **not continuous**, though it sort of looks continuous when you write

$$T_\delta[\varphi+\delta\varphi] = (\varphi+\delta\varphi)(0) \;\; = \varphi(0)+\delta\varphi(0) \;\; = T_\delta[\varphi] + \delta\varphi(0)$$

$$T_\delta[\varphi+\delta\varphi] - T_\delta[\varphi] \;\; = \;\; \delta\varphi(0)$$

I guess the idea is that you can't really show that $\delta\varphi(0) \to 0$ as $\delta\varphi \to 0$ (in the conventional sense). I might imagine some test functions where $\delta\varphi_n \to 0$ in L_2 norm, but $\delta\varphi_n(0) = 1$ for the entire sequence.

My Contradiction #1 was going to be this:

(1) $T_\delta[\varphi]$ is linear and continuous (the continuous part I thought a while ago)
(2) Therefore Riesz says $T_\delta[\varphi] = \langle h,\varphi \rangle$ where h is some member of the HS D ! And the inner product here is the usual L^2 inner product but we sort of restrict to $D \subset L^2$. So $T_\delta[\varphi] = \int_{-\infty}^{\infty} dx\, h(x)\varphi(x)$.

(3) But we know that h = δ which is NOT an element of HS D or even of L^2.

So this contradiction is resolved because the assumption that $T_\delta[\varphi]$ was continuous was wrong!

General Comments on Stakgold's Approach

One big problem is that he treats distributions in Chapter 1, but treats Hilbert Spaces and functionals in Chapter 2. This causes various notational confusions that I will now try to clarify:

1. Later in the book, the notation <a,b> always means an inner product associated with a Hilbert Space. One example is the HS = L^2 on the reals, with its definition as $\langle a,b\rangle = \int dx\ a(x)b(x)$. However, in Chapter 1 he uses this same notation <t,φ> to describe "the action of t on φ". He does this with <t,φ> NOT being in all cases an inner product.

Also later in the book, he uses T[x] to define on operator T:X→R where the range space is the reals. Such a thing is called a functional. But he does not use this notation at all in Chapter 1.

Also later in the book he proves the Riesz theorem which says

"Any continuous linear functional T[x] can be written in the form T[x] = <f,x> where f is some vector in the Hilbert Space associated with the inner product. "

I like to call this vector f "a marker function", and I might then write $T_f[x] = \langle f,x\rangle$. This says there is a homomorphism between the Hilbert Space and the space of linear continuous functions acting on that Hilbert Space.

2. Here then is what I propose to do:

(a) I will use the following three equivalent notations for a functional:

$$" f " = T_f[\varphi] = (f,\varphi)$$

where (f,φ) is in general NOT an inner product in a Hilbert Space. The notations (f,φ) and " f " are just another way to "notate" the functional $T_f[\varphi]$. We can call either of the second two notations "the action of f on φ". I will reserve the notation <f,g> to refer to the usual function integration inner product which works on space L^2 and on any subspace of that space. According to Riesz

$$T_f[\varphi] = (f,\varphi) = \langle f,\varphi\rangle \qquad \text{if } T_f \text{ is a continuous (ie, bounded) linear functional}$$

(b) I shall now define the words "distribution" and "generalized function".

Definition: A "distribution f" is any linear functional " f " = $T_f[\varphi] = (f,\varphi)$ which is defined on a certain subspace S of the Hilbert Space "L^2 on some interval (a,b)". That certain subspace S is the space of all "test functions" -- the space of all C^∞ functions having compact support, which means zero outside some finite range. This space might be written $S = C^\infty_c$. Notice there is no inner product used in this definition of a distribution, although there does of course exist an inner product for L^2 which we could write as <a,b> if a and b are in L^2. We know that the test functions φ are all in L^2, but the "distribution f" is not

necessarily associated with a function f in L^2 or in fact with a function of any kind. Our object " f " is a functional, not a function. Notice that the *value* of any "distribution f" when acting on a test function is $T_f[\varphi] = (f,\varphi)$ = a real number.

3. Question: Are normal "functions" also "distributions" ?

Suppose f(x) is a "normal function". We can define " a distribution f associated with this function f " as follows:

$$T_f[\varphi] = (f,\varphi) \equiv \langle f,\varphi \rangle = \int_a^b dx\, f(x)\varphi(x)$$

Thus, we could say as an example for f(x) = x,

$$\text{"x"} = T_x[\varphi] = (x,\varphi) \equiv \langle x,\varphi \rangle = \int_a^b dx\, x\, \varphi(x)$$

So this is a fine distribution which we might call "x". If we happened to select our Hilbert Space such that (a,b) = (-∞,∞), then the function f(x) = x is in fact NOT an L^2 function because the integral of x^2 diverges. However, the above integral $\int_{-\infty}^{\infty} dx\, x\, \varphi(x)$ is still finite for any test function due to the compact support idea of a test function. Thus, the above defines a fine distribution " f " = "x" even though the function f(x) = x is not L^2.

Here is another example where the function $f(x) = \exp(-x^2)$ is in fact L^2 for (-∞,∞)

$$T_f[\varphi] = (\exp(-x^2),\varphi) \equiv \langle \exp(-x^2),\varphi \rangle = \int_{-\infty}^{\infty} dx\, \exp(-x^2)\, \varphi(x)$$

The name of this distribution is " $\exp(-x^2)$ " which we wrote above as " f ". Here we see an advantage of the (f,g) notation because we can write out the function more easily. We might also use

$$T_f[\varphi] = T[\varphi; f]$$

then we could write our distribution as $T[\varphi; \exp(-x^2)]$.

In these examples and in our definition, we see that "a normal function f" and its "distribution f " are NOT the same thing. The "distribution f " is "*associated with*" a "normal function f" via the integral.

$$\text{" f "} = T_f[\varphi] = (f,\varphi) \equiv \langle f,\varphi \rangle = \int_a^b dx\, f(x)\varphi(x)$$

" distribution f " ⟂ "normal function f "

The distribution is a functional which is labeled by the marker function f. The distribution is an integral of f against any and all test functions. For every marker function f in L^2 (quite a lot of functions!), there exists a distribution " f " as shown above. So, every "normal function f " is *associated with* (or "has") a

distribution "f", but we would never say that normal functions are also distributions because they are different animals. A distribution for a normal function f is a set of integrals of the function f against various test functions φ, whereas a function f is a set of values of f at different points x on the real axis. You can think of each of these objects as a "table", but they are different tables (each with ∞ # of entries).

Here is just a quick example in passing. Heaviside H(x) is a "normal function" so we can say

$$\text{" H "} = T_H[\varphi] = (H,\varphi) \equiv \langle H,\varphi\rangle = \int_{-\infty}^{\infty} dx\, H(x)\varphi(x) = \int_0^{\infty} dx\, \varphi(x)$$

Again, the function is H(x) and the distribution is "H(x)" and all is well. There is no need here for any kind of "generalized function" as defined below.

Suppose we consider f(x) = 1/x . This function is not L^2 if (a,b) straddles x = 0 because $\int dx/x^2$ diverges. But the real problem is that, if we try our integral above $\int_a^b dx\, f(x)\varphi(x)$, the integral has to be finite for all test functions. Suppose a = -b + 1 and we use our standard issue symmetric test function on this interval (-b+1,b) where we assume a straddle. This integral diverges for such a test function because the test function itself is reasonable and finite at x = 0, value φ(0), but 1/x is non-integrable there and we are "off center" with our symmetric test function. If we happen to choose a symmetric test function on (-a,a) you might argue that that *particular* integral is 0, but it is easy to find test functions as we just did which have a diverging integral. Therefore we cannot use our integral form $\int_a^b dx\, f(x)\varphi(x)$ to represent our " distribution 1/x ". But we can still talk about T[φ; 1/x] = (1/x,φ) as being a perfectly happy distribution named " 1/x ". Therefore, in this example,

$$\text{" 1/x "} = T[\varphi;\, 1/x] = (1/x,\varphi) \neq \langle 1/x,\varphi\rangle$$

The reason that 1/x fails to have the <1/x,φ> representation is that there is a place inside the interval (a,b) where the function 1/x is not integrable over arbitrary sub-intervals. This place is x = 0 and for any subinterval that hits or overlaps this x = 0 point, we have a problem. One says that at x = 0, the function f(x) = 1/x is "not locally integrable". Of course we could also say that the functional T[φ; 1/x] is therefore not bounded (since it is ∞ for certain φ), and thus T is not continuous, and thus Riesz does not apply, and thus we are not surprised that T[φ; 1/x] ≠ <1/x,φ>.

4. Locally integrable (LI) functions. A LI function defined on the interval (a,b) is one that can be integrated in the normal classical fashion over any finite subinterval in that interval. Certainly at any point where a function is continuous we will have the LI property. A function with a finite jump like H(x) at x=0 is still LI at x=0. Where you might have a problem is if your function is infinite at some point in the interval. Another case might be some weird pathological function definition such that the integral does not exist, but I don't care about that case (I think the Lebesgue integral solves this problem anyhow). We have seen that f(x) = 1/x is *not* LI at x = 0. The reason is that the integral of this function is ln(x) which blows up at x = 0. The function f(x) = ln(x) blows up at x = 0, but its integral is x ln(x) - x which is finite at x = 0, and so ln(x), although infinite at x = 0, is still LI there. If we consider a set of nested intervals I_n which shrink around a point of interest like x = a, then we expect that as the interval gets very small, we

can say (if f is continuous at a) that $\int_{In} dx\ f(x) \approx f(a)(\int_{In} dx)$ where f(a) is some finite value of the function, and then ($\int_{In} dx$) is the width of the interval. If we take the limit that this interval goes to 0 width, then ($\int_{In} dx$)→ 0. So yes, this is a "property" of LI functions which Stak quotes in his book, and which I am now fine with (a different proof would be needed for f = ln(x-a) but conclusion is still true). It is clear that the "function" δ(x) does not have this property and is therefore not LI, just as Stak says. The problem is that δ(x) is infinite at x=0 with finite area. Of course the Heaviside function H(x) is LI.

The upshot of all this discussion is the following page 32 (top)Theorem of Stakgold which should be completely obvious to the reader so far of the current document,

If the function f(x) is LI over the interval (a,b) of interest, then we can associate with it a distribution " f " which can be represented in the usual integral fashion:

$$\text{" f "} = T_f[\varphi] = (f,\varphi) = \langle f,\varphi\rangle = \int_a^b dx\ f(x)\varphi(x)$$

" distribution f " |.. "a LI f "

The integral is finite for all test functions.

Just because a function like 1/x is not LI does not mean there is no "distribution 1/x" . It just means that the "distribution 1/x " cannot be represented as the above classical integral. So the existence of a distribution associated with a function is not at question, it is the integral form that is in question.

5. Definition of a generalized function.

We know that for LI functions, we can say

$$\text{" f "} = T_f[\varphi] = (f,\varphi) = \langle f,\varphi\rangle = \int_a^b dx\ f(x)\varphi(x)$$

" distribution f " |.. "a LI f "

and we know that for functions that are not LI, we can still have a distribution

$$\text{" f "} = T_f[\varphi] = (f,\varphi)$$

" distribution f "

We may be able to find an "expression" for the "distribution f" where f is not an LI function, such as

$$\text{" f "} = T_f[\varphi] = (f,\varphi) = \text{"some expression involving f and } \varphi\text{"}$$

" distribution f "

Let's immediately take an example.

Example 1:

We consider the function 1/x which is not LI. It is shown in Stak on page 49 that there is in fact an expression you can write for the "distribution 1/x " :

$$\text{" } 1/x \text{ " } = T_{1/x}[\varphi] = (1/x,\varphi) = \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) dx\,(1/x)\,\varphi(x) = \fint dx\,(1/x)\,\varphi(x)$$

Here we have a rule for expressing the "distribution 1/x " as a limit of a sum of two integrals, which limit we often represent using the tick integral sign. This expression is NOT an inner product in the L^2 sense, it is not even an "integral". It is the limit of the sum of two integrals, but we cannot write that as a single classical integral. Notice that as $\varepsilon\to 0$, each of the two integrals diverges, the first one gets large negative because $x < 0$ on that side, so that 1/x = large negative number. But the second integral is a correspondingly large positive number. The two infinities cancel out leaving a finite result.

Now, we can "pretend" there is a classical integral, and define a certain thing to replace the 1/x above and have it *look like* a classical integral. We thus say:

$$\fint dx\,(1/x)\,\varphi(x) = \int dx\ \text{GeneralizedFunctionVersionOf}(1/x)\,\varphi(x) = \int dx\ \text{GFVof}(1/x)\,\varphi(x)$$

The thing on the right is just a notation! There is no such classical integral. The meaning of the thing on the right is the thing on the left! So anytime you encounter the thing on the right, you mean the thing on the left. If you encounter GFVof(1/x) all by itself, not in an integral, it has no meaning as a classical function until you put it into an integral. However, by looking at the thing on the left in the above, we may come up with some "manipulation rules" for GFVof(1/x). We can then validly "manipulate" GFVof(1/x) in expressions where it appears without an integral, but we only recover the meaning of the thing when we apply the classical integration to it, and then the meaning is what is shown on the left above. Stakgold gives this particular generalized function a certain name pf(1/x), we we have

$$\text{" } 1/x \text{ " } = T_{1/x}[\varphi] = (1/x,\varphi) \fint dx\,(1/x)\,\varphi(x) = \int dx\ \text{pf}(1/x)\,\varphi(x)$$

where he says pf means pseudofunction, which is of course just our notion of generalized function.

Now here is an example of an equation involving pf(1/x) in isolation (ie, un-integrated)

$$\lim_{R\to\infty}(1/x)(1 - \cos(Rx)) = \text{pf}(1/x) = \lim_{R\to\infty}(2/x)(\sin^2(Rx/2))$$

where pf(1/x) is our generalized function. The *meaning* of this equation appears only when we put it into an integration against a test function,

$$\lim_{R\to\infty}\int_{-\infty}^{\infty} dx\,(1/x)(1 - \cos(Rx))\varphi(x) = \int_{-\infty}^{\infty} dx\ \text{pf}(1/x) = \fint dx\,(1/x)\,\varphi(x)$$

The three equivalent expressions on this line are all finite numbers, there is no mystery.

Now there is a **little question** here that comes up: is the above limit possibly true for φ(x) not being a test function? The problem area seems to be in the neighborhood of x = 0, so perhaps the above is true for any φ(x) that is reasonable in the neighborhood of x = 0, perhaps any φ(x) that is continuous there. For example, if we try φ(x) = 1, we have $\int dx\ (1/x) = 0$ which is pretty reasonable. The LHS is also 0 since its integrand for any R is odd, and $\lim_{R\to\infty} 0 = 0$. This is a bit of a special case. The important thing would be that the integral on the right converge to a finite value, and the R integral on the left as well. Probably this happens for functions φ(x) which are not official test functions. Stak has not commented on this issue as of page 50. [If this is true, then we are "continuing" the equality from the test function subspace to some larger part of L_2.]

Distinction between a classical function and a generalized function.

Classical functions include those which can be constructed from elementary functions. For example, f(x) = 5/x + sin(a+x)*exp(4x²)* (1-ln(x)). But we know there are fancier functions like Γ(x) that are also perfectly fine classical functions, but which cannot be written in terms of elementary functions except perhaps as a limit or a series.

Plan A. Maybe we can say that a classical function f:R→R is a piecewise continuous (PC) function which has perhaps some singular points. The piecewise sections of the function can be **graphed** in that for every x, we can plot f(x) on a piece of paper (or two for complex). In the neighborhood of any singular point, we can approximate the classical function in terms of elementary functions. This sounds pretty good to me right now. It might be better to think of f:C→C and then talk about Re(f) and Im(f).

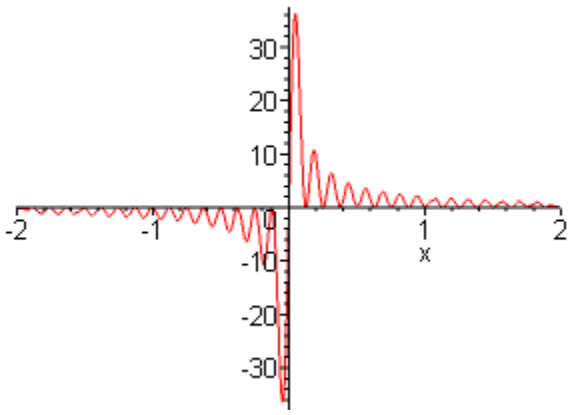
How could you approximate δ(x) near x=0 with an elementary function?

Also, if you make a graph of δ(x) with 10 billion points, it cannot be distinguished from the graph of the function f(x) = 0. So δ(x) really has no distinguishing "graph".

A generalized function can always (I think) be written as a limit of a sequence of classical functions. We know how this works for the δ(x) case. Here is another example relating to the example above:

$$pf(1/x) = \lim_{R\to\infty} (1/x)(1 - \cos(Rx)) = \lim_{R\to\infty} (2/x)\sin^2(Rx/2)$$

The sequence functions (2/x)sin²(Rx/2) are finite everywhere in this case. Here is R = 50:



As R→∞, the oscillation gets faster without limit, and we then *cannot even draw* the graph of the function in that limit. It has no graph. For this function, we cannot approximate pf(1/x) near *any* value of x with an elementary function. If you had to draw something, you would draw a shaded area under each of the 1/x envelopes. Every point in the shaded area lies on the generalized function. The slope of the function is infinite at every value of x.

Example 2: We consider the function shown page 47 and called $f(x) = \log(x)_+$, a variant of the regular log(x) which is zero on the left side. We find the following fact:

$$T[\varphi; \log(x)_+] = (\log(x)_+, \varphi) = \int_{-\infty}^{\infty} dx \log(x)_+ \varphi(x)$$

It happens that the function $\log(x)_+$ is in fact LI, so the classical integral on the right is finite and nothing strange happens. Notice that $[\log(x)_+]' = (1/x)_+$ as shown page 48 E.

We then do the usual idea that the derivative of a distribution is a distribution and find that

$$\text{" } [\log(x)_+]' \text{ "} = T[\varphi; [\log(x)_+]'] = ([\log(x)_+]', \varphi)$$

$$= \text{" } (1/x)_+ \text{"} = T[\varphi; (1/x)_+] = ((1/x)_+, \varphi) = \lim_{\varepsilon \to 0} \left[\int_{\varepsilon}^{\infty} dx\, (1/x)_+ \varphi(x) + \varphi(\varepsilon)\log(\varepsilon) \right]$$

which is a finite quantity, by the way, though not totally obvious at first. Notice that we could put either 1/x or $(1/x)_+$ inside this integral, since x > 0. We want to put $(1/x)_+$ because $[\log(x)_+]' = (1/x)_+$ and this makes it more closely match the LHS notation. So we have some kind of "expression" on the right, and we define a Generalized Function to model it:

$$= \int_{-\infty}^{\infty} dx \;\; \text{GeneralizedFunctionVersionOf}(1/x)\, \varphi(x)$$

$$= \int_{-\infty}^{\infty} dx \;\; \text{pf}(1/x)_+ \varphi(x) \equiv \lim_{\varepsilon \to 0} \left[\int_{\varepsilon}^{\infty} dx\, (1/x)_+ \varphi(x) + \varphi(\varepsilon)\log(\varepsilon) \right]$$

Again, the notation $\int_{-\infty}^{\infty} dx \;\; \text{pf}(1/x)_+ \varphi(x)$ is not a classical integral nor even an integral of any kind. It is the limit of a true integral added to another term as shown. As before we can endow $\text{pf}(1/x)_+$ with a life of its own.

General rule: when we don't have some special name for a Generalized function, use we the name of the distribution, remove our double quotes for distribution name, and add pf:

$$\text{" } (1/x)_+ \text{"} = \int_{-\infty}^{\infty} dx \;\; \text{pf}(1/x)_+ \varphi(x)$$

Example 3:

$$" \delta " = T_{\delta}[\varphi] = (\delta,\varphi) = \varphi(0)$$

This is a "perfectly fine" distribution, linear as can be. But in this case, our "expression" is not the limit of some combination of integrals as in our first example. In this example, we are not really starting with some function f(x) like 1/x or log(x). We are just defining the above linear functional and giving it the name δ. At this point there is no function δ(x), just a distribution named "δ". The action of " δ " on any test function is to produce its value at x = 0. I have shown earlier (with those φ_n) that this functional is unbounded and non-continuous and thus does not have a normal <δ,φ> representation as an inner product with some function δ(x) in L^2.

As in all the above cases where we have one of these strange objects, we "pretend" we can represent it as a classical integral, and we insert a Generalized Function. Since we did not start with a function in this example, we cannot say it is pf(some function(x)) as in previous examples. So we give it the Dirac historical name δ(x) and write

$$" \delta " = T_{\delta}[\varphi] = (\delta,\varphi) = \varphi(0) = \int_{-\infty}^{\infty} dx \ \delta(x)\,\varphi(x)$$

As usual, there is no real integral here. The δ(x) only has true meaning when it is inside an integral, and then its meaning is exactly as shown: φ(0). As usual, we might be able to do "manipulations" with δ(x) if we can prove they are valid with the integral present. For example, consider:

$$\delta(ax) = \delta(x)/a \qquad a>0$$

Is this true? Well:

$$\int_{-\infty}^{\infty} dx \ \delta(ax)\,\varphi(x) = (1/a) \int_{-\infty}^{\infty} dy \ \delta(y)\,\varphi(y) = (1/a)\,\varphi(0)$$

$$\int_{-\infty}^{\infty} dx \ [\delta(x)/a\,]\ \varphi(x) = (1/a) \int_{-\infty}^{\infty} dx \ \delta(x)\,\varphi(x) = (1/a)\,\varphi(0)$$

So yes, the two generalized functions are equal. Notice that we are allowed to formally "manipulate" our classical integral shell here such as doing a change of integration variable (or parts integration) . The integration mechanism is real, but the generalized function inside is not real, so as a whole we don't have "a classical integral of a classical function".

So in this example, we have a name δ(x) for the generalized function, but the name of the distribution is just " δ ". In an earlier example, we displayed a "function" [1/x] in both our distribution and generalized function notations: " 1/x " and pf(1/x). But in the current example, the thing that corresponds to pf(1/x) is δ(x) and there is no function we can use for the double quote name of the distribution. It is just " δ ".

Question: Can you add generalized functions? Can you add distributions?

Answer: yes, because you can add functionals. Thus

$$\text{"a"} = T_{\mathbf{a}}[\varphi] = (a,\varphi) = \int_{-\infty}^{\infty} dx\ A(x)\,\varphi(x)$$

$$\text{"b"} = T_{\mathbf{b}}[\varphi] = (b,\varphi) = \int_{-\infty}^{\infty} dx\ B(x)\,\varphi(x)$$

where here A and B are either real functions or generalized functions. Then

$$\text{"a+b"} = T_{\mathbf{a+b}}[\varphi] = (a+b,\varphi) = \int_{-\infty}^{\infty} dx\ (A+B)(x)\,\varphi(x) = \int_{-\infty}^{\infty} dx\ \{ A(x) + B(x) \}\,\varphi(x)$$

$$= \int_{-\infty}^{\infty} dx\ A(x)\,\varphi(x) + \int_{-\infty}^{\infty} dx\ B(x)\,\varphi(x) = \text{"a"} + \text{"b"}$$

where again we have used standard "integration machinery". We knew we could do this because we knew that we had a *linear* functional. Notice that the correct Generalized Function for "a+b" is the sum of the individual generalized functions A(x) and B(x).

Question: Can you multiply two generalized functions.

Answer: No. Suppose A(x) and B(x) above are both generalized functions. Then we know what $\int_{-\infty}^{\infty} dx\ A(x)\,\varphi(x)$ means because it is some (a,φ) that we presumably know something about. Same for B. But we have no "meaning" assigned to this integral: $\int_{-\infty}^{\infty} dx\ A(x)\,B(x)\varphi(x)$.

Similarly, consider the functionals. We could try to talk about $(T_{\mathbf{a}}T_{\mathbf{b}})[\varphi] = T_{\mathbf{a}}T_{\mathbf{b}}[\varphi]$ but this makes no sense (has no meaning) because $T_{\mathbf{b}}[\varphi]$ is a real number, and then Ta[real number] is not even the right operating class for our symbols. We can talk about $T_{\mathbf{a}}[\varphi]\ T_{\mathbf{b}}[\varphi]$ = product of two real numbers. This certainly is a mapping S→R so this product would be a functional we could call $T_{\mathbf{a*b}}[\varphi]$. But

$$T_{\mathbf{a*b}}[\varphi] = T_{\mathbf{a}}[\varphi]\ T_{\mathbf{b}}[\varphi] = \int_{-\infty}^{\infty} dx\ A(x)\,\varphi(x) * \int_{-\infty}^{\infty} dx\ B(x)\,\varphi(x)$$

$$\neq \int_{-\infty}^{\infty} dx\ A(x)\,B(x)\varphi(x)$$

so this thing is not our desired integral with the generalized functions multiplied together. This last thing is more a direct product idea , could maybe say $T_{\mathbf{a}\otimes\mathbf{b}} = (T_{\mathbf{a}} \otimes T_{\mathbf{b}})\,[\varphi]$ mapping S→ R.

A good example is $\delta(x)\delta(x) = \delta(0)\,\delta(x)$ so that

$$\int_{-\infty}^{\infty} dx\ \delta(x)\delta(x)\varphi(x) = \int_{-\infty}^{\infty} dx\ A(x)\,B(x)\varphi(x) = \delta(0)\int_{-\infty}^{\infty} dx\ \delta(x)\varphi(x) = \delta(0)\,\varphi(0) = \text{infinite}$$

Remember the whole point is that the integral of a generalized function is supposed to give a finite result, but here the integral of $\delta(x)\delta(x)$ gives an infinite result.

Question: Can you multiply a generalized function by a regular function?

In the above example, let A(x) be a GF and let B(x) = g(x) be a C^∞ regular function. Then we get

$$T_{ag}[\varphi] = (ag,\varphi) = \int_{-\infty}^{\infty} dx\ A(x)\, g(x)\varphi(x) = \int_{-\infty}^{\infty} dx\ A(x)\, \varphi_1(x) = (a,\varphi_1) = T_a[\varphi_1] = T_a[g\varphi]$$

So here the integral $\int_{-\infty}^{\infty} dx\ A(x)\, g(x)\varphi(x)$ *does* have a meaning. That meaning is $T_a[\, g\varphi\,]$. We could not do this trick with two generalized functions because then gφ would not be a test function and then we have no meaning.

Warning: I think some people incorrectly use the word "distribution" to describe a "generalized function". We have seen how "1/x" is a distribution, while pf(1/x) is the corresponding generalized function. In the case of the δ, we have "δ" as the distribution, and δ(x) as the generalized function, so the notation is pretty close here. Of course "δ"

More Notes on the meaning of (f,g) and application to distribution ODEs.

This is just my take on things. I think the meaning of the singular distributions and their associated "generalized functions" is given by sequence limits.

1. First, back on pages 35-36 we took note of three "operations with distributions". These properties were derived for distributions involving LI functions, for which the inner product exists. The properties then follow trivially from the mechanical properties of the integral form: translation, scaling, and multiplication by an LI function. Later a fourth property is mentioned: a differentiation rule. Here we shall write these four properties using our generic distribution notation.

translation:	$(f(x+a),\varphi) = (f,\varphi(x-a))$		redefine int variable
scaling:	$(f(\alpha x),\varphi) = \lvert\alpha\rvert^{-1}(f(x),\varphi(x/\alpha))$	$\alpha \neq 0$	redefine int variable
mult by LI g:	$(gf,\varphi) = (f,g\varphi)$		must move f from A to B
diff:	$(f\,',\varphi) = -(f,\varphi')$		integration by parts

In the last item, the compactness of the test function space makes the parts vanish. These four rules are clear for non-singular distributions because you just write out the inner product integral. But what can we say about the above rules of the distributions are singular so the integral form does not exist?

2. Starting on page 42, Stak talks about the notion of $t_n \rightarrow t$ where both symbols here mean distributions (I drop the double quotes for now) not functions. Let's assume that when $n < \infty$, we can associate with t_n a function $t_n(x)$ which is LI. Then we can write:

$$(t_n,\varphi) = \langle t_n,\varphi\rangle = \int_{-\infty}^{\infty} dx\ t_n(x)\, \varphi(x)$$

The above four properties apply here. Let's just write all four of them:

$(t_n (x+a),\varphi) = (t_n,\varphi(x-a))$
$(t_n (\alpha x),\varphi) = |\alpha|^{-1}(t_n (x),\varphi(x/\alpha))$
$(g\ t_n,\varphi) = (t_n,g\varphi)$
$(t_n ', \varphi) = - (t_n, \varphi')$

3. Now we consider the singular case. We take $t_n \rightarrow t$, but we find that $t_n(x) \rightarrow t(x)$ which is not a real function, only a generalized function. I think the big idea is that we can take the limit through the integration of the inner product "in the magical world of generalized functions". Then we can restate all the above rules replacing t_n with the limit t. Then we claim that all four of the above rules are still true for a singular distribution, with the understanding that t(x) is now a generalized function and not a regular function. So we then have these rules which then apply to both regular and singular distributions

$(t(x+a),\varphi) = (t,\varphi(x-a))$
$(t(\alpha x),\varphi) = |\alpha|^{-1}(t(x),\varphi(x/\alpha))$
$(gt,\varphi) = (t,g\varphi)$
$(t', \varphi) = - (t, \varphi')$

If we combine the last two rules together, and if we assume that the functions in operator L are C^∞, then we come up with a "fifth rule" which is this:

$(Lt,\varphi) = (t, L^*\varphi)$

where L* is the adjoint and is also exactly given by formula p 55 B or p 40 A. For example, consider

$(g\ D^m t, \varphi) = (D^m t, g\varphi)$ // by rule 3 above
$= (-1)^m (t, D^m(g\varphi))$ // by applying rule 4 m times

To bring things a little down to earth on this $t_n \rightarrow t$ idea, let's do a few of "our own" examples.

4. First example, the delta function. We start with

$$(t_n,\varphi) = \langle t_n,\varphi \rangle = \int_{-\infty}^{\infty} dx\ t_n(x)\ \varphi(x)$$

where t_n is any smooth unit-area sequence we like which approaches δ. We even know a sequence of test functions that does it if we want. We then take the limit on both sides

$\lim_{n\rightarrow\infty} (t_n,\varphi) \equiv (t,\varphi)$ where $\lim_{n\rightarrow\infty} t_n = t$

So far we are just defining t in our little parens notation. We could also say $\lim_{n\rightarrow\infty} T_{tn}[\varphi] = T_t[\varphi\ ]$.

How do we know that the limit exists? We don't in general, but here we just assume it exists and then if we can find what it is, we will then know that it exists.

So now we have

$$(t,\varphi) = \lim_{n\to\infty} \{ \int_{-\infty}^{\infty} dx\, t_n(x)\, \varphi(x) \}$$

We know that the integral converges for every finite n. The official "interchange is allowed" theorem requires that $| t_n(x)| <$ some f(x) for all n (see chap 2 notes "interchange"), but we don't have that property here for t_n going to a delta function. So instead, let's consider very large n so that $t_n(x)$ is very strongly peaked. We can then say

$$\int_{-\infty}^{\infty} dx\, t_n(x)\, \varphi(x) \approx \varphi(0) \int_{-\infty}^{\infty} dx\, t_n(x) = \varphi(0)$$

As n gets larger, this ≈ approaches = (we know that φ(x) is continuous at x = 0). The area of any $t_n(x)$ is arranged to be 1 of course, so we have the far right above. So there is not much objection I think to saying that

$$(t,\varphi) = \lim_{n\to\infty} \{ \int_{-\infty}^{\infty} dx\, t_n(x)\, \varphi(x) \} = \varphi(0)$$

Since we have found the limit, we now know that the limit t exists, which we only assumed earlier.

At this point, we "pretend" that we can do the order interchange and then try to make sense out of the result in terms of "generalized functions" . So we get

$$(t,\varphi) = \int_{-\infty}^{\infty} dx\, \{ \lim_{n\to\infty} t_n(x) \}\varphi(x) = \int_{-\infty}^{\infty} dx\, \delta(x)\, \varphi(x) \qquad // \equiv (\delta,\varphi) = \varphi(0)$$

where "the meaning" of $\int_{-\infty}^{\infty} dx\, \delta(x)\, \varphi(x)$ is simply φ(0). So we simply have a "bookkeeping" method by doing this "pretention".

5. Next example is δ' . Start as above with, where $t_n(x)$ is still our delta function approaching sequence,

$$(t_n,\varphi) = \int_{-\infty}^{\infty} dx\, t_n(x)\, \varphi(x)$$

Apply the D operator so we then have

$$(t_n',\varphi) = \int_{-\infty}^{\infty} dx\, t_n(x)'\, \varphi(x) = - \int_{-\infty}^{\infty} dx\, t_n(x)\, \varphi'(x) \qquad // \text{ by rule 4}$$

Now take the same limit as above, and assume that t' exists. We then have

$$\int_{-\infty}^{\infty} dx\, t_n(x)\, \varphi'(x) \approx \varphi'(0) \int_{-\infty}^{\infty} dx\, t_n(x) = \varphi'(0)$$

$$(t',\varphi) = \lim_{n\to\infty} \{- \int_{-\infty}^{\infty} dx\, t_n(x)\, \varphi'(x) \} \quad = \varphi'(0)$$

and now we have shown that t' exists. Now we go into the pretend world and write

$$(t',\varphi) = \int_{-\infty}^{\infty} dx \{ \lim_{n\to\infty} t_n'(x) \}\varphi(x) \quad = \quad \int_{-\infty}^{\infty} dx\, \delta'(x)\, \varphi(x)$$

where "the meaning" of $\int_{-\infty}^{\infty} dx\, \delta'(x)\, \varphi(x)$ is simply $\varphi'(0)$. So we simply have a "bookkeeping" method once again.

6. On p 37, doing our approach above, Stak shows that if t exists, then t' exists (ie, t' is a distribution). And so is the n^{th} derivative of t. He shows this in the world of LI functions, and then we extend the result to singular functions making use of the generalized functions as a bookkeeping device. In our work to come, we often start with some t that is regular, but in the process of doing n derivatives, at some point we hit a singular distribution. This is the "weak" situation. If all derivatives are regular distributions, we have the "classical" situation, and if we start off with t being singular, we are in the "distributional case".

7. Now let's jump to page 55 and ponder Stak's activities there. We are trying to solve the generalized function ODE which is $L\, t = \delta(x-\xi)$. This is just bookkeeping, and means $(Lt,\varphi) = \varphi_\xi(0)$ where both sides are finite and under good control. Stak comes up with a candidate solution p 55 1.40 where u(x) is any homo solution to $L\, u = 0$ (function ODE), and then $v_\xi(x)$ is the same as u(x) except there is a difference in the $n\text{-}1^{th}$ derivative at $x=\xi$ as shown in page 55 A.

He wants to prove that $(Lt,\varphi) = \varphi_\xi(0) = \varphi(\xi)$. If we can show this, then in bookkeeping language we have shown that t solves $L\, t = \delta(x-\xi)$.

We start off the proof with the first step which we discussed above in item 3:

$$(Lt,\varphi) = (t, L^*\varphi) = \langle t, L^*\varphi \rangle = \int_{-\infty}^{\infty} dx\, t_\xi(x)\, L^*\varphi(x)$$

which we know is true even for singular t as long as we are allowed to use the GF bookkeeping device. Due to the left/right nature of our candidate solution 1.40, we have to write the above integral as the sum of two integrals, one covering the region of x to the left of ξ and the other to the right of ξ. Technically each of these things is really a limit as $\xi-\varepsilon \to \xi$, but we don't want to bury our notation in detail. The difference in the two integrals is that the first has u(x) and the second has $v_\xi(x)$.

Now we know that the need to break the integral into two parts like this is only necessary for terms in L^* which are non-continuous at ξ. Let's look at the important term involved here

$$\langle t, L^*\varphi \rangle = \int_{-\infty}^{\xi-} dx\, u(x)\, L^*\varphi + \int_{\xi+}^{\infty} dx\, v_\xi(x)\, L^*\varphi$$

$$= \int_{-\infty}^{\xi-} dx\, u(x)\, (-1)^n D^n(a_0\varphi) + \int_{\xi+}^{\infty} dx\, v_\xi(x)\, (-1)^n D^n(a_0\varphi) \qquad \text{// important term}$$

As we swing one D over to u(x) or $v_\xi(x)$, we pick up "parts" at the ξ end (only) of each of these integrals. Here are those two parts contributions:

$$\{ u(x) (-1)^n D^{n-1}(a_0\varphi) \}|^{x=\xi-} - \{ v_\xi(x) (-1)^n D^{n-1}(a_0\varphi) \}|^{x=\xi+}$$

But this difference is 0 because the function inside {...} is the same on both sides, ie, it is continuous. We know that a_o and φ are both C^∞ so that part is continuous, and in 55 A we show $u(\xi) = v_\xi(\xi)$ so it too is continuous.

Now we want to carry out the process of swinging all the D's over to the LHS in each integral. As we do this, if we completely ignore any non-zero parts we might pick up, we know we will end up with the following integrals

$$<t, L^*\varphi> = \int_{-\infty}^{\xi-} dx\, Lu(x)\, \varphi + \int_{\xi+}^{\infty} dx\, Lv_\xi(x)\, \varphi$$

In each integral we are avoiding the point $x=\xi$ (the whole point of having two integrals), and so in each integrand we can set $Lu(x) = 0$ and $Lv_\xi(x) = 0$ since we know both functions are homo solutions. Then we get $<t, L^*\varphi> = 0 + 0 = 0$. So the true result will be

$(Lt,\varphi) = (t, L^*\varphi) = 0$ + any "parts" we might pick up in our process.

In the above, we examined the act of "swinging over the first D. After doing that we had

$$\int_{-\infty}^{\xi-} dx\, Du(x) (-1)^{n-1} D^{n-1}(a_0\varphi) + \int_{\xi+}^{\infty} dx\, Dv_\xi(x) (-1)^{n-1} D^{n-1}(a_0\varphi)$$

and we threw out the parts, assuming n is "large". Notice we have adjusted for the minus sign we pick up in doing the swing. Now let's do the next swing. The non-parts part will be this:

$$\int_{-\infty}^{\xi-} dx\, D^2u(x) (-1)^{n-2} D^{n-2}(a_0\varphi) + \int_{\xi+}^{\infty} dx\, D^2v_\xi(x) (-1)^{n-2} D^{n-2}(a_0\varphi)$$

while the parts part will be

$$\{ Du(x) (-1)^{n-1} D^{n-2}(a_0\varphi) \}|^{x=\xi-} - \{ Dv_\xi(x) (-1)^{n-1} D^{n-2}(a_0\varphi) \}|^{x=\xi+}$$

The rule for parts is that you knock out the "in transit D" and write down what is left. Let's now compare this parts contribution to the previous one:

$$\{ u(x) (-1)^n D^{n-1}(a_0\varphi) \}|^{x=\xi-} - \{ v_\xi(x) (-1)^n D^{n-1}(a_0\varphi) \}|^{x=\xi+}$$

$$\{ Du(x) (-1)^{n-1} D^{n-2}(a_0\varphi) \}|^{x=\xi-} - \{ Dv_\xi(x) (-1)^{n-1} D^{n-2}(a_0\varphi) \}|^{x=\xi+}$$

Here are some rules which will allow us to "write any parts":

1) The sum of the D powers is n-1
2) the (-1) exponent is one more than the right D power.

So eventually we are going to get this parts contribution

$$\{ D^{n-1}u(x)\,(-1)^1\,D^0(a_0\varphi)\,\}|^{x=\xi-} - \{ D^{n-1}v_\xi(x)\,(-1)^1\,D^0(a_0\varphi)\,\}|^{x=\xi+}$$

$$= [\ \{ D^{n-1}u(x)\}\ |^{x=\xi-} - \{ D^{n-1}v_\xi(x)\ \}|^{x=\xi+}\ ]\,(-1)^1(a_0\varphi)\ |^{x=\xi}$$

$$= [\ \{ D^{n-1}\,v_\xi(x)\}\ |^{x=\xi-} - \{ D^{n-1}\,u(x)\ \}|^{x=\xi+}\ ]\,(a_0\varphi)\ |^{x=\xi}$$

But this is the case where we have a "jump" so we write it as

$$= [\ \{ D^{n-1}\,v_\xi(x)\}\ |^{x=\xi-} - \{ D^{n-1}\,u(x)\ \}|^{x=\xi+}\ ] = 1/a_o(\xi)$$

Then our parts contribution is this:

$$\{ D^{n-1}u(x)\,(-1)^1\,D^0(a_0\varphi)\,\}|^{x=\xi-} - \{ D^{n-1}v_\xi(x)\,(-1)^1\,D^0(a_0\varphi)\,\}|^{x=\xi+}$$

$$= \ 1/a_o(\xi)\,(a_0\varphi)\ |^{x=\xi}\ = \varphi\ |^{x=\xi}\ = \varphi(\xi)$$

Since this is the only parts contribution, we have shown that

$$(Lt,\varphi) = \ (t,\ L^*\varphi)\ = \varphi(\xi)$$

$$\Rightarrow \qquad Lt = \delta_\xi \qquad \text{// no GF's involved yet}$$

$$\Rightarrow \qquad Lt(x) = \delta(x-\xi) \quad \text{// now we are talking GF's.}$$

So we have proven that our candidate Green's like solution 1.40 really is a solution.

The reason then that we write $v_\xi(x)$ with the label ξ is this. This function is equal to u(x) in all derivatives but the n-1st where we have a jump of $1/a_o(\xi)$, and thus, although u(x) does not depend on ξ, the function v(x) must depend on ξ.

The most general solution to our ODE $Lt = \delta(x-\xi)$ is our Green's Like solution + the n homo solutions each with an undetermined constant. Our ODE "system" is then completed by specifying n BC's which determine those n constants (assuming that the BC's allow you to find those n constants).

We arrive then at this important conclusion: for n reasonable BC's, there is exactly one solution to $Lt = \delta(x-\xi)$ and it can be written as 1.40 + n properly constanted homo solutions. But, there are n ways to write this unique solution because the function u(x) in 1.40 can be any one of the n homo solutions. It is possible that one of these n ways will result in all the n homo constants being 0.