Appendix A: Spherical Harmonics PhL 7.26.09

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This appendix is Stakgold's "take" on the spherical harmonics. He writes $\nabla^2 u = 0$ in spherical coordinates (making a typo shown p 393 in red), separates variables, but then instead of using operator L^2 , he uses operator $S = \sin\theta\, L^2$. So A.1 is our familiar result that $L^2 Y = \lambda Y$ where λ is the separation constant in the RY separation, eventually to become $\ell(\ell+1)$. Since $Y(\theta,\varphi)$ is defined on a sphere, we have conditions p 394 A first of all, which say value and gradient at the day boundary must be the same whichever way you measure them.

Then p 394 B says that Y is not singular at $z = \pm 1$. Why is this necessary? In 1D, if we were studying Legendre polynomials, we want to have "smooth" orthogonal polynomials so we can use them to expand smooth functions $f(\theta)$ which are not singular at these values of z. So I guess if we want to expand $f(\theta,\varphi)$ which are not singular at $z = \pm 1$ we should restrict to non-singular basis functions. More soon.

Now we separate Y into ΘΦ with separation constant $\mu = m^2$ and our rules A above lead at once to quantization of $m = 0, \pm 1, \pm 2$ .... There are no other solutions of Φ that work. This of course leaves us with a Θ equation which contains m, as in A.4.

Comments on the Θ equation wrt "singular points".

See "Frobenius Method Applied to the Associated Legendre Equation.doc" for Frobenius comments on the ODE for Θ. The indicial equation gives the Frobenius exponents as ±m/2, solutions to the indicial equation. For positive m, we can write the w_1 solution in this manner:

$$P_\nu^m(z) = (z^2 - 1)^{m/2}\, \partial_z^m P_\nu(z) = (\sin\theta)^m\, \partial_z^m P_\nu(z) \qquad \text{where } \nu(\nu+1) = \lambda$$

The associated Legendre ODE has regular singular points at $z = 1$ and $z = -1$.

Question: why do we require that ν = n ?

If we don't do this, then the functions $P_\nu(z)$ and $\partial_z^m P_\nu(z)$ are not polynomials. Also, they diverge at either $z = 1$ or $z = -1$.

I have not proven that, but think I could do so. We know for example that

$$P_\nu(z) = F(-\nu,\nu+1;\, 1;\, (1-z)/2); \qquad P_\nu(-1) = F(-\nu,\nu+1;\, 1;\, 1);$$

In general, $F(a,b,c,z)$ only converges for $|z| < 1$. On the circle $z = 1$, there are some rules quoted p 556 AS which say this:

(a) if Re(c-a-b) ≤ -1, you diverge. We have c-a-b = 1 +ν - ν - 1 = 0, so not this case.
(b) We do not have convergence either, since we don't have Re(c-a-b) > 0
(c) etc.

Basically the series diverges at z = -1 unless you can make it truncate, and that is why need ν = n.

So now back to question: what is wrong with having divergence at z = -1? No one gives a clean explanation of why this is bad. Yes, it would mean we don't have polynomial in sinθ and cosθ.

Here is my answer: We are trying to expand f(θ,φ) on our spherical harmonics which include $P_\nu^m(\theta)$. If we allow ν to have bad values, then we have an expansion like this:

$$f(\theta,\varphi) = \Sigma_{\text{all } \nu} \Sigma_m \; f_{n,\nu} \, e^{im\varphi} \, P_\nu^m(\theta)$$

If we don't restrict ν, then every single term in this sum diverges at z = -1, so it cannot be used to represent a smooth function f(θ,φ), and is therefore of no use to us.

I think one would make the same argument for expanding f(θ) on the $P_\nu(z)$.

Resume page 395

OK, after the above huge digression and a whole separate document to solve my temporary contradiction I was getting about the indicial equation, I accept that we must have ν = n and λ = n(n+1) in order to have a set of useful basis functions. I think the condition |m| ≤ n is also needed to avoid divergence at z = -1.

Normalization: Stakgold normalizes his Y as shown in p 395 C'. With this choice, we must throw in the ugly factor $N_{m,n}$ into the orthogonality condition which is E + F + A.5. Jackson and Messiah prefer to include this mess into the $Y_{\ell m}$ definitions. There is also the question of $(-1)^m$ factor in the definition of the P^m_ℓ which Jackson and Messiah do differently from each other. On page 395 Stakgold sidesteps this issue by not quoting any of the specific functions! The connection for us is this

$$Y_{\ell m}(\text{Jackson}) = Y_{\ell m}(\text{Stakgold})/\sqrt{N_{m\ell}} \qquad \text{// see Jackson p 65 3.53}$$

Page 396. So our conclusions for this subsection are statements of the spherical harmonic expansion transforms, with and without φ dependence, all fine.

Fancy Facts involving Spherical Harmonics (396)

Equation A.7 is just a statement of $\nabla^2 E = \delta$, in spherical coordinates, note the sinθ on the right.

Expand E (the 1/4πr Green's thing) on the spherical harmonics using A.6 as the projector. Here are the many steps for doing this: (result is shown in p 397 E)

Derive p 397 A: The radial term in A.7 has its "E" projected by the action he mentions on bottom page 396 hence the first term in 397 A. The SE term sits as you see it in 397 A where the sinθ is cancelled away. The right side is delta'd as shown, the sines cancel here too, but he has a typo here where he has omitted $\delta(r-r_0)$ but that delta does appear in B.

Derive p 397 C: The L^2 type operator S is self-adjoint so you can parts it over onto Y* in the second term of A with a + sign overall, and then S acting on Y* does as shown in B, so we pick up n(n+1) and the sinθ is back and that all acts on E, to give just E_{mn} as shown p 396. So this then gives us the LHS of B.

Derive p 397 D: Well C is an ODE in variable r of the form $Lu = k\delta(r-r_0)$. This is just a usual Green's function equation. You solve away from r_0 and get $u_n = A_n(r/r_0)^n + B_n(r/r_0)^{-n-1}$. Integrate both sides over tiny interval surrounding r_0 to get:

$-\partial_r(r^2\partial_r u)+ n(n+1)\, u = k\delta(r-r_0)$ // before integration

$-(r_0^2\partial_r u)|^+_- = k$

$(\partial_r u)|^+_- = -k/r_0^2$

which says the slope has the jump shown at $r = r_0$. So let's build up our function.

$u(r) = A_n(r/r_0)^n$ for $r < r_0$ $\partial_r u^- = nA_n r^{n-1} r_0^{-n}$
$u(r) = B_n(r/r_0)^{-n-1}$ for $r > r_0$ $\partial_r u^+ = -(n+1)B_n\, r^{-n-2} r_0^{n+1}$

Match condition at $r = r_0$ then says

$-(n+1)\, B_n\, r^{-n-2} r_0^{n+1} - nA_n r^{n-1} r_0^{-n} = -k/r_0^2$

$-(n+1)\, B_n\, r_0^{-n-2} r_0^{n+1} - nA_n r_0^{n-1} r_0^{-n} = -k/r_0^2$

$-(n+1)\, B_n\, r_0^{-1} - nA_n r_0^{-1} = -k/r_0^2$

$-(n+1)\, B_n - nA_n = -k/r_0$ $(n+1)\, B_n = k/r_0 - nA_n$

Now looking at result D, we would like to have

$u = k/(2n+1) * r^n\, r_0^{-n-1}$ for $r > R_0$

$= k/(2n+1) * (r/r_0)^n\, r_0^{-1}$

which tells us to set $A = k/[(2n+1)r_0]$. Then we compute

$B_n = \{k/r_0 - n\ k/[(2n+1)r_0]\}\ /(n+1)$

$= (k/r_0)\ \{\ 1 - n/(2n+1)\ \}/(n+1)$

$= (k/r_0)\ \{\ 2n+1 - n\ \}/[(n+1)(2n+1)]$

$= (k/r_0)\ \{\ n+1\ \}/[(n+1)(2n+1)]$

$= (k/r_0)\ /(2n+1)]$

$= A_n$

Therefore we get

$u_n = k/[(2n+1)r_0]\ (r/r_0)^n \qquad r < r_0$
$u_n = k/[(2n+1)r_0]\ (r/r_0)^{-n-1} \qquad r > r_0$

$u_n = k/[(2n+1)]\ r^n r_0^{-n-1} \qquad r < r_0$
$u_n = k/[(2n+1)]\ r^{-n-1}\ r_0^n \qquad r > r_0$

$u_n = k/[(2n+1)]\ r_>^{-n-1}\ r_<^n \qquad$ and r and r_0

and finally we arrive at result D. There of course must be a simpler way! I think operator L is symmetric, so you have to get $u_n = f(r, r_0) = f(r_0, r)$.

Another approach would have been to simply require that $A_n = B_n$ which would have told us that

$(n+1)\ A_n = k/r_0 - nA_n \quad => \quad (2n+1)A_n = k/r_0 \qquad k = Y/N$ by the way

Derive p 397 E and H: Use p 396 G for E, install p 397 D for E_{mn}, and you have it, QED. This is then the famous expansion I will now write out:

$$1/|\mathbf{r} - \mathbf{r}_0| = \Sigma_{n,m}\ r_<^n\ r_>^{-n-1}\ Y_{nm}(\Omega)\ Y_{nm}(\Omega_0)\ *\ (n+|m|)!\ /\ (n-|m|)!$$

See other doc about fact G.

Then if we pick $\theta_0 = 0$, the m stuff goes away and we get

$$1/|\mathbf{r} - \mathbf{r}_0| = \Sigma_n\ r_<^n\ r_>^{-n-1}\ P_n(z) \qquad H$$

Derive I. Notice that $z = \cos\theta$ and $\cos\theta = \hat{r}_0 \bullet \hat{r}$ so θ is the angle between the two vectors. If we take r_o off the z axis, result is still true, and we call this angle γ as shown in the usual way,

$\hat{\mathbf{r}} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$
$\hat{\mathbf{r}}' = (\sin\theta'\cos\varphi',\ \sin\theta'\sin\varphi',\cos\theta')$

$\hat{\mathbf{r}}' \bullet \hat{\mathbf{r}} = \sin\theta\cos\varphi\ \sin\theta'\cos\varphi' + \sin\theta\sin\varphi\ \sin\theta'\sin\varphi' + \cos\theta\ \cos\theta' = \cos\gamma$

$= \cos\theta\ \cos\theta' + \sin\theta\ \sin\theta'\ \{\ \cos\varphi\ \cos\varphi' + \sin\varphi\ \sin\varphi'\ \}$

$= \cos\theta\ \cos\theta' + \sin\theta\ \sin\theta'\ \cos(\varphi-\varphi')$ // as claimed in p 110 A

$= \cos(\gamma)$

Thus our general result J is verified from H and saying the right words as we did above.

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Derivation of A.14 on page 398.

This is a real bitcheroo. There surely is a simpler way, but here at least is A way:

The Taylor series expansion A.12 is familiar to me. Diff this wrt r and wrt z:

$-1/2\ [...]^{-3/2}\ (2r-2z) = \Sigma_{n=0}\ n\ r^{n-1}\ P_n(z) = \Sigma_{n=0}(n+1)\ r^n P_{n+1}(z)$ wrt r

$-1/2\ [...]^{-3/2}\ (-2r) = \Sigma_{n=0}\ r^n P_n'(z)$ wrt z

Restate:

$[...]^{-3/2}\ (z-r) = \Sigma_{n=0}\ n\ r^{n-1}\ P_n(z) = \Sigma_{n=0}(n+1)\ r^n P_{n+1}(z)$ (1)

$[...]^{-3/2}\ r = \Sigma_{n=0}\ r^n P_n'(z)$ (2)

Add to get

$[...]^{-3/2} z = \Sigma_{n=0}\ r^n\ [(n+1)\ P_{n+1}(z) + P_n'(z)\ ]$

Divide by z to get

$[...]^{-3/2}\ 1 = \Sigma_{n=0}\ r^n\ [(n+1)\ P_{n+1}(z) + P_n'(z)\ ]/z$ (3)

Multiply (1) by r and (2) equation by z

$[...]^{-3/2}\ (rz - r^2) = \Sigma_{n=0}\ n\ r^n\ P_n(z)$

$[...]^{-3/2}\ rz = \Sigma_{n=0}\ r^n\ z\ P_n'(z)$

Subtract to get

$[...]^{-3/2}\ (-r^2) = \Sigma_{n=0}\ r^n\ (\ n\ P_n(z) - z\ P_n'(z)\ )$ (4)

Now add (3) and (4) to get

$$[...]^{-3/2}\,(1-r^2) = \Sigma_{n=0}\, r^n \;\{\;\; [(n+1)\,P_{n+1}(z) \;+ P_n'(z)\,]/z \;+ n\,P_n(z) - z\,P_n'(z) \;\}$$

The LHS is the thing we are looking for. It must then be true that

$$[(n+1)\,P_{n+1}(z) \;+ P_n'(z)\,]/z \;+ n\,P_n(z) - z\,P_n'(z) \;=\; (2n+1)\,P_n(z)$$

which we rewrite as (mult thru by z)

$$(n+1)\,P_{n+1}(z) \;+ P_n'(z) \;+ nz\,P_n(z) - z^2\,P_n'(z) \;=\; (2n+1)\,zP_n(z)$$

and now combine like terms to get

$$(n+1)\,P_{n+1}(z) \;+ P_n'(z) \;- z^2\,P_n'(z) \;=\; (n+1)\,zP_n(z)$$

or

$$(n+1)\,P_{n+1}(z) \;- (z^2-1)\,P_n'(z) \;=\; (n+1)\,zP_n(z)$$

Now use last p 147 Schaum which says

$$(z^2-1)\,P_n'(z) = nzP_n(z) - nP_{n-1}(z)$$

We then have

$$(n+1)\,P_{n+1}(z) \;- nzP_n(z) + nP_{n-1}(z) \;=\; (n+1)\,zP_n(z)$$

Gather like terms

$$(n+1)\,P_{n+1}(z) \;+ nP_{n-1}(z) \;=\; (2n+1)\,zP_n(z)$$

and this agrees with the first identity on page Schaum 147, QED.

OK, now that we have the details, our final result is 6.26 which amazingly gives the potential everywhere inside our sphere as an integral against the surface imposed value $f(\theta,\varphi)$. If we set r = 0, we get

$$u(0,\theta,\varphi) = (1/4\pi) \int d\Omega'\, f(\theta',\varphi')$$

Conjecture about A.14'

I don't know what A.14 is for r > 1. I would start with A.13 and redo all the steps. Not obvious to me what the result would be.

But here is a conjecture. First , we can get A.13 from A.12 by making the change $r \rightarrow 1/r$:

$$\text{A.12} = \Sigma_{n=0} r^{-n} P_n(z) \;\; = 1/[1 + r^{-2} - 2r^{-1}\cos\theta]^{1/2} \;=\; r/[r^2 + 1 - 2\cos\theta]^{1/2}$$

$\Sigma_{n=0} r^{-n-1} P_n(z) = 1/[r^2 + 1 - 2\cos\theta]^{1/2}$ = A.13

So my conjecture would be that A.14' comes by this same change from A.14. Then

A.14 = $\Sigma_{n=0}(2n+1)\, r^{-n} P_n(z) = (1-r^{-2})/[1 + r^{-2} - 2r^{-1}\cos\theta]^{3/2}$ // mult r^3/r^3

$= r\,(r^2-1)/[r^2 + 1 - 2r\cos\theta]^{3/2}$

$\Sigma_{n=0}(2n+1)\, r^{-n-1} P_n(z) = -(1-r^2)/[r^2 + 1 - 2r\cos\theta]^{3/2}$

So the change to A.14 is change r^n to r^{-n-1} and add an overall minus sign.

Notes added. In later Exercise 6.21 I had to derive this sum rule:

$[...]^{-3/2}(rz - 1) = -\Sigma_{n=0}\, r^n (n+1) P_n(z)$

so we can add that to our batch.

$[...]^{-3/2}(rz - r^2) = \Sigma_{n=0}\, n\, r^n P_n(z)$

$[...]^{-3/2}\, rz = \Sigma_{n=0}\, r^n\, z\, P_n'(z)$

$[...]^{-3/2}(1-r^2) = \Sigma_{n=0}(2n+1)\, r^n P_n(z)$

$[...]^{-3/2}(rz - 1) = -\Sigma_{n=0}\, r^n (n+1) P_n(z)$

In retrospect, I could have gotten the last one by adding the first and third.