Stakgold Vol 2 Appendix B PhL 4.6.11

I don't know where else you would see a section like this. Stak does it right.

A. Getting the large t behavior of a Laplace-Transform-Like integral. 1
B. Special case of (B.1): Large s limit of the Laplace Transform 2
C. Large t limit of the Inverse Laplace Transform 3

A. Getting the large t behavior of a Laplace-Transform-Like integral.

B.1 shows our integral type of interest where we want behavior for large t.

$$I(t) = \int_a^b dx\, e^{-h(x)t} f(x)$$

As noted, this is mostly going to come from region of small h(x) within the integration range (a,b).

Suppose h(x) has a minimum in this middle of this range at some x = c. Then model as

$h(x) = h(c) + (1/2!)\, h''(c)\, (x-c)^2$ ` $h''(c) > 0$ cupping up we assume

So the idea here is that h(x) is positive and thus we care about its minimum. Then get

$$\int_{c-\delta}^{c+\delta} dx\, f(x) \exp[-t \{h(c) + (1/2!)\, h''(c)\, (x-c)^2\}]$$

$$= e^{-th(c)} \int_{c-\delta}^{c+\delta} dx\, f(x) \exp[-(t/2)\, h''(c)\, (x-c)^2\}]$$

$$\approx e^{-th(c)}\, f(c) \int_{c-\delta}^{c+\delta} dx \exp[-(t/2)\, h''(c)\, (x-c)^2\}]$$

Change variables to y = x-c so this becomes

$$\approx e^{-th(c)}\, f(c) \int_{-\delta}^{+\delta} dy \exp[-(t/2)\, h''(c)\, y^2\}]$$

$$= 2\, e^{-th(c)}\, f(c) \int_{0}^{+\delta} dy \exp[-(t/2)\, h''(c)\, y^2\}]$$

$$\approx 2\, e^{-th(c)}\, f(c) \int_{0}^{\infty} dy \exp[-(t/2)\, h''(c)\, y^2\}]$$ // since t large, makes little diff.

$= 2\, e^{-th(c)}\ f(c)\ (1/2)\sqrt{\pi}\ [(t/2)\, h''(c)]^{-1/2}$ // Maple

$= e^{-th(c)}\ f(c)\sqrt{\pi}\ [(t/2)\, h''(c)]^{-1/2} = e^{-th(c)}\ f(c)\sqrt{2\pi}\ 1/\sqrt{t\, h''(c)}$

$= e^{-th(c)}\ f(c)\sqrt{2\pi/t\, h''(c)}$ // agrees with page 399 A.

Then for this case in which h(x) is positive and has a single minimum at c inside interval (a,b) we get

$$\int_a^b dx\ f(x)\ e^{-th(x)} \approx e^{-th(c)}\ f(c)\sqrt{2\pi/t\, h''(c)}$$

Suppose we have a minimum instead at endpoint a and there is no interior minimum. This means that $h'(a) \geq 0$ since h(x) has to be rising at the left edge if it is the intervals minimum. Then smallest h(x) is at x = a where

$h(x) = h(a) + (x-a)\, h'(a)$

$I(t) \approx \int_a^b dx\ f(x) \exp(-t\,[h(a) + (x-a)\, h'(a)])$

$\approx \int_a^{a+\delta} dx\ f(x)\ e^{-th(a)}\ e^{-th'(a)(x-a)}$

$\approx e^{-th(a)}\ f(a) \int_a^{a+\delta} dx\ e^{-th'(a)\ (x-a)}$

Now let y = x-a to get

$= e^{-th(a)}\ f(a) \int_0^{\delta} dy\ e^{-th'(a)y}$

$\approx e^{-th(a)}\ f(a) \int_0^{\infty} dy\ e^{-th'(a)y}$

$= e^{-th(a)}\ f(a)\ [1/th'(a)\ ]$ // agrees with (B.3)

Stak says that if h'(a) = 0 at this endpoint, then pretend it is the first case and add 1/2, I agree.

B. Special case of (B.1): Large s limit of the Laplace Transform

[s = t here] Suppose h(x) = x and interval is (a,b) = (0,∞) as arises in Laplace transform. Then we have our second case and we get h'(x) = 1 so

$$\int_0^{\infty} dx\ f(x)\ e^{-tx} \approx f(0)\ /t$$ // agrees with p 400 A.

If f(x) is analytic about x = 0, expand in the usual Taylor series p 400B (when you expand around the point x = 0, it is called a McLaurin series for some reason). Then we go look up the Laplace of x^n and find it to be [Stak is using t in place of the usual Laplace variable s]

$x^n/n! \quad \rightarrow \quad 1/t^{n+1}$

Thus the corrected B.5 should read

$f(x) = \Sigma_{n=0}^{\infty} f^{(n)}(0)\, x^n/n! \quad \Rightarrow$

$L(t) = \Sigma_{n=0}^{\infty} f^{(n)}(0) / t^{n+1}$ // Stak has a typo in his B.5

I cannot find this on the web anywhere, no good search handle.

Suppose instead that f(x) has an expansion in non-integer powers as in p 400 C (perhaps those powers are eigenvalues in some problem). Then ET I tells us that

(1)	t^{ν}	$\mathrm{Re}\ \nu > -1$	$\Gamma(\nu+1)p^{-\nu-1}$	$\mathrm{Re}\ p > 0$

$f(x) = \Sigma_n\ a_n\, x^{\alpha n}$

$x^{\alpha n} \quad \rightarrow\ \Gamma(\alpha_n+1)\, t^{-\alpha n-1}$

and then

$F(t) = \Sigma_n\ a_n\, \Gamma(\alpha_n+1)\, t^{-\alpha n-1} \quad = \quad \Sigma_n\ a_n\, (\alpha_n)!\, t^{-\alpha n-1}$ // agrees with B.6

and I checked that if $\alpha_n = n$, this gives my version of B.5.

C. Large t limit of the Inverse Laplace Transform

Good grief. Usual Laplace uses t = time and s = Laplace. In the above, Stak uses x = time and t = Laplace. Now in page 401 a we are back to the usual t = time and s = Laplace. So p 401A is just the usual Laplace inversion formula. He shows a sample case where there are two poles and two cuts and drags the contour to the left and assumes the left side great half circle vanishes. This gives equation B which goes with the figure.

A pole of order m. [402] He actually intends s_2 to have a pole of order m, not just order 1. This means that the function $F(s)(s-s_1)^m$ is analytic in s, so F(s) could have for example a term $(s-s_1)^{-m+1}$. Thus we get the strange looking power series of p 402 A where the largest negative power is $(s-s_1)^{-m}$. P 402 B is correct, but I will do this my way instead, We have

$$f(t) = (1/2\pi i) \oint ds \ \Sigma_{k=-m}^{\infty} a_k (s-s_2)^k \ e^{ts}$$

$$= \Sigma_{k=-m}^{\infty} a_k \ (1/2\pi i) \oint ds \ (s-s_2)^k \ e^{ts}$$

But only the negative powers of k can make contributions, so

$$= \Sigma_{k=-m}^{-1} a_k \ (1/2\pi i) \oint ds \ (s-s_2)^k \ e^{ts}$$

$$= \Sigma_{k=1}^{m} a_{-k} (1/2\pi i) \oint ds \ (s-s_2)^{-k} \ e^{ts}$$

$$= \Sigma_{k=1}^{m} a_{-k} (1/k-1)!) \ \partial_t{}^{(k-1)} (e^{s2t})$$

$$= \Sigma_{k=1}^{m} a_{-k} (t^{k-1}/k-1)!) (e^{s2t})$$

$$= e^{s2t} \Sigma_{k=1}^{m} a_{-k} \ t^{k-1}/(k-1)! \qquad \text{// agrees with B.7}$$

So this would be a contribution from a "pole of order m" and of course you need all the a_{-k} coefficients in order to really know what you've got. He notes that we don't care about the s_4 pole since it is to the left and will give a contribution $\exp(s_4 t) \sim \exp(-|\mathrm{Re}(s_4)|\ t)$ and is down by large ratio from the s_2 term, it is down by an exponential $\exp(-t\Delta s)$ that is to say, so we ignore s_4.

A Cut. Now we have to face the cut and the one with branch point s_1 dominates. Now we will do the discontinuity integral. I agree with modeling F(s) as shown in B.8 since this is a Frobenius form for a solution of an ODE in s near a singular point, though I question the α range. We then have for example

$$\text{term} = -(1/2\pi i) \int_{C2} e^{st} F(s) \, ds$$

$$= -(1/2\pi i) \int_{-\infty}^{a} ds \ e^{st} \, \mathrm{disc}(F(s)) \qquad \text{// a is Re}(s_1)$$

Now $R = |s-s_1| = s_1 - s$ for s on the cut,

$$\mathrm{disc}(s-s_1)^{\alpha} = R^{\alpha} (e^{i\pi a} - e^{-i\pi\alpha}) = R^{\alpha} \, 2i\sin(\pi\alpha) \qquad R = s_1 - s > 0 \text{ on the integral}$$

$$\mathrm{disk}(F(s)) = (s_1-s)^{\alpha} 2i\sin(\pi\alpha) \ \Sigma_{n=0}^{\infty} b_n (s-s_1)^n$$

so get (assume that $\mathrm{Im}(s_1) = r$ and $\mathrm{Re}(s_1) = a$, this latter used in the page 401 picture)

$$\text{term} = -(1/2\pi i) \int_{-\infty+ir}^{a+ir} ds \ e^{st} \ (s_1-s)^{\alpha} 2i\sin(\pi\alpha) \ \Sigma_{n=0}^{\infty} b_n (s-s_1)^n$$

$$= -(1/\pi)\sin(\pi\alpha)\,\Sigma_{n=0}^{\infty} b_n (-1)^n \int_{-\infty+ir}^{a+ir} ds\ e^{st}(s_1-s)^{\alpha+n}$$

Now, let s' ≡ s-ir be a complex variable which runs along the cut with Im(s') = 0. Then ds = ds' and $e^{st} = e^{t(s'+ir)}$ and $(s_1-s) = \mathrm{Re}(s_1-s) = a - s'$ and endpoints are then -∞ and a, so we have

$$= -(1/\pi)\sin(\pi\alpha)e^{irt}\,\Sigma_{n=0}^{\infty} b_n (-1)^n \int_{-\infty}^{a} ds'\ e^{s't}(a - s')^{\alpha+n}$$

But of course s' is now just a real variable, so let's first replace it with -y to get

$$= -(1/\pi)\sin(\pi\alpha)e^{irt}\,\Sigma_{n=0}^{\infty} b_n (-1)^n \int_{-a}^{\infty} dy\ e^{-yt}(a+y)^{\alpha+n}$$

Now let x = y+a and we then get

$$= -(1/\pi)\sin(\pi\alpha)e^{irt}\,\Sigma_{n=0}^{\infty} b_n (-1)^n \int_{0}^{\infty} dx\ e^{-(x-a)t}\,x^{\alpha+n}$$

$$= -(1/\pi)\sin(\pi\alpha)e^{irt}\,e^{at}\,\Sigma_{n=0}^{\infty} b_n (-1)^n \int_{0}^{\infty} dx\ e^{-xt}\,x^{\alpha+n}$$

Now notice that $s_1 = a + ir$ so we really have

$$= -(1/\pi)\sin(\pi\alpha)e^{ts1}\,\Sigma_{n=0}^{\infty} b_n (-1)^n \int_{0}^{\infty} dy\ e^{-xt}\,x^{\alpha+n}$$

Now consider this integral

$$L(t) = \int_{0}^{\infty} dy\ e^{-xt}\,x^{\alpha+n}$$

Think of this as one term in the series p 400C where $\alpha_n = \alpha+n$ and $a_n = 1$. We can then claim

$$L(t) \approx (\alpha+n)!\ t^{-\alpha-n-1}$$

so our result is now

$$\text{term} = -(1/\pi)\sin(\pi\alpha)e^{ts1}\,\Sigma_{n=0}^{\infty} b_n (-1)^n (\alpha+n)!\ t^{-\alpha-n-1} \qquad \text{// agrees with (B.9)'}$$

and that certainly took a long time! Once again, this "term" is what you get in the inverse Laplace transform if you have a cut of this Frobenius cut form:

$$F(s) = (s-s_1)^{\alpha}\,\Sigma_{n=0}^{\infty} b_n (s-s_1)^n$$

Now what about the little half turn around the branch point?

$$\text{term}' = -(1/2\pi i) \int_{\text{halfturnccw}} e^{st} F(s)\, ds$$

The usual thing is to set $s-s_1 = Re^{i\theta}$ and then we have $ds = iRe^{i\theta}d\theta$ so

$$\text{term}' = -(1/2\pi) \int_{-\pi}^{\pi} d\theta\, Re^{i\theta} F(s_1+Re^{i\theta})$$

But

$$F(_1+Re^{i\theta}) = (Re^{i\theta})^{\alpha} \Sigma_{n=0}^{\infty} b_n (Re^{i\theta})^n$$

and the lowest power of R is with n = 0 and as $R\to 0$ and this power is R^{α}. So for small R we have

$$\text{term}' \approx -(1/2\pi) \int_{-\pi}^{\pi} d\theta\, Re^{i\theta} R^{\alpha} e^{i\alpha\theta} b_0$$

and this vanishes as long as $R^{1+\alpha} \to 0$ which means $1+\alpha > 0$ which means $\alpha > -1$, that is, $\text{Re}(\alpha) > -1$. This is how I would state things. He puts an upper limit on $\text{Re}(\alpha)$ of 0 which seems unnecessarily restrictive on this result.

Summary of the cut result:

Assume there is a branch point at s_1 and that in a disk centered at this branch point we have

$$F(s) = (s-s_1)^{\alpha} \Sigma_{n=0}^{\infty} b_n (s-s_1)^n \qquad \text{where } \text{Re}(\alpha) > -1$$

True, if α were 1 + 1/3, say, we could absorb the integer part into the series and say $b_0 = 0$. Now given this branch point form for F(s), its contribution to the inverse Laplace transform is this:

$$f(t) = -(1/\pi) \sin(\pi\alpha) e^{ts1} \Sigma_{n=0}^{\infty} b_n (-1)^n (\alpha+n)!\, t^{-\alpha-n-1}$$

Application: Suppose our F(s) has a branch point at $s = s_1 = 0$ with $\alpha = -1/2$ and at the same time suppose it has a pole of order 1 also at s = 0, as shown in page 402 D. Then the pole gives a_{-1} and the cut gives

$$f_{cut}(t) = +(1/\pi) \Sigma_{n=0}^{\infty} b_n (-1)^n (n-1/2)!\, t^{-n-1/2} \qquad \sin(\pi\alpha) = -1$$

I guess this example is going to arise somewhere.

Comment: Notice that we assume the form shown F(s) *just near the branch* point. This is not going to be the form for F(s) all along the infinite cut. But just as in other cases, we know that most of the

contribution is going to come from the integration region "just to the left of the branch point". We are then allowed to extend this region to -∞ without making too much difference.

A. Redo Part A above in the case that h(x) is complex.

B.1 shows our integral type of interest where we want behavior for large t.

$$I(t) = \int_a^b dx\, e^{-h(x)\,t}\, f(x)$$

As noted, this is mostly going to come from region of small Re(h(x)) within the integration range (a,b).

Suppose Re(h(x)) has a minimum in this middle of this range at some x = c. Then model as

$h(x) = h(c) + (1/2!)\, h''(c)\, (x-c)^2$ $\quad$ $\mathrm{Re}\{h(c)\}'' > 0$ cupping up we assume

Note in passing that $h(x) = u(x) + iv(x)$ so $h''(x) = u''(x) + iv''(x)$ etc.

So the idea here is that Reh(x) is positive and thus we care about its minimum. Then get

$$\int_{c-\delta}^{c+\delta} dx\, f(x) \exp[\, -t\, \{h(c) + (1/2!)\, h''(c)\, (x-c)^2\}\,]$$

$$= e^{-th(c)} \int_{c-\delta}^{c+\delta} dx\, f(x) \exp[\, -(t/2)\, h''(c)\, (x-c)^2\}\,]$$

$$\approx e^{-th(c)}\, f(c) \int_{c-\delta}^{c+\delta} dx \exp[\, -(t/2)\, h''(c)\, (x-c)^2\}\,]$$

Change variables to y = x-c so this becomes

$$\approx e^{-th(c)}\, f(c) \int_{-\delta}^{+\delta} dy \exp[\, -(t/2)\, h''(c)\, y^2\}\,]$$

$$= 2\, e^{-th(c)}\, f(c) \int_{0}^{+\delta} dy \exp[\, -(t/2)\, h''(c)\, y^2\}\,]$$

$$\approx 2\, e^{-th(c)}\, f(c) \int_{0}^{\infty} dy \exp[\, -(t/2)\, h''(c)\, y^2\}\,] \qquad \text{// since t large, makes little diff.}$$

$$= 2\, e^{-th(c)}\, f(c)\, (1/2) \sqrt{\pi}\, [(t/2)\, h''(c)]^{-1/2} \qquad \text{// Maple}$$

$$= e^{-th(c)}\, f(c) \sqrt{\pi}\, [(t/2)\, h''(c)]^{-1/2} = e^{-th(c)}\, f(c) \sqrt{2\pi}\, 1/\sqrt{t\, h''(c)}$$

$$= e^{-th(c)}\, f(c) \sqrt{2\pi/t\, h''(c)} \qquad \text{// agrees with page 399 A.}$$

Then for this case in which h(x) is positive and has a single minimum at c inside interval (a,b) we get

$$\int_a^b dx\, f(x)\, e^{-t\,h(x)} \approx e^{-t\,h(c)}\, f(c) \sqrt{2\pi/t\, h''(c)}$$

Suppose we have a minimum instead at endpoint a and there is no interior Re(h) minimum. This means that Reh(a)' ≥ 0 since Reh(x) has to be rising at the left edge if it is the intervals minimum. Then smallest h(x) is at x = a where

$$h(x) = h(a) + (x-a)\, h'(a) \qquad \Rightarrow \qquad Reh(x) = Reh(a) + (x-a)\, Reh'(a)$$

$$I(t) \approx \int_a^b dx\, f(x) \exp(-t\,[h(a) + (x-a)\, h'(a)])$$

$$\approx \int_a^{a+\delta} dx\, f(x)\, e^{-t\,h(a)}\, e^{-t\,h'(a)\,(x-a)}$$

$$\approx e^{-t\,h(a)}\, f(a) \int_a^{a+\delta} dx\, e^{-t\,h'(a)\,(x-a)}$$

Now let y = x-a to get

$$= e^{-t\,h(a)}\, f(a) \int_0^{\delta} dy\, e^{-t\,h'(a)\,y}$$

$$\approx e^{-t\,h(a)}\, f(a) \int_0^{\infty} dy\, e^{-t\,h'(a)\,y}$$

$$= e^{-t\,h(a)}\, f(a)\, [1/t\,h'(a)\,] \qquad \text{// agrees with (B.3)}$$

Stak says that if h'(a) = 0 at this endpoint, then pretend it is the first case and add 1/2, I agree.

B. Redo the single endpoint case with a full power series and see what happens

We start with

$$I(t) = \int_a^b dx\, e^{-h(x)\,t}\, f(x)$$

Assume the min of h(x) occurs at endpoint a, and that one of two situations arises, both of which are legal for a minimum at this left endpoint

h'(a) > 0
h'(a) = 0 and h''(a) > 0
etc.

The claim is that for very large t, only the region very close to the endpoint contributes. We then integrate over a to a+δ, and then we replace δ by ∞ saying that by adding a part far from the endpoint, we don't make any difference.

So I would argue that we want to model h(x) *as best we can* in the region of the endpoint. So let's put in a full series like this

$$h(x) = h(a) + \Sigma_{n=1}^{\infty} h^{(n)}(a)/n! * (x-a)^n$$

Then we have

$$I(t) = \int_a^b dx\, e^{-h(x)t} f(x)$$

$$\approx f(a) \int_a^{a+\delta} dx \exp(-t [h(a) + \Sigma_{n=1}^{\infty} h^{(n)}(a)/n! * (x-a)^n])$$

$$\approx \exp(-t h(a)) f(a) \int_a^{a+\infty} dx \exp(-t \Sigma_{n=1}^{\infty} h^{(n)}(a)/n! * (x-a)^n)$$

Now take z = x-a to get

$$= \exp(-t h(a)) f(a) \int_0^{\infty} dz \exp(-t \Sigma_{n=1}^{\infty} h^{(n)}(a)/n! * z^n)$$

The problem with keeping *multiple* terms is that this integral is a bit ugly. If we try to keep just two terms, for example, we get this (see Maple below)

$$\int_0^{\infty} dz \exp(-t [Az+Bz^2) = (1/2) \sqrt{\pi/tB}\ \exp(tA^2/[4B]) \operatorname{erfc}(t^{1/2}A/[2\sqrt{B}\ ])$$

where we now define α so that

$$\alpha = A/[2\sqrt{B}\ ] \qquad A = h'(a) \qquad B = h''(a)/2 \qquad \alpha^2 = A^2/[4B]$$

$$\alpha = h'(a)/ [\ 2 \sqrt{h''(a)/2}\ ] = h'(a) / \sqrt{2h''(a)} \qquad \alpha^2 = h'(a)^2/ [2\ h''(a)]$$

and then

$$\int_0^{\infty} dz \exp(-t [Az+Bz^2) = (1/2) \sqrt{\pi/tB}\ \exp(t\alpha^2) \operatorname{erfc}(\alpha\, t^{1/2})$$

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assume(A>0,t>0,B>0);
Int(exp(-t*(A*z+B*z^2)), z=0..infinity);
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$$\int_0^{\infty} e^{(-t\,(A\,z+B\,z^2))}\,dz$$

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int(exp(-t*(A*z+B*z^2)), z=0..infinity);
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$$-\frac{1}{2}\,\frac{\sqrt{\pi}\;e^{\left(\frac{1}{4}\,\frac{t\,A^2}{B}\right)}\left(-1+\operatorname{erf}\left(\frac{1}{2}\,\frac{t\,A}{\sqrt{t\,B}}\right)\right)}{\sqrt{t\,B}}$$

In this case, to find the large t limit, we are forced to study the error function. So we need to know about the erf for large argument, and we call upon new AS 2010 to get

7.12(i) Complementary Error Function

As $z \to \infty$

7.12.1

$$\operatorname{erfc} z \sim \frac{e^{-z^2}}{\sqrt{\pi}z}\sum_{m=0}^{\infty}(-1)^m\frac{1\cdot 3\cdot 5\cdots(2m-1)}{(2z^2)^m},$$

so that

$$\operatorname{erfc}(\alpha\, t^{1/2}) \sim [\exp(-\alpha^2 t)/(\sqrt{\pi}\,\alpha t^{1/2})]\;\; \Sigma_{m=0}^{\infty}(-1)^m\,(2m-1)!!\,/\,(2\alpha^2 t)^m$$

Now above we can write

$$\int_0^{\infty} dz\,\exp(-t\,[Az+Bz^2]) = (1/2)\sqrt{\pi/tB}\;\; \exp(t\alpha^2)\,[\exp(-\alpha^2 t)/(\sqrt{\pi}\,\alpha t^{1/2})]\,\Sigma_{m=0}^{\infty}(-1)^m\,(2m-1)!!\,/\,(2\alpha^2 t)^m$$

$$= (1/2)\;\; [\sqrt{\pi/tB}\,/(\sqrt{\pi}\,\alpha t^{1/2})]\;\; \Sigma_{m=0}^{\infty}(-1)^m\,(2m-1)!!\,/\,(2\alpha^2 t)^m$$

But

$$[\sqrt{\pi/tB}\,/(\sqrt{\pi}\,\alpha t^{1/2})] = [\sqrt{1/tB}\,/(\alpha t^{1/2})] = t^{-1/2}B^{-1/2}\,\alpha^{-1}\,t^{-1/2}$$

But

$$\alpha = A/[2\sqrt{B}] = A\,2^{-1}\,B^{-1/2} \qquad \Rightarrow \qquad \alpha^{-1} = A^{-1}\,2^{1}\,B^{1/2}$$

so that

$$[\sqrt{\pi/tB}\,/(\sqrt{\pi}\,\alpha t^{1/2})] = t^{-1/2}B^{-1/2}\,[A^{-1}\,2^{1}\,B^{1/2}]\,t^{-1/2} = t^{-1}\,A^{-1}2^{1} = 2/(At)$$

so we then have

$$\int_0^\infty dz\, \exp(-t\,[Az+Bz^2]) = (1/2)\ [2/(At)]\ \Sigma_{m=0}^{\infty} (-1)^m (2m-1)!!\ /\ (2\alpha^2 t)^m$$

$$= [1/(At)]\ \Sigma_{m=0}^{\infty} (-1)^m (2m-1)!!\ /\ (2\alpha^2 t)^m$$

Then our single endpoint result keeping just these two terms is

$$I(t) = \exp(-t\,h(a))f(a)\ [1/(At)]\ \Sigma_{m=0}^{\infty} (-1)^m (2m-1)!!\ /\ (2\alpha^2 t)^m$$

$$\approx \exp(-t\,h(a))f(a)\ [1/(At)]\ [\,1\ - 3/(2\alpha^2)\, t^{-1} + ...]$$

$$\approx \exp(-t\,h(a))f(a)\ [1/(t\,h'(a))]\ [\,1\ - 3/(2\alpha^2)\, t^{-1} + ...]$$

Then

$$3/(2\alpha^2) = 3\ 2^{-1}\ \alpha^{-2} = 3\ 2^{-1}\ 2\ h''(a)/\ h'(a)^2 = (3\ h''(a)/\ h'(a)^2)$$

so our result is then

$$I(t) = \exp(-t\,h(a))f(a)\ [1/(t\,h'(a))]\ [\,1\ - (3\ h''(a)/\ h'(a)^2)\, t^{-1} + ...]$$

Now is this consistent with Stak's B.3 for example? Yes it is! (finally). But now I have a second term as well.

Conclusion: if we keep just the linear and quadratic terms in our model for h(x) near its minimum, we get the last result above as our approximation for the integral I(t) for large t. If we try to keep 3 terms, the dz integral becomes difficult. For example, Maple cannot do it, nor can Wolfram. In any event, our first term agrees exactly with B.3.

What if at the other endpoint? We just copy and edit from above, and I show changes in red except as all a→b, I won't make all the b's red.

We start with

$$I(t) = \int_a^b dx\, e^{-h(x)\,t} f(x)$$

Assume the min of h(x) occurs at endpoint b, and that one of two situations arises, both of which are legal for a minimum at this left endpoint

$h'(b) < 0$
$h'(b) = 0$ and $h''(b) > 0$
etc.

The claim is that for very large t, only the region very close to the endpoint contributes. We then integrate over b-δ to b, and then we replace δ by ∞ saying that by adding a part far from the endpoint, we don't make any difference.

So I would argue that we want to model h(x) *as best we can* in the region of the endpoint. So let's put in a full series like this

$$h(x) = h(b) + \Sigma_{n=1}^{\infty} h^{(n)}(b)/n! * (x-b)^n$$

Then we have

$$I(t) = \int_a^b dx\, e^{-h(x)t} f(x)$$

$$\approx f(b) \int_{b-\delta}^{b} dx \exp(-t [h(b) + \Sigma_{n=1}^{\infty} h^{(n)}(b)/n! * (x-b)^n])$$

$$\approx \exp(-t h(b)) f(b) \int_{b-\infty}^{b} dx \exp(-t \Sigma_{n=1}^{\infty} h^{(n)}(b)/n! * (x-b)^n)$$

Now take z = x-b to get

$$= \exp(-t h(b)) f(b) \int_{-\infty}^{0} dz \exp(-t \Sigma_{n=1}^{\infty} h^{(n)}(b)/n! * z^n)$$

and then take z→-z to get

$$= \exp(-t h(b)) f(b) \int_{0}^{\infty} dz \exp(-t \Sigma_{n=1}^{\infty} h^{(n)}(b)/n! * (-z)^n)$$

So the changes are a→b and z→-z. In our 2-term calculation above, we accomplish this by A→ -A (but $\alpha^2{}^2 = A^2/[4B]$ stays the same). We then get this intermediate result

$$\int_0^{\infty} dz \exp(-t [-Az+Bz^2) = - [1/(At)] \Sigma_{m=0}^{\infty} (-1)^m (2m-1)!! / (2\alpha^2 t)^m$$

which is just an overall sign change. Then we get

$$I(t) = - \exp(-t h(b)) f(b) [1/(t h'(b))] [1 - (3 h''(b)/ h'(b)^2) t^{-1} + ...]$$

and we can in effect say that we take our "a" result and take a→b and h'(a)→ - h'(b) to get the "b" result.

C. Redo Part A above in the case that h(x) leading endpoint terms cancel

Assume that the minimum of h(x) occurs at the endpoints and is the same at both endpoints. So there is a max of h(x) out in the interval somewhere. In this case, we want to add the two minima contributions. I will assume for the moment that the endpoint minima are not quite equal.

$$I(t) = \int_a^b dx\, e^{-h(x)\,t} f(x)$$

Expand the exponent

$h(x) = h(a) + h'(a)(x-a) + (1/2)h''(a)(x-a)^2$ at endpoint a
$h(x) = h(b) + h'(a)(x-b) + (1/2)h''(b)(x-b)^2$ at endpoint b

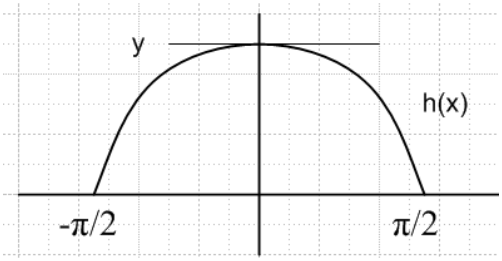
We have solved this problem above and we know the contribution from each endpoint:

$I(t) = \exp(-t\,h(a))f(a)\ [1/(t\,h'(a))]\ [\,1 - (3\,h''(a)/\,h'(a)^2)\,t^{-1} + ...]$ // from endpoint a
$I(t) = -\exp(-t\,h(b))f(b)\ [1/(t\,h'(b))]\ [\,1 - (3\,h''(b)/\,h'(b)^2)\,t^{-1} + ...]$ // from endpoint b

If $h(a) \neq h(b)$, one of these two terms dominates in the expo factor, so we forget the other term. But suppose $h(a) = h(b)$. Then we have to add the two terms.

Example: Consider this integral, where $y > 0$

$$I(t) = \int_{-\pi/2}^{\pi/2} dx\, e^{-ytcosx} \qquad h(x) = y\cos(x)$$



Here we have an example of two equal minimums, but in this case the integrand is even in x so we can write

$$I(t) = 2\int_0^{\pi/2} dx\, e^{-ytcosx}$$

and now we have only one minimum at endpoint $b = \pi/2$. Our large t approximation is then

$$I(t) = -\exp(-t\,h(b))f(b)\ [1/(t\,h'(b))]\ [\,1 - (3\,h''(b)/\,h'(b)^2)\,t^{-1} + ...]$$

We compute (note that $f(x) = 1$)

$h(x) = y\cos(x)$ $h(b) = h(\pi/2) = 0$

$h'(x) = -y\sin(x)$ $\qquad$ $h'(b) = -y$

$h''(x) = -y\cos(x)$ $\qquad$ $h''(b) = 0$

so that

$$I(t) = -\exp(-t\,h(b))f(b)\,[1/(t\,h'(b))]\,[\,1 - (3\,h''(b)/\,h'(b)^2)\,t^{-1} + \ldots]$$

$$= +\,f1\,[1/(t\,y]\,[\,1 - 0\,t^{-1} + O(t^{-2})\,]$$

So we conclude that

$$\int_{-\pi/2}^{\pi/2} dx\, e^{-ytcosx} \approx (2/y)t^{-1} \qquad \text{as } t \to \infty$$