Stakgold Chapter 1 meta meta summary PhL 4.19.09

This is needed because the meta notes themselves are so heavily detail-laden that you cannot "get your arms around" the larger picture. So here is an attempt to give a high level review. The text is 90 pages, my raw notes are 66 pages, the meta notes are 29 pages, and these meta-meta notes fill 9 pages.

Overview. The main subject of this chapter is dealing with the Lu=0 homo and Lu=f inhomo ODE problems, first without and later with BC's. The accessory problem Lg=δ is very useful and, when BC's are included, g is called "the Green's Function" and it is symmetric when L is self-adjoint. If g is found, then the solution to Lu=f with the same BC's is given by $u(x) = \int d\xi\ g(x|\xi)\ f(\xi)$ or u = Gf. As with any linear operator, associated with operator L is an Alternative Theorem which relates non-trivial solutions of Lu=0 with consistency conditions on f in the problem Lu=f. When Lu=0 has non-trivial solutions, the normal Green's Function does not exist, so you use a certain Modified Green's Function. In the middle of this 90 page chapter is a 30 page digression on "the theory of distributions" which is relevant since the Green's Function involves a delta function which is the generalized function of the "δ" distribution. The prototype example is a "string under transverse pressure", though heat conduction and beam deflection problems are also used as examples. Some of the chapter deals with n^{th} order ODE's, while the rest assumes n = 2. Reference is made to Appendices 1 and 2 for physics problems like heat and beams, and for Bessel functions. The Wronskian plays a role as a tool in proving various theorems and solving problems.

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Chapter 1. "The Green's Function"

1.1 The string under transverse pressure f(x) (1)

Appendix 1 develops the ODE for the string, adding some bells and whistles. A square-shaped force is applied to the string and the problem solved, then a delta function limit is taken. The solution under a delta transverse pressure is the Green's Function g in Lg = δ. Solution of Lu = f is then u = Gf = $\int$ dy g(x|y)f(y) which is verified by "careful differentiation" but is also intuitive by "superposition. In a sense, $G = L^{-1}$. The Green's method "builds in" the BC's.

1.2 The Dirac Delta Function (18)

Kicks the myth that δ(0) = +∞. Sequences called $s_k(x)$. The sifting property. At this point, the δ is just the limit of any sequence with unit area that does sifting in the limit.

1.3 The Theory of Distributions (28)

Dirac invented and used the delta function in his famous 1930 book Principles of QM. Later in 1935-1950 workers **Sobolev** and **Schwartz** came up with "the theory of distributions" which provided a strict mathematical basis for "generalized functions" like the delta function. This long chapter section presents this theory.

Definition of a distribution: A "distribution f" is any linear functional " f " = $T_f[\varphi] \equiv (f,\varphi)$ which is defined on a certain subspace S of the Hilbert Space "L^2 on some interval (a,b)". That certain subspace S is the space of all "test functions" -- the space of all C^∞ functions having compact support, which means zero outside some finite range. This space might be written $S = C^\infty_c$. Notice there is no inner product used in this definition of a distribution, although there does of course exist an inner product for L^2 which we could write as <a,b> if a and b are in L^2. We know that the test functions φ are all in L^2, but the "distribution f" is not necessarily associated with a function f in L^2 or in fact with a function of any kind. Our object " f " is a functional, not a function. Notice that the *value* of any "distribution f" when acting on a test function is $T_f[\varphi] = (f,\varphi)$ = a real number. NB: our notation (f,φ) is merely another way to write $T_f[\varphi]$ and is not the same as the inner product $\langle f,\varphi\rangle$. As noted, there may exist no function f.

Question: "Are normal functions also distributions? " The answer is semantically no. However, with every normal function f(x) we may *associate* a distribution T_f as follows:

$$\text{" f "} = T_f[\varphi] = (f,\varphi) \equiv \langle f,\varphi\rangle = \int_a^b dx\, f(x)\varphi(x)$$

" distribution f " ⟂ "normal function f "

Recall that the Riesz Representation Theorem says that every bounded (=continuous) linear functional like "f" above can be written as the inner product $\langle f,\varphi\rangle$ where f is some function in the HS. The theory of distributions extends this idea to unbounded functionals like δ for which the "function f" is a GF.

Question: " What is a normal function? " It turns out that the notion of a "normal function" means a "locally integrable" (LI) function, a pretty clearly defined animal.

Definition of a generalized function. We know that for LI functions, we can say

$$\text{" f "} = T_f[\varphi] = (f,\varphi) = \langle f,\varphi\rangle = \int_a^b dx\, f(x)\varphi(x)$$

" distribution f " |.. "a LI f "

and we know that for functions that are *not* LI, we can still have a distribution

$$\text{" f "} = T_f[\varphi] = (f,\varphi) \qquad \text{Example: "}\delta\text{"} = T_\delta[\varphi] = (\delta,\varphi) = \varphi(0)$$

" distribution f "

We may be able to find an "expression" for the "distribution f " when f is not an LI function, such as

$$\text{" f "} = T_f[\varphi] = (f,\varphi) = \text{"some expression involving f and } \varphi\text{"}$$

" distribution f "

Example 1: We consider the function 1/x which is not LI. It is shown in Stak on page 49 that there is in fact an expression you can write for the "distribution 1/x " :

$$\text{" 1/x "} = T_{1/x}[\varphi] = (1/x,\varphi) = \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) dx\,(1/x)\,\varphi(x) = \fint dx\,(1/x)\,\varphi(x)$$

$$= \int dx\ \mathrm{pf}(1/x)\,\varphi(x) \quad \text{// where pf(1/x) is here the "generalized function" or "symbolic function"}$$

The delta function fits into this picture as follows:

$$\text{" }\delta\text{ "} = T_\delta[\varphi] = (\delta,\varphi) = \varphi(0) = \int dx\,\delta(x)\,\varphi(x) \quad \text{where } \delta(x) = \text{the GF}$$

where "the meaning" of $\int_{-\infty}^{\infty} dx\,\delta(x)\varphi(x)$ is simply $\varphi(0)$. So we simply have a "bookkeeping" method by doing this "pretention". The integral machinery acting on the generalized function and test function $\varphi(x)$ maintains all its usual properties, such as parts integration and change of variable. This allows us to develop certain "properties" of generalized functions, such as $\delta(ax) = \delta(x)/a$, $a>0$

When there is no LI function for a distribution, it is called a **singular distribution**, such as for the delta above. Else called a **regular distribution**.

Properties of distributions (singular or regular)

translation:	$(f(x+a),\varphi) = (f,\varphi(x-a))$		redefine int variable
scaling:	$(f(\alpha x),\varphi) = \lvert\alpha\rvert^{-1}(f(x),\varphi(x/\alpha))$	$\alpha \neq 0$	redefine int variable
mult by LI g:	$(gf,\varphi) = (f,g\varphi)$		must move f from A to B (?)
diff:	$(f', \varphi) = -(f, \varphi')$		integration by parts

Remember, in the above rules (x,y) is *not* a scalar product, but all the rules are as they would be if it *were* a scalar product. This is how the whole distribution idea is floated off the inner product platform.

Theorem 0. A distribution is infinitely differentiable. $(t^{(n)},\varphi) = (-1)^n (t ,\varphi^{(n)})$. The fact that the test functions are C^∞ comes into play here. This is true for t being regular or singular.

Theorem 1: If LI $f_n(x)$ converges *uniformly* to f(x) over all finite intervals, then $f_n \to f$ as a distribution.

Examples of the Convergence of Distributions (43-47)

Here are some sequences whose limits are generalized functions (except fourth one)

$$\lim_{k\to\infty} \{ (1/\pi)\, k/(1+k^2x^2) \} = \delta(x)$$
$$\lim_{\alpha\to\infty} \{ \sin(\alpha x)/(\pi x) \} = \delta(x)$$
$$\lim_{\alpha\to\infty} \{ \int_{-\alpha}^{\alpha} d\omega\, e^{i\omega x} \} = 2\pi\delta(x)$$

$$\lim_{\alpha\to\infty} \{ \sin(nx) \} = 0$$
$$\lim_{\varepsilon\to 0} \{ 1/(x\pm i\varepsilon) \} = \mp\, i\pi\delta(x) + pf(1/x)$$

Differential equations in Distributions (with examples) (51)

I found it useful to introduce a hat notation to talk about an operator acting on a distribution,

$$\hat{D}\, T_f[\varphi] \equiv T_{Df}[\varphi] = \hat{D}\ \text{" f "} = (Df,\varphi) = -(f,\varphi') \qquad D = d/dx$$
$$\hat{L}\, T_t[\varphi] \equiv T_{Lt}[\varphi] = \hat{L}\ \text{" t "} = (Lt,\varphi) = (t,L^*\varphi) \qquad L = \Sigma_{i=0}^{n}\, a_{n-i}(x) D^i$$

Here then is an "ODE in distributions"

$$\hat{L}\ \text{" t "} = \text{" f "} \qquad \text{or} \qquad (Lt,\varphi) = (f,\varphi)$$

We can classify the solution " t " of our distributional ODE as follows: (n = order of ODE)

- $t, t^{(1)}, \ldots t^{(n)}$ are all classical functions // t = **classical** solution
- one or more of the above *derivatives* are symbolic functions // t = **weak** solution
- t itself is a symbolic function // t = **distributional** solution
- any of the above cases, perhaps a sum of them // t = **generalized** solution

Theorem: If f is LI, and if $a_0(x) \neq 0$, both on our interval (a,b), then all solutions of $\hat{L}$ " t " = " f " are classical solutions. [Stakgold does not prove this.]

Big Theorem for n^{th} order linear ODE:

(1) Lu=0 has exactly n independent solutions, even if L is not symmetric.
(2) Assume you can find A solution of Lu=f. Call this solution "a particular solution". Any solution!
(3) Then the most general solution of Lu=f is particular + homo solutions with N constants.
(4) if you impose the "initial-value" BC set, the solution to Lu=f exists and is unique.

Solving L "g" = " δ_ξ " or $(Lg,\varphi) = \varphi(\xi)$ // This is the Green's problem !
or $L\, g(x|\xi) = \delta(x-\xi)$

This is solved in the form $g(x|\xi) = \{ u(x), v_\xi(x) \}$ where $Lu=Lv_\xi= 0$. Here u(x) is *any* Laplace solution. The function $v_\xi(x)$ matches u(x) at the point $x = \xi$ in all derivatives except $v^{(n-1)}(\xi) = u^{(n-1)}(\xi) + 1/a_0(\xi)$. The technical name for this $g(x|\xi)$ is a **fundamental solution with pole at** ξ, and really is only called a **Green's Function** when you have added some BC's to nail down constants. If you take u(x) = 0, you get "the **causal** fundamental solution". Notice that each u(x) gives a different $g(x|\xi)$.

1.4 Preliminary Results on 2nd order linear ODEs (58)

Theorem 1 (**initial value theorem**). Lu=f has exactly one solution for "initial value" BC's.
Corollary: If the initial value BC's are all 0, then u = 0 is that one solution
Corollary: If you scale all the BC's by γ, new solution is γu.
(Theorem 1 will be proven in Exercise 3.10 in Chapter 3, far in the future!)

The order-n **Wronskian** W is the determinant of a matrix whose first row is $f_1....f_n$ (any set of functions) and whose remaining n-1 rows are the incrementing derivatives of these functions, p 59. Unrelated to Jacobian which involves functions of multiple variables.

Abel's formula says $W(u,v:x) = C\, e^{-m(x)}$ where $m'(x) = a_1(x)/a_o(x)$ and u,v are two solutions of Lu=0. From this we know that:

Theorem 2: if $W(u,v:x=x_0) = 0$, then W(u,v:x) = 0 everywhere in the interval.

Theorem 3: solutions u and v are *dependent* $\Leftrightarrow$ $W(u,v:x_0) = 0$ for some point x_0 in interval
$\Leftrightarrow$ W(u,v:x) = 0 for all x in interval (Thm 2)

The ∞ issue and the Volterra-form solution to Lu=f.

$z(x) = \int_a^x d\xi\, v_\xi(x) f(\xi)$ is a solution to Lz = f on the interval $(-\infty,\infty)$ with the boundary conditions z(a) = 0 and z'(a) = 0, where $v_\xi(x)$ is the solution of Lv = 0 with the (different) boundary conditions that $v_\xi(\xi) = 0$ and $v_\xi'(\xi) = 1/a_o(\xi)$. There is only one such solution $v_\xi(x)$ to Lv = 0 with those BC's. We have found a

particular solution of Lz = f. We know there are no solutions to Lz = 0 with BC's z(a) = 0 and z'(a) = 0 (our corollary earlier, ie, only solution is z ≡ 0), so there are no homo solutions to add, so our z(x) is then THE ONLY solution of Lz = f. This is the **Theorem p 63**.

1.5 Boundary Value Problems (64)

We are now going to formally add "boundary conditions" (BC's) to the soup, still n = 2. These then turn our ODE problems into "boundary value problems", the title of this book and section. For each of our two BC's we are going to say that a certain linear combination of u(a), u'(a), u(b),u'(b) = 0. I suppose we could have picked some other points inside the interval (a,b), but we don't. These two linear combinations are called $B_{1,2}(u)$ and the two equations $B_{1,2}(u) = 0$ are called "**homogeneous BC's**". Later we will set these to constants so $B_{1,2}(u) = \alpha,\beta$ and these would be "**inhomogeneous BC's**". Notice that we use the descriptors homo and inhomo *independently* to describe the BC's and to distinguish Lu=0 and Lu=f. This is a bit confusing. The homo word always means " = 0 ".

There are **two special cases** always of interest: (1) the **initial-value** case, where for example we only retain in $B_{1,2}$ the terms referring to a and not b (or the reverse, or even out in the middle at c). (2) the **unmixed** (=pure) case where B_1 has terms only for a, and B_2 has terms only for b.

Fact: An unmixed BC has the form Au(a) + Bu'(a) = 0. Notice that if you go with the initial-value conditions at endpoint a, which is to say u(a) = u'(a) = 0, then you have satisfied the unmixed condition.

definitions:

- Complete Homo: Lu = 0 with $B_1(u) = 0$ and $B_2(u) = 0$ 1.54
- Complete Inhomo: Lu = f with $B_1(u) = \alpha$ and $B_2(u) = \beta$
- Green's Problem and Function: Lg = δ with $B_1(u) = 0$ and $B_2(u) = 0$ 1.55

Theorem: Suppose u_1 and u_2 are solutions of Lu = f with inhomo BC's $B_i(u) = k_i$. Then the difference function $w = u_1 - u_2$ must be a solution of Lu=0 with $B_i(w) = 0$. That is to say, the difference between any two solutions of the complete inhomo problem is a solution of the complete homo problem. (the proof is obvious in 10 seconds).

Corollary: If the complete homo has no solutions, there is at most one solution to complete inhomo.

Fact: If complete homo Lu=0 has solutions, then operator L has a nullspace of some dimension N>0, and we know this will restrict the range of Lu=f (from our knowledge of later chapters in Stakgold). If f is not in the range, there will be *no* solutions to Lu=f. If f is in the range, there will be N+1 solutions given by a particular solution + the nullspace solutions. So "none or many" if N > 0. This is the Alternative Theorem idea.

Big Theorem: If complete homo has no solutions, then Lg = δ with $B_i(g) = 0$ has a unique solution. Notice that we are talking homo BC's here. This solution is called The Green's Function.

Stakgold spends a lot of time proving this Big Theorem in various situations such as unmixed BC's and initial value BC's, but not for the general BC case. He then gives two situations:

(a) Inhomo ODE with *homo* BC's: $Lu=f$ $B_i = 0$

The solution to this problem is just the straight integral u = Gf (1.61) that we always use, because there are no homo solutions.

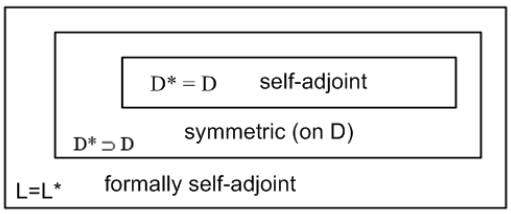
(b) Inhomo ODE with *inhomo* BC's: $Lu=f$ $B_i = \alpha,\beta$

We try here a solution of the form $u(x) = \int g(x|\xi)\, f(\xi) + c_1u_1 + c_2u_2$ where the u_i are certain Lu=0 functions, and we then find the right constants c_i, see 1.63.

Adjoint, Symmetric, and Self-Adjoint Systems (69)

The opening gambit here is to do parts to show that $\langle v,Lu\rangle = \langle L^*v,u\rangle + J(u,v)|_a^b$. The domain of L is called D and is determined by the BC's $B_i(u) = 0$. The domain of L* is called D* and is determined by some generally different BC's called $B^*_i(v) = 0$. If u is in D, then the $B^*_i(v) = 0$ are determined fully by requiring that $J(u,v)|_a^b = 0$. If L = L*, operator is **formally self-adjoint** and L* is the **adjoint** of L. But only if L = L* and D = D* is the ODE system said to be **self-adjoint** (or **fully self-adjoint**). If it happens that $D^* \supset D$, then L is not fully self-adjoint, but it is called **symmetric** on D.

Note in the above that if you are given some L and some $B_i(u) = 0$, the $B^*_i(v) = 0$ are completely determined. I did some raw notes work deriving results for $B^*_i(v) = 0$ given $B_i(u) = 0$.



Fact: If the L system is fully self-adjoint, the Green's function is symmetric

In the above fact "symmetric" means $g(x|\xi) = g(\xi|x)$. Earlier we are saying that a differential operator L is "symmetric" which means $\langle v,Lu\rangle = \langle Lv,u\rangle = \langle u,Lv\rangle$ for real for u,v in D. Both these situations have a symmetric matrix interpretation involving either $g_{x,\xi}$ or $L_{v,u}$. { we are talking real functions here }

Fact: If L=L* formally, then Abel tells us $a_o(x)W(u,v;x) = C$ (easy, p 72 1.76)

The above fact is used a lot in proving things.

Fact: If $B_i(u)= 0$ are *unmixed*, then D=D* and any L=L* then becomes fully self-adjoint.

This is shown by me as a Theorem 1 in the raw notes, and it is easy to prove, 1 page or so of algebra.

Theorem: Suppose we have initial-value BC's u(a) = u'(a) = 0 defining domain D. Then it is easy to show [by studying the J thing] that the equations defining D* are u(b) = u'(b) = 0; these are the $B_i^*(u) = 0$. But these BC's are NOT the same as $B_i(u) = 0$. That is u(a) = 0 differs from u(b) = 0, for example, since a ≠ b . Therefore:

Fact: A system with initial-value BC's can never be self-adjoint since D ≠ D*, regardless of whether or not L=L* formally.

This section contains lots of Examples and Exercises involving specific ODE's. Some of these ODE's apply to **heat conduction** and to **beam deflection** of a **"rod"**, since we are doing 1D work. Some examples involve **Bessel** functions. These three subjects are dealt with in Appendices A and B. In heat conduction, the ODE is usually for u = the temperature. In beam deflection, the ODE is 4^{th} order.

The terms **impulse response** and **step response** are mentioned.

1.6 Alternative Theorems and Modified Green's Functions (79)

Alternative Theorems

The subject of "alternative theorem" is addressed in three worlds: matrix world, integral equation world, and differential equation world. When an operator is taken as B = (λK-1) for integral operator K, say, then the homo problem Bu=0 is connected to the eigenvalue problem for K. In all cases, if the nullspace has non-trivial solutions, then the range is restricted by "consistency conditions".

As a major alternative theorem example, Stakgold solves for "the string in motion" driven at ω. If ω is not an eigenvalue, then nullspace of $L = -D^2 - \omega^2$ is empty, so solution LX=F always exists and is unique and he finds this solution. But if $\omega = \omega_k$, an eigenvalue (in light of the string BC's), then L has a non-trivial solution, and the Alt Theorem puts a restriction on f(x) which is $F_k \equiv \langle F,u_k \rangle = 0$. For $F_k \neq 0$, our ODE LX=F has no solution. The overall PDE in t and x does have a solution, but not in the separable form which gave rise to our ODE LX=F. That solution includes a linear term in t, and we get the solution "blow up" associated with the term **resonance**.

The Modified Green's Function

If, for some given BC's, Lu=0 has non-trivial solutions, it turns out that Lg=δ has *no* solution, ie, there is no Green's Function! To see this, let a non-trivial solution of Lu=0 be $u_1(x)$; the Alternative Theorem's consistency condition on Lg=δ is therefore $\int dx\, u_1(x)\, \delta(x-\xi) = 0$. But this says $u_1(\xi) = 0$ on the interval, a contradiction. This situation is remedied by consideration of an equation which replaces Lg=δ:

$$Lg_M = \delta(x-\xi) - \Sigma_{i=1}^{m}\, u_i(x)u_i(\xi) \qquad (*)$$

where on the RHS we subtract off products for all non-trivial solutions of Lu=0 (which we assume are orthonormalized). In this equation for g_M, the consistency conditions are all met so there is always a

solution and that solution g_M is called the "modified Green's Function". Because Lu=0 has homo solutions, the resulting g_M is not unique. [It may not be easy to solve for g_M in the general case.]

Stakgold assumes that L is self-adjoint at this point. If you impose extra conditions $\int dx\ g_M(x|\xi)\ u_i(x) = 0$, then g_M's free constants are nailed down and you get a $g_M(x|\xi)$ which is symmetric, which is what we would expect for $g(x|\xi)$ with a self-adjoint L.

The solution to the problem Ly=f is then given by

$y(x) = \int d\xi\ g_M(x|\xi)\ f(\xi) + \Sigma_{i=1}^{m} K_i\ u_i(x)$ // if we use the symmetric g_M

The text introduces this subject with a long string example where BC's are u' = 0 at both ends which allows for the non-trivial solution of Lu= 0 in the form u_1 = constant:

$L = -D^2$	with unmixed, so self-adjoint	// p 86 (1.89)
$u_1(x) = \sqrt{1/\ell}$	so that u_1 is normalized on $(0,\ell)$	// p 87 A
$Lg_M = \delta(x-\xi) - 1/\ell$		// p 87 (1.91)
$\int dx\ f(x) = 0$	consistency	// p 87 B
$\int dx\ g_M(x\|\xi) = 0$	condition to make g_M be symmetric	// p 88 A
$y(x) = \int d\xi\ g_M(x\|\xi)\ f(\xi) + K$	// if we use the symmetric g_M	// p 89 A

$g_M(x|\xi)$ is computed and shown explicitly in p 88 B (symmetric)

Last full reading of these meta-meta notes: 5.29.09, 3.12.11