Stakgold Chapter 1 Meta Notes PhL 4.8.09

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Appendix A.1: The strings (and membrane in part (d)). (323)

(a) A physics string problem is set up and Stak shows that the ODE describing it is -Tu"(x) = f(x) where T is a constant, the string tension, and where f(x) is the applied force distribution along the string, which he likes to call a pressure. He arrives at this equation by doing F = ma = 0 static physics.

We can visually see at once that a static "delta function force" at some point out in the middle of the string causes a simple static deflection, and this is the Green's Function for this problem. On either side of the force, the string obviously assumes a straight line u(x) = Ax + B which is a solution of -Tu"(x) = 0, the homo equation. We can see that the string location u(x) is continuous at the delta force point, but the derivative has a jump there. This is the basic Green's idea. Then $u(x) = \int g(x|\xi)f(\xi)d\xi$ would be the string's response to a distributed pressure f(x) and automatically meets the BC's like u(0) = 0. We shall always have this simple example in mind!!!

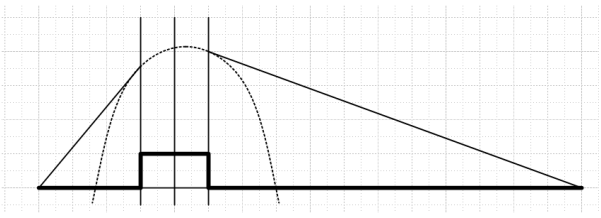
(b) You can model a "string **restoring force**" toward the center line as -Tu"(x) = f(x) - ku(x), where then k is a spring constant per unit length.

(c) If f(x) varies in time, and we add a velocity viscous force, we get $\rho u_{tt} - T u_{xx} - \gamma u_t = f(x,t)$ where ρ is the string's mass per unit length. The **dissipative wave equation**. This is now a **PDE**, not an ODE. Here we are considering only transverse modes of the string and we ignore longitudinal action.

(d) For a rubber sheet (**membrane**), this becomes $\rho u_{tt}(x,y;t) - T \nabla^2 u(x,y;t) - \gamma u_t(x,y;t) = f(x,y;t)$ where now ρ is mass per area and T is tension per area and γ is viscous force per area and f is real pressure. This is then an example of a **PDE with 2 space and 1 time dimension**.

1.1 The string under pressure f(x) (1)

Right off the bat we solve the string for when there is a box-shaped pressure applied out in the middle. I did a Visio plot of the solution, which is parabolic where there is force,



The box curve of course is also the plot of curvature u" since -Tu"(x) = f(x). Since no δ forces here, u and u' are both continuous. We get our first formal "statement of problem" which includes saying f(x) is PC, u and u' are C, and u" is PC. Limit as box gets narrow is the Green's below.

The Green's Function for the String with Fixed end points (5) Stak solves the problem -Tu"(x) = δ(x-ξ) using Appendix 1 results and we obtain the string Green's function written out as 1.6 page 9. He used continuity of g and jump in g' at the delta point. In this problem, L is symmetric so g is symmetric, but that is not mentioned.

Uses of the Green's Function for the String (10) . The superposition argument for $u(x) = \int dy\, g(x|y)f(y)$ is presented.

At this point, I go off on a wiki digression learning the meaning of PC and of C^k for functions, the general subject is different levels of continuity. See raw notes. I then prove this theorem:

Theorem: (careful differentiation) (but not really a general theorem of use)

(a) Consider F(x,y) which has some kind of "problem" at x = y, something is perhaps not continuous or not differentiable there. Let $F_-(x,y) \equiv F(x,y)$ when $y < x$ and $F_+(x,y) \equiv F(x,y)$ when $y > x$. If we assume that $F_\pm$ are both C^1 in variable x ($\partial_x F_\pm$ = continuous, meaning $F_\pm$ = differentiable), then we find that

$$d/dx\ [\ \int dy\ F(x,y)] \quad = \int dy\ (\partial F\ (x,y)/\partial x) \ + \quad \{\ F_-(x,x)\ -\ F_+(x,x)\ \}$$

Notice that we have not assumed that F is C^1 at the point y=x, nor that F is C^0 there. That is to say, F might be discontinuous and/or non-differentiable at y = x. If discontinuous, then $F_-(x,x) \neq F_+(x,x)$.

(b) If we now ALSO assume that F is in fact continuous at x = y, AND that $\partial_x F_\pm$ are both C^1 in variable x ($\partial^2_x F_\pm$ = continuous, meaning $\partial_x F_\pm$ = differentiable) then we get

$$d^2/dx^2\ [\ \int dy\ F(x,y)] \quad = \int dy\ (\partial^2{}_x F\ (x,y)) \quad + \quad \{\ \partial_x F_-(x,x)\ -\ \partial_x F_+(x,x)\ \}$$

You show these things by breaking the original integral into two pieces separated at y = x. The general question here is when can you "interchange order" of ∂_x and $\int dy\ F(x,y)$ as shown. The answer is that for ∂_x interchange is OK if F is continuous, and for ∂_x^2 it is OK if F *and* $\partial_x F(x,x)$ are continuous.

Application of Theorem to Green's. We then *apply* this theorem to the following situation. We have -Tu" = f(x) and we conjecture from superposition that $u(x) = \int dy\, g(x|y)f(y)$ where g is our string Green's Function. So we apply theorem part (b) above. We know that $\partial^2_x\, g(x|y) = 0$ in the integral so only the {..} is left. We thus find:

$$\partial^2_x\, u(x) = \partial^2_x \int dy\; g(x|y)f(y) = \{\, \partial_x\, g_-(x|y) \;-\; \partial_x\, g_+(x|y)\}\; f(y) \;= -T\, f(y)$$

and this says Lu = f which is what we want. The point is that this is all done without delta functions. We have directly proven that $u(x) = \int dy\; g(x|y)f(y)$ is a solution of Lu = f.

Comments: Note the following in the sense of classical functions:

$$\partial^2_x \int dy\; g(x|y)f(y) \;\neq\; \int dy\; [\partial^2_x\, g(x|y)]f(y) \quad = 0$$

where the y integral is the two part integral with limit $\varepsilon \to 0$. However, when we later get into δ and distributions, then we talk about $\partial^2_x\, g(x|y) = \delta(x-y)$ and then we have left the realm of classical functions. We then have $\int dy\; \delta(x-y)\; f(y) \;= (\delta_x,\, f) = f(x)$ where δ_x is then a distribution $T_{\delta x}[f] = (\delta_x, f) = f(x)$. But then we have moved into generalized functions and an extended meaning for $\int dy$ which does not apply in our work above. In the classical world, we have $\int dy\; [\partial^2_x\, g(x|y)]f(y) = 0$ and we need those extra parts terms {...}. In the distribution world, we have $\int dy\; [\partial^2_x\, g(x|y)]f(y) \quad = f(x)$ and there are no extra terms. You can work in either world. [don't worry about this paragraph]

Notion that L and G are inverses.

Start with $u(x) = \int dy\; g(x,y)f(y) \equiv Gf$. We know that Lu = f, so we get

$$Lu = L(Gf) = (LG)f \;= f \qquad \Rightarrow \qquad LG = 1 \text{ when acting on f-class functions}$$

But we can also say

$$(GL)u \;= G(Lu) = Gf = u \qquad \Rightarrow \qquad GL = 1 \text{ when acting on u-class functions}$$

What are these "classes"? Going back to page 4, I think they are these:

u-class =functions where u" is piecewise continuous. Then u' = C^0 and u = C^0 so both u' and u are continuous. This agrees with the words there on page 4. Thus, u-class is C^0 and C^1 class and is in fact C^2 except at a few points.

f-class = piecewise continuous so f = weak C^0 (not continuous and not differentiable at some points) So f-class is C^0 except at a few points.

So u-class is a stricter class by two levels! The u-class is smoother than the f-class. Any function in u-class is also in f-class! Thus we have (LG)u = u and (GL)u = u and for u-class functions it seems that we can say LG = GL = 1 so that $G = L^{-1}$ and $L = G^{-1}$.

Advantages of the Green's Function method.

1. The BC's are built into g, so as you change f, no need to recompute constants.

2. We have the interpretation of $u(x) = \int dy\, g(x,y)f(y)$ as a superposition of responses to point loads.

3. We can combine a continuous load (pressure) with point pressures as p 13 A.

4. If we shift the BC's to be more general, u(0)=a and u(ℓ) = b instead of 0 and 0, the same g for the 0,0 problem can be used but we get a simple extra term as in p 14B. This result is derived more simply I think in Exercise 1.1 using superposition. [and appears later in the chapter in the n=2 stuff]

5. The g function allows us to relate an ODE to an integral equation. Here, he talks only about the EV equation situation. The ODE is $Lu = \lambda u$ and the IE is $u = \lambda Gu$. Use $L = G^{-1}$ to move back and forth.

$Lu = \lambda u \Rightarrow GLu = \lambda Gu \Rightarrow u = \lambda Gu$ so ODE $\Rightarrow$ IE

$u = \lambda Gu \Rightarrow Lu = \lambda LGu \Rightarrow Lu = \lambda u$ so IE => ODE

Note added 3.12.11. The Richard Price question: Given G, when can you find L such that LG=1? Only if you can find such a L can you find an ODE that "goes with" your integral equation. The question is: given a kernel g(x,y), how can you tell if it is the Green's Function of some L such that LG=1? If such an L exists, then it must be that $L = G^{-1}$. So one condition is that operator G must have an inverse. Is the Love kernel invertible? Save this question for later in this Stak review.

1.2 The Dirac Delta Function (18)

δ sequences $s_k(x)$ (19) The myth that $\delta(0) = +\infty$ is thrown out by s_k counterexamples where $\delta(0) = -\infty$ and $\delta(0) = 0$. The value must somehow be undefined for $\delta(x)$ at x=0. Similarly, the value of H(0) is undefined, though it happens to be 1/2 in the page 20 example.

The sifting property (22). $A_k[\varphi] = \int dx\, s_k(x)\varphi(x)$ is the "action" of delta sequence s_k on φ. In limit he shows you get $A[\varphi] = \int dx\, \delta(x)\varphi(x) = \varphi(0)$. Can shift from 0 to x_0 and this is called the sifting property. Limitation is that φ(x) must be continuous and bounded.

1.3 The Theory of Distributions (28)

I add some history (1935-1950) with names **Sobolev** and **Schwartz.** I was confused on first reading and went off and wrote separate notes "distributions.doc" and I will quote here from that doc. I quote a lot here, so this section of meta notes is going to be long.

General Comments on Stakgold's Approach and Notation

One big problem is that he treats distributions in Chapter 1, but treats Hilbert Spaces and functionals in Chapter 2. This causes various notational confusions that I will now try to clarify:

(1) Later in the book, the notation <a,b> always means an inner product associated with a Hilbert Space. One example is the HS = L^2 on the reals, with its definition as $\langle a,b\rangle = \int dx\ a(x)b(x)$. However, in Chapter 1 he uses this same notation <t,φ> to describe "the action of t on φ". He does this with <t,φ> NOT being in all cases an inner product.

(2) Also later in the book, he uses T[x] to define on operator T:X→R where the range space is the reals. Such a thing is called a functional. But he does not use this notation at all in Chapter 1.

(3) Also later in the book he proves the Riesz theorem which says:

"Any continuous linear functional T[x] can be written in the form T[x] = <f,x> where f is some vector in the Hilbert Space associated with the inner product. "

I like to call this vector f "a marker function", and I might then write $T_f[x] = \langle f,x\rangle$. This says there is a homomorphism between the Hilbert Space and the space of linear continuous functions acting on that Hilbert Space.

So I will use the following three equivalent notations for a functional:

$$\text{" f "} = T_f[\varphi] = (f,\varphi)$$

where (f,φ) is in general NOT an inner product in a Hilbert Space. The notations (f,φ) and " f " are just another way to "notate" the functional $T_f[\varphi]$. We can call either of the second two notations "the action of f on φ". I will reserve the notation <f,g> to refer to the usual function integration inner product which works on space L^2 and on any subspace of that space. According to Riesz

$$T_f[\varphi] = (f,\varphi) = \langle f,\varphi\rangle \qquad \text{if } T_f \text{ is a continuous (ie, bounded) linear functional}$$

I show earlier in distributions.doc that $T_\delta[\varphi] = \varphi(0)$ is not bounded, so it is therefore not continuous, and therefore Riesz does not apply and therefore you *cannot* use the = <f,φ> for this δ functional. That is, you cannot use the inner product in the sense of classical functions.

Definition of a distribution: A "distribution f" is any *continuous* (ie, bounded) linear functional " f " = $T_f[\varphi] = (f,\varphi)$ which is defined on a certain subspace S of the Hilbert Space "L^2 on some interval (a,b)". That certain subspace S is the space of all "test functions" -- the space of all C^∞ functions having compact support, which means zero outside some finite range. This space might be written $S = C^\infty_c$. Notice there

is no inner product used in this definition of a distribution, although there does of course exist an inner product for L^2 which we could write as <a,b> if a and b are in L^2. We know that the test functions φ are all in L^2, but the "distribution f" is not necessarily associated with a function f in L^2 or in fact with a function of any kind. Our object " f " is a functional, not a function. Notice that the *value* of any "distribution f" when acting on a test function is $T_f[\varphi] = (f,\varphi)$ = a real number.

Question: "Are normal functions also distributions? " The answer is semantically no. However, with every normal function f(x) we may *associate* a distribution T_f as follows:

$$\text{" f "} = T_f[\varphi] = (f,\varphi) \equiv \langle f,\varphi\rangle = \int_a^b dx\, f(x)\varphi(x)$$

" distribution f " |.. "normal function f "

The distribution is a functional which is labeled by the marker function f. The distribution is an integral of f against any and all test functions φ. For every marker function f in L^2 (quite a lot of functions!), there exists a distribution " f " as shown above. So, every "normal function f " is *associated with* (or "has") a distribution "f", but we would never say that normal functions *are* also distributions because they are different animals. A distribution for a normal function f is a set of integrals of the function f against various test functions φ, whereas a function f is a set of values of f at different points x on the real axis. You can think of each of these objects as a "table", but they are different tables (each with ∞ # of entries).I guess alternatively you can think of a distribution as an operator or mapping which acts on the space of text functions to give a real number: $T_f : S \rightarrow R$.

Here is just a quick example in passing. Heaviside H(x) is a "normal function" so we can say

$$\text{" H "} = T_H[\varphi] = (H,\varphi) \equiv \langle H,\varphi\rangle = \int_{-\infty}^{\infty} dx\, H(x)\varphi(x) = \int_0^{\infty} dx\, \varphi(x)$$

Question: " What is a normal function? " It turns out that the notion of a "normal function" means a "locally integrable" function.

4. Locally integrable (LI) functions. A LI function defined on the interval (a,b) is one that can be integrated in the normal classical fashion over any finite subinterval in that interval. Certainly at any point where a function is continuous we will have the LI property. A function with a finite jump like H(x) at x=0 is still LI at x=0. Where you might have a problem is if your function is infinite at some point in the interval. As an example, f(x) = 1/x is not LI at x = 0, whereas f(x) = ln(x) is LI even though it is singular at x = 0. Certainly δ(x) is not LI as the "method of nested intervals" shows. For a LI function, our functional $T_f[x]$ shown above takes on a finite real value.

Definition of a generalized function. We know that for LI functions, we can say

$$\text{" f "} = T_f[\varphi] = (f,\varphi) = \langle f,\varphi\rangle = \int_a^b dx\, f(x)\varphi(x)$$

" distribution f " |.. "a LI f "

and we know that for functions that are *not* LI, we can still have a distribution

$$\text{" f "} = T_f[\varphi] = (f,\varphi)$$
$$\text{" distribution f "}$$

We may be able to find an "expression" for the "distribution f " when f is not an LI function, such as

$$\text{" f "} = T_f[\varphi] = (f,\varphi) = \text{"some expression involving f and } \varphi\text{"}$$
$$\text{" distribution f "}$$

Example 1: We consider the function 1/x which is not LI. It is shown in Stak on page 49 that there is in fact an expression you can write for the "distribution 1/x " :

$$\text{" 1/x "} = T_{1/x}[\varphi] = (1/x,\varphi) = \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) dx\,(1/x)\,\varphi(x) = \fint dx\,(1/x)\,\varphi(x)$$
$$\text{"expression involving 1/x and } \varphi\text{ "}$$

Here we have a rule for expressing the "distribution 1/x " as a limit of a sum of two integrals, which limit we often represent using the tick integral sign. This expression is NOT an inner product in the L^2 sense, it is not even an "integral". It is the limit of the sum of two integrals, but we cannot write that as a single classical integral. Notice that as $\varepsilon\to 0$, each of the two integrals diverges, the first one gets large negative because $x < 0$ on that side, so that 1/x = large negative number. But the second integral is a correspondingly large positive number. The two infinities cancel out leaving a finite result.

Now, we can **pretend** (symbolically) there is a classical integral, and define a certain thing to replace the 1/x above and have it *look like* a classical integral of a classical function. We thus say, in this particular example, that "expression"

$$\fint dx\,(1/x)\,\varphi(x) = \int dx\ \text{GeneralizedFunctionVersionOf}(1/x)\,\varphi(x) = \int dx\ \text{GFVof}(1/x)\,\varphi(x)$$

The thing on the right is just a notation! There is no such classical integral of a classical function. The meaning of the thing on the right is the thing on the left! So anytime you encounter the thing on the right, you *mean* the thing on the left. If you encounter GFVof(1/x) all by itself, not in an integral, it has no meaning as a classical function until you put it into an integral. However, by looking at the thing on the left in the above, we may come up with some "manipulation rules" for GFVof(1/x). We can then validly "manipulate" GFVof(1/x) in expressions where it appears without an integral, but we only recover the meaning of the thing when we apply the classical integration to it, and then the meaning is what is shown on the left above.

Notice that we are allowed to formally "manipulate" our classical integral shell here such as doing a change of integration variable (or parts integration). The integration mechanism is real, but the generalized function inside is not real (not a classical function), so as a whole we don't have "a classical integral of a classical function".

Stakgold gives this particular generalized function a certain name pf(1/x), we we have

$$\text{" } 1/x \text{ "} = T_{1/x}[\varphi] = (1/x,\varphi) = \text{⨍} dx\ (1/x)\ \varphi(x) = \int dx\ \ pf(1/x)\ \varphi(x)$$

Distinction between a classical function and a generalized function.

Classical functions include those which can be constructed from elementary functions. For example, f(x) = 5/x + sin(a+x)*exp(4x²)* (1-ln(x)). But we know there are fancier functions like Γ(x) that are also perfectly fine classical functions, but which cannot be written in terms of elementary functions except perhaps as a limit or a series.

Plan A. Maybe we can say that a classical function f:R→R is a piecewise continuous (PC) function which has perhaps some singular points. The piecewise sections of the function can be **graphed** in that for every x, we can plot f(x) on a piece of paper (or two for complex). In the neighborhood of any singular point, we can approximate the classical function in terms of elementary functions. This sounds pretty good to me right now. It might be better to think of f:C→C and then talk about Re(f) and Im(f).

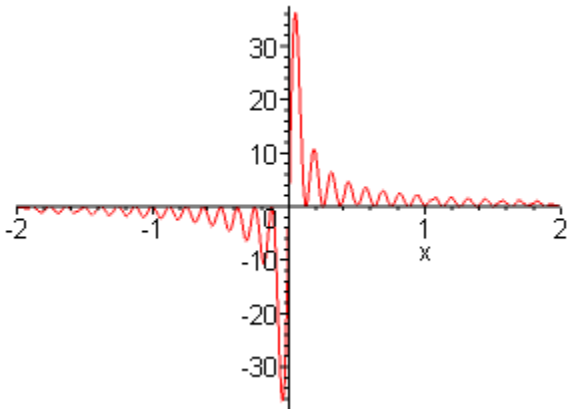
How could you approximate δ(x) near x=0 with an elementary function?

Also, if you make a graph of δ(x) with 10 billion points, it cannot be distinguished from the graph of the function f(x) = 0. So δ(x) really has no distinguishing "graph".

A generalized function can always (I think) be written as a limit of a sequence of classical functions. We know how this works for the δ(x) case. Here is another example relating to the example above:

$$pf(1/x) = \lim_{R\to\infty} (1/x)(1 - \cos(Rx)) = \lim_{R\to\infty} (2/x)\sin^2(Rx/2)$$

The sequence functions $(2/x)\sin^2(Rx/2)$ are finite everywhere in this case. Here is R = 50:



As R→∞, the oscillation gets faster without limit, and we then *cannot even draw* the graph of the function in that limit. It has no graph. For this function, we cannot approximate pf(1/x) near *any* value of x with an elementary function. If you had to draw something, you would draw a shaded area under each of the 1/x envelopes. Every point in the shaded area lies on the generalized function. The slope of the function is infinite at every value of x.

In distributions.doc and in the raw notes and in the text we have various other examples of generalized functions. The most famous of course is δ(x) which goes with this distribution

" δ " $= T_{\delta}[\varphi] = (\delta,\varphi) = \varphi(0)$ // $= \int dx\ \delta(x)\ \varphi(x)$

Here are some examples of "manipulation rules" for the generalized function δ(x)
(h(x) is a classical function)

$\delta(ax) = \delta(x)/a$ $a>0$
$\delta(x)\ h(x) = \delta(x)\ h(0)$
$\delta'(x)h(x) = \delta'(x)\ h(0) - h'(0)\delta(x)$
$3\delta(x+1) - 5\delta(x-a)$ // this is a generalized function
$\delta(ax^2 + bx + c)$ // this is a generalized function
$\delta(ax^2 + bx + c) + 2x + 1$ // this is a "mixed" function, part gen, part class
$5\delta(x-a)\delta(2x)$ // this is NOT a generalized function

Question: Can you multiply two generalized functions. Let A and B be gen functions,

"a" $= T_a[\varphi] = (a,\varphi) = \int_{-\infty}^{\infty} dx\ A(x)\ \varphi(x)$

"b" $= T_b[\varphi] = (b,\varphi) = \int_{-\infty}^{\infty} dx\ B(x)\ \varphi(x)$

How could you express $\int_{-\infty}^{\infty} dx\ [A(x)\ B(x)]\ \varphi(x)$ in terms of (a,φ) and (b,φ) ? Taking the product of

$(a,\varphi)(b,\varphi) = \int_{-\infty}^{\infty} dx\ A(x)\ \varphi(x)\ \int_{-\infty}^{\infty} dx\ B(x)\ \varphi(x)$

does not work, for example. If A and B are *classical* functions, we know that A(x)B(x) is another classical function C(x) and we then have $T_c = (c,\varphi) = \langle c,\varphi\rangle = \int_{-\infty}^{\infty} dx\ [A(x)\ B(x)]\ \varphi(x)$ and so we know that $T_c[\varphi] = T_{ab}[\varphi]$ exists.

Question: Can you multiply a generalized function by a regular function?

In the above example, let A(x) be a GF and let B(x) = g(x) be a C^{∞} regular function. Then we get

$T_{ag}[\varphi] = (ag,\varphi) = \int_{-\infty}^{\infty} dx\ A(x)\ g(x)\varphi(x) = \int_{-\infty}^{\infty} dx\ A(x)\ \varphi_1(x) = (a,\varphi_1) = T_a[\varphi_1] = T_a[g\varphi]$

So here the integral $\int_{-\infty}^{\infty} dx\ A(x)\ g(x)\varphi(x)$ *does* have a meaning. That meaning is $T_a[\ g\varphi\ ]$. We could not do this trick with two generalized functions because then gφ would not be a test function and then we have no meaning.

I am now going to try to segue back into my Chapter 1 raw notes in terms of headings, but I still take the actual words here more from the distributions.doc notes.

Operations with Distributions (35)

1. These properties were derived for distributions involving LI functions, for which the inner product exists. The properties then follow trivially from the mechanical properties of the integral form: translation, scaling, and multiplication by an LI function. Later a fourth property is mentioned: a differentiation rule. Here we shall write these four properties using our generic distribution notation.

translation:	$(f(x+a),\varphi) = (f,\varphi(x-a))$		redefine int variable
scaling:	$(f(\alpha x),\varphi) = \lvert\alpha\rvert^{-1}(f(x),\varphi(x/\alpha))$	$\alpha \neq 0$	redefine int variable
mult by LI g:	$(gf,\varphi) = (f,g\varphi)$		must move f from A to B
diff:	$(f', \varphi) = -(f, \varphi')$		integration by parts

In the last item, the compactness of the test function space makes the parts vanish. These four rules are clear for non-singular distributions because you just write out the inner product integral. But what can we say about the above rules of the distributions are singular so the integral form does not exist?

Convergence of Distributions (42)

2. Starting on page 42, Stak talks about the notion of $t_n \rightarrow t$ where both symbols here mean distributions (I drop the double quotes for now) not functions. Let's assume that when $n < \infty$, we can associate with t_n a function $t_n(x)$ which is LI. Then we can write:

$$(t_n,\varphi) = \langle t_n,\varphi \rangle = \int_{-\infty}^{\infty} dx\, t_n(x)\, \varphi(x)$$

The above four properties apply here. Let's just write all four of them from above,

$(t_n (x+a),\varphi) = (t_n,\varphi(x-a))$
$(t_n (\alpha x),\varphi) = \lvert\alpha\rvert^{-1}(t_n (x),\varphi(x/\alpha))$
$(g\, t_n,\varphi) = (t_n,g\varphi)$
$(t_n', \varphi) = -(t_n, \varphi')$

3. Now we consider the singular case. We take $t_n \rightarrow t$, but we find that $t_n(x) \rightarrow t(x)$ which is not a real function, only a generalized function. I think the big idea is that we can take the limit through the integration of the inner product "in the magical world of generalized functions". Then we can restate all the above rules replacing t_n with the limit t. Then we claim that all four of the above rules are still true for a singular distribution, with the understanding that t(x) is now a generalized function and not a regular function. So we then have these rules which then apply to both regular and singular distributions

$(t(x+a),\varphi) = (t,\varphi(x-a))$
$(t(\alpha x),\varphi) = |\alpha|^{-1}(t(x),\varphi(x/\alpha))$
$(gt,\varphi) = (t,g\varphi)$
$(t', \varphi) = - (t, \varphi')$

If we combine the last two rules together, and if we assume that the functions in operator L are C^∞, then we come up with a "fifth rule" which is this:

$(Lt,\varphi) = (t, L^*\varphi)$

where L* is the adjoint and is also exactly given by formula p 55 B or p 40 A. For example, consider

$(g\, D^m t, \varphi) = (D^m t, g\varphi)$ // by rule 3 above
$= (-1)^m (t, D^m(g\varphi))$ // by applying rule 4 m times

To bring things a little down to earth on this $t_n \rightarrow t$ idea, let's do a few of "our own" examples.

4. First example, the delta function. We start with

$$(t_n,\varphi) = \langle t_n,\varphi\rangle = \int_{-\infty}^{\infty} dx\, t_n(x)\, \varphi(x)$$

where t_n is any smooth unit-area sequence we like which approaches δ. We even know a sequence of test functions that does it if we want. We then take the limit on both sides

$\lim_{n\rightarrow\infty} (t_n,\varphi) \equiv (t,\varphi)$ where $\lim_{n\rightarrow\infty} t_n = t$

So far we are just defining t in our little parens notation. We could also say $\lim_{n\rightarrow\infty} T_{t_n}[\varphi] = T_t[\varphi]$.

How do we know that the limit exists? We don't in general, but here we just assume it exists and then if we can find what it is, we will then know that it exists.

So now we have

$$(t,\varphi) = \lim_{n\rightarrow\infty} \{ \int_{-\infty}^{\infty} dx\, t_n(x)\, \varphi(x) \}$$

We know that the integral converges for every finite n. The official "interchange is allowed" theorem requires that $| t_n(x)| <$ some f(x) for all n (see chap 2 notes "interchange"), but we don't have that property here for t_n going to a delta function. So instead, let's consider very large n so that $t_n(x)$ is very strongly peaked. We can then say

$$\int_{-\infty}^{\infty} dx\, t_n(x)\, \varphi(x) \approx \varphi(0) \int_{-\infty}^{\infty} dx\, t_n(x) = \varphi(0)$$

As n gets larger, this ≈ approaches = (we know that φ(x) is continuous at x = 0). The area of any $t_n(x)$ is arranged to be 1 of course, so we have the far right above. So there is not much objection I think to saying that

$$(t,\varphi) = \lim_{n\to\infty} \{ \int_{-\infty}^{\infty} dx\, t_n(x)\, \varphi(x) \} = \varphi(0)$$

Since we have found the limit, we now know that the limit t exists, which we only assumed earlier.

At this point, we "pretend" that we can do the order interchange and then try to make sense out of the result in terms of "generalized functions" . So we get

$$(t,\varphi) = \int_{-\infty}^{\infty} dx\, \{ \lim_{n\to\infty} t_n(x) \}\varphi(x) = \int_{-\infty}^{\infty} dx\, \delta(x)\, \varphi(x) \qquad // \equiv (\delta,\varphi) = \varphi(0)$$

where "the meaning" of $\int_{-\infty}^{\infty} dx\, \delta(x)\, \varphi(x)$ is simply φ(0). So we simply have a "bookkeeping" method by doing this "pretention".

5. In this similar example in the distribution.doc notes, I show that

$$(\delta',\varphi) = -(\delta,\varphi') = \lim_{n\to\infty} \{- \int_{-\infty}^{\infty} dx\, \delta_n(x)\, \varphi'(x) \} = \int_{-\infty}^{\infty} dx\, \{ \lim_{n\to\infty} \delta_n'(x) \}\varphi(x) = \varphi'(0)$$

$$= \text{ " } \int_{-\infty}^{\infty} dx\, \delta'(x)\, \varphi(x) \text{ " } \qquad \text{where } \delta'(x) \text{ is a generalized function}$$

Here are some theorems related to the above topic:

Theorem 0. A distribution is infinitely differentiable. $(t^{(n)},\varphi) = (-1)^n (t, \varphi^{(n)})$. The fact that the test functions are C^∞ comes into play here. This is true for t being regular or singular.

Theorem 1: If LI $f_n(x)$ converges *uniformly* to f(x) over all finite intervals, then $f_n \to f$ as a distribution.

Remark: It might be true that $f_n \to f$ as a distribution even if, as a function, we not only fail to have uniform convergence, but we also fail to have *pointwise* convergence. In our Big Example below, we have $f_n = \sin(nx)$ which pretty clearly does not have pointwise convergence at any x! But we will show that in fact $f_n \to 0$ as a distribution.

Term by term: You would not use the phrase "differentiate term by term" with regard to a simple sequence, but suppose the sequence is a partial sum thing:

$$t_k(t) = \Sigma_{n=0}^{k} a_n(t) = \text{some distribution}$$

Suppose $t_k \to t = \Sigma_{n=0}^{\infty} a_n(t)$. Then consider $t_k' = \Sigma_{n=0}^{k} a_n'(t)$. We have a finite number of terms here so no issue of whether this term by term differentiation is legal. But now our Theorem 1 tells us that

$t_k' \rightarrow t' = \Sigma_{n=0}^{\infty} a_n'(t)$. So NOW we can say that for a distribution that is an infinite sum like t, we can in fact *differentiate term by term* and we get the right answer, despite doubts about doing so with an ∞ sum.

Big Example. We consider this distribution:

$$\text{" } \sin(nx) \text{ " } = (\sin(nx),\varphi) = \int_{-\infty}^{\infty} dx \sin(nx)\varphi(x)$$

which is finite for any n and is as happy as can be. What happens as $n\rightarrow\infty$? We learn from the raw notes analysis that in fact $\lim_{n\rightarrow\infty}$ " sin(nx) " = 0. We can then ask what this tells us about the function inside the integral. This function sin(nx) for any finite n is just a classical function, not a generalized function. But suppose we "think of it as" a generalized function, meaning the integral of it is really defined by what is on the LHS. Since we know that limit of LHS = 0, we can "think of" the generalized function → 0 and we can write that $\lim_{n\rightarrow\infty} \sin(nx) = 0$ in a "generalized function" sense. I guess we could say that in fact this limit 0 IS a generalized function S(x) = 0. Again: before the limit, sin(nx) is a classical function. But the limit is a generalized function equal to 0. If you think of this limit in a classical sense, it is like the case mentioned above, the function approaches a "shaded area" of height 1. This is another example of a function that you cannot graph.

Examples of the Convergence of Distributions (43-47)

Here are some sequences whose limits are generalized functions.

$$\lim_{k\rightarrow\infty} \{ (1/\pi)\, k/(1+k^2x^2) \} = \delta(x)$$

$$\lim_{\alpha\rightarrow\infty} \{ \sin(\alpha x)/(\pi x) \} = \delta(x)$$

$$\lim_{\alpha\rightarrow\infty} \{ \int_{-\alpha}^{\alpha} d\omega\, e^{i\omega x} \} = 2\pi\delta(x)$$

$$\lim_{\alpha\rightarrow\infty} \{ \sin(nx) \} = 0$$

$$\lim_{\varepsilon\rightarrow 0} \{ 1/(x\pm i\varepsilon) \} = \mp\, i\pi\delta(x) + pf(1/x)$$

Exercises (47-51)

This section talks about the "expression" which defines the generalized function $pf(1/x)_+$ and other examples of this type, including the one quoted above involving $\oint dx\ (1/x)\varphi(x)$

Differential equations in Distributions (with examples) (51)

First, we need to clean up some notation that up to now has been hazy. Consider:

$$(Df,\varphi) = (f', \varphi) = -(f, \varphi') = \int dx\, f'(x)\, \varphi(x)$$

Here we allow that f(x) and/or g(x) = Df(x) might be classical or symbolic functions. The object g = Df is some new "function" of some kind, so we would write (g,φ) and then $T_{g}[\varphi] = T_{Df}[\varphi]$. The subscript of the linear functional symbol T is the marker of the linear functional. Notice that it makes no sense whatsoever to apply d/dx to $T_{f}[\varphi]$ which is applying d/dx to a real number which is surely 0. So in this sense, there is no interesting meaning to the notation D ($T_{f}[\varphi]$). However, we can endow this with a more interesting and useful meaning by saying this:

$$\hat{D}\, T_{f}[\varphi] \equiv T_{Df}[\varphi] = \hat{D}\ \text{" f "} = (Df,\varphi) = -(f,\varphi')$$

We when we talk about "differentiating a distribution" we mean the above: we mean differentiating the distribution's marker function, even if it is a symbolic function.

We can of course extend this idea of a general order-n linear differential operator L with C^{∞} coefficient functions and we can then write (here we change from letter f to letter t , and letter will do),

$$\hat{L}\, T_{t}[\varphi] \equiv T_{Lt}[\varphi] = \hat{L}\ \text{" t "} = (Lt,\varphi) = (t,L^{*}\varphi)$$

Armed with this notation, we could ponder this interesting equation:

$$\hat{L}\ \text{" t "} = \text{" f "} \qquad \text{or} \qquad (Lt,\varphi) = (f,\varphi)$$

The things in the differential equation here are not functions, they are distributions. On the left, we make it "look like" a differential equation all in classical functions like Lu = f, but when we want to do anything, we of course use the notation on the right. So now we are studying a differential equation in which the things appearing on the two sides are distributions, not functions.

At once, we classify the solution " t " of our distributional ODE as follows:

- t, $t^{(1)}$, ... $t^{(n)}$ are all classical functions // t = **classical** solution
- one or more of the above derivatives are symbolic functions // t = **weak** solution
- t is a symbolic function // t = **distributional** solution
- any of the above cases , perhaps a sum of them // t = **generalized** solution

Of course once something is a generalized function, all its derivatives will be as well.

Theorem: If f is LI, and if $a_0(x) \neq 0$ both on our interval (a,b), then all solutions of $\hat{L}$ " t " = " f " are classical solutions. [Stakgold does not prove this.]

Since this subject is so strange to me, here are three examples of ODE's in distributions:

Example 1: x d/dx " t " = " f " = 0. (homo)

We have $(xt',\varphi) = (t', x\varphi) = 0$. But we could slyly write this 0 as $[x\varphi(x)]^{x=0}$. Then we find that $t = \delta$ works here as a solution, since $(\delta,\psi) = \psi(0)$ where $\psi(x) = x\varphi(x)$ = a test function. Since $t' = \delta$, we then know that $t = H$. If we write out $(t', x\varphi) = \int dx\, \delta(x)\, x\, \varphi(x) = 0$, we can obtain one of our little symbolic manipulation rules which is that $x\delta(x) = 0$, which is no big mystery. Our most general solution to this example must then be $t(x) = c_1 H(x) + c_2$ where k = constant. We know that

$$d/dx\,(c_2,\varphi) = d/dx\,(\, c_2 \int dx\varphi(x)\,) = d/dx\,(\text{ some number}) = 0.$$

so it is certainly true that x d/dx " c_2 " = 0. Our solution to this problem is a **weak** solution because, although t(x) is itself a classical function, $t'(x) = \delta(x)$ is a generalized function.

Example 2: x^2 d/dx " t " = " f " = 0. (homo)

I show that both H and δ are solutions so we get $t = c_1 H(x) + c_2\delta(x) + \delta_3$. Solution is distributional.

Example 3: d^2/dx^2 " t " = " δ" " = 0. (inhomo)

The general solution here is $\delta(x) + Bx + C$ where we add in the homo general solution. The particular solution is just $\delta(x)$. The resulting solution is distributional, but could be called generalized since a mix.

Notice that in all of the above, we have said nothing about any "boundary conditions" for the ODE.

Fundamental Solutions and Green's Functions (54)

Preliminaries. We assume that if L is order n, then the nullspace of L has dimension n, so there are n independent solutions to Lu=0. I don't know *how* we know this fact at this point in the book. Maybe this is only really true if L is self-adjoint? I seem to recall comments on this from chapters 2 and 3, but cannot find the references. Maybe chapter 4 will have more to say. I am used to this being true for n=2 -- always two solutions.

Here in ODEreview.pdf which I downloaded, we have this theorem stated (though not proved) which is exactly what I am looking for:

Theorem 2 *Assume that the coefficients $a_0(t)$, $a_1(t)$, $\cdots$, $a_{n-1}(t)$, $a_n(t)$ and $F(t)$ are reasonably smooth, bounded functions and that $a_0(t)$ is not zero.*

(a) The general solution to the ODE (1) is of the form

$$y(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t) + \cdots + C_n y_n(t)$$

where

- *n is the order of the ODE (1)*
- *the particular solution, $y_p(t)$, is any solution to (1)*
- *C_1, C_2, $\cdots$, C_n are arbitrary constants*
- *y_1, y_2, $\cdots$, y_n are n independent solutions to the homogenous equation*

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_{n-1}(t)y'(t) + a_n(t)y(t) = 0$$

"Independent" just means that no y_i can be written as a linear combination of the other y_j's. For example, $y_1(t)$ cannot be expressed in the form $d_2y_2(t) + \cdots + d_ny_n(t)$.

(b) Given any constants $b_0, \cdots, b_{n-1}$ there is exactly one function $y(t)$ that obeys the ODE (1) and the initial conditions

$$y(0) = b_0 \qquad y'(0) = b_1 \qquad \cdots \qquad y^{(n-1)}(0) = b_{n-1}$$

Here is my summary of the points made in this theorem:

(1) Lu=0 has exactly n independent solutions, even if L is not symmetric
(2) Assume you can find A solution of Lu=f. Call this solution "a particular solution". Any solution!
(3) Then the most general solution of Lu=f is particular + homo solutions with N constants.
(4) if you impose the "initial-value" BC set, the solution to Lu=f exists and is unique.

Notice that there cannot somehow be, say, *two* particular solutions and you then write

$$u = u_{1,\text{particular}} + K\, u_{2,\text{particular}} + \Sigma_{i=1}^{n} C_i \varphi_i$$

in which case we have n+1 free constants. You would find that the n+2 solutions shown here are not independent.

So the above theorem which Stakgold fails to illuminate clearly allows the following solution of an ODE. First, construct ANY solution of Lu=f by any means at your disposal. Then add to it the homo solutions and you have the most general solution.

Solving L "t" = " δ_ξ"

So, with the above examples as warmup, we now consider L "t" = " δ_ξ", a bit like example 3. This is the Big Deal Green's-defining equation on which we need to focus our attention. If we can solve it, then we can also solve L "t" = " f ". We have here a differential equation in distributions! In terms of symbolic functions we can write L t(x) = $\delta(x-\xi)$.

On pages 54-55 Stakgold takes a fascinating approach to constructing the solution t. He assumes a certain form is possible, and then he proves that this form is in fact a solution. Then as outlined in

preliminaries above, he adds the n homo solutions to get a most general solution. So the above theorem seems to "carry through" into the situation where some of your functions are symbolic functions.

He works at general order n throughout. The form he assumes is this: (1) put u(x) to the left of x = ξ, where Lu = 0 so u is one of the n independent homo solutions; (2) put homo solution v(x) to the right of x=ξ. So we are saying t(x) = { u(x), v(x) } broken at the point x=ξ. (3) Now assume (just a guess) that the solution t(x) and all its derivatives are continuous at x=ξ up to but not including the n-1st derivative. So we assume that $t^{(n-1)}(x) = \{ u^{(n-1)}(x), v^{(n-1)}(x) \}$ has a "jump" at x= ξ. [In our string example, we have n=2 and we have a "jump" in the first derivative at x=ξ where the string takes a bend.] . (4) If we now integrate both sides of L t(x) = δ(x-ξ) over the point x=ξ, we get 1 on the RHS, and we pick up a contribution only from the "jump" on the LHS and we find that we must have

$$t^{(n-1)}(\xi+\varepsilon) - t^{(n-1)}(\xi-\varepsilon) = v^{(n-1)}(\xi) - u^{(n-1)}(\xi) = 1/a_0(\xi)$$

So assuming we picked u(x) blindly on the LHS, we have these n conditions on v(x):

$$\begin{aligned} v(\xi) &= u(\xi) \\ v^{(1)}(\xi) &= u^{(1)}(\xi) \\ v^{(2)}(\xi) &= u^{(2)}(\xi) \\ &.... \\ v^{(n-2)}(\xi) &= u^{(n-2)}(\xi) \\ v^{(n-1)}(\xi) &= u^{(n-1)}(\xi) + 1/a_0(\xi) \end{aligned}$$

We can then think of v(ξ) as the solution to the problem Lv = 0 with the above list as forming an initial-value BC set at the point x = ξ, and we know from Preliminaries above that there is a unique solution v(x). Thus, for any given u(x) and for any point ξ, there is a unique solution v(x). Since v(x) is different for different values of ξ, Stakgold calls it $v_\xi(x)$.

So, with the (non-justified) assumption on the general form of t(x), Stakgold has come up with what appears to be a particular solution. But maybe our assumption was no good, and then this particular solution is not valid. To remove this possibility, Stak then shows that the particular solution obtained here, $t(x) = \{ u(x), v_\xi(x) \}$, does in fact solve L t(x) = δ(x-ξ), or rather $(Lt,\varphi) = (\delta_\xi,\varphi)$. We know of course that $(\delta_\xi,\varphi) = \varphi(\xi)$, and we know that $(Lt,\varphi) = (t,L^*\varphi)$. So Stak needs to show $(t,L^*\varphi) = \varphi(\xi)$. If he can show this, then he has shown that our particular solution solves our distributional ODE L "t" = " δ_ξ".

I discuss this proof in distributions.doc, and outline those notes in the raw notes, which outline I replicate here, items (2) and (3):

(2) I then discuss the proof Stak is doing on page 55. He wants to prove that the candidate solution shown in 1.40 does in fact solve Lt = δ(x-ξ) which means $(Lt,\varphi) = \varphi_\xi(0) = \varphi(\xi)$. He then carries out the proof. The first step in the proof is result 5 above so we have $(Lt,\varphi) = (t, L^*\varphi)$. He then writes this as two separate integrals in order to avoid the problem point ξ, and he properly uses u(x) on the left and $v_\xi(x)$ on the right. This is shown in page 55 B and in my notes. He then starts "swinging" the D operators from right to left. I show that, if we ignore all parts contributions, he ends up with $(Lt,\varphi) = 0$ which is the wrong answer. There is in fact one place where you get parts contributions, and I show in my notes that that parts contribution is exactly φ(ξ) and thus our proof is complete. The one place parts contributes is

where the ξ parts contribution in the two terms does not cancel, and that happens when you have $D^{n-1}u$ sitting in the parts expressions, because this parts term then gives ($D^{n-1}v_\xi$ – $D^{n-1}u$) which, instead of being zero, is given as shown in p 55A because this function has a non-zero "jump".

(3) The most general solution to our ODE Lt = δ(x-ξ) is our Green's Like solution 1.41 + the n homo solutions each with an undetermined constant. Our ODE "system" is then completed by specifying n BC's which determine those n constants (assuming that the BC's allow you to find those n constants). We arrive then at this important conclusion: for n reasonable BC's, there is exactly one solution to Lt = δ(x-ξ) and it can be written as 1.40 + n properly constanted homo solutions. But, there are n ways to write this unique solution because the function u(x) in 1.40 can be any one of the n homo solutions. It is possible that one of these n ways will result in all the n homo constants being 0. However you write it, the thing you get which matches all the n BC's is called "the Green's Function" for the ODE system including the BC's, and it is unique. [We have not yet written this thing as g(x|ξ) nor have we yet claimed that it is symmetric.] Notice that this Green's Function concept is general to our n^{th} order ODE, though we will probably only use if in the case n = 2 since that includes most problems of interest.

Definition: Any solution of Lt(x) = δ(x-ξ) is called a "**fundamental solution** with pole at ξ ".

Definition: If you add n BC's, this fundamental solution is unique, and is called a **Green's Function**.

Example 1: L = $-D^2$, n = 2: he writes two different Green's solutions, then shows they are related by a solution of Lu=0.

The causal fundamental solution. (Example 2) In the above, we said u(x) was one of the n independent solutions of Lu=0. However, it is also possible to take u(x) = 0 which solves Lu=0. Then t(x) = { 0, $v_\xi(x)$ } and we then have this list of conditions on the function $v_\xi(x)$

$v(\xi) = 0$
$v^{(1)}(\xi) = 0$
$v^{(2)}(\xi) = 0$
....
$v^{(n-2)}(\xi) = 0$
$v^{(n-1)}(\xi) = 1/a_0(\xi)$

For our string problem, this solution would look like  which could be a solution if u(a)=0 and u(b) = 5, but cannot be a solution with u(b) = 0.

With the exact same list of conditions above, another causal solution is t(x) = {$-v_\xi(x)$, 0 } where we now put the horizontal segment on the right instead of the left.

1.4 Preliminary Results on 2nd order linear ODEs (58)

The first subject here is a theorem and some obvious corollaries.

The Initial-value BC Theorem; Wronskians; Abel's Formula; the W=0 Theorem.

Theorem 1 (**initial value theorem**). Lu=f has exactly one solution for "initial value" BC's
Corollary: If the initial value BC's are all 0, then u = 0 is that one solution
Corollary: If you scale all the BC's by γ, new solution is γu.

The order-n **Wronskian** is the det of a matrix whose first row is $f_1....f_n$ (any set of functions) and whose remaining n-1 rows are the incrementing derivatives of these functions, p 59. Unrelated to Jacobian.

Abel's formula says $W(u,v:x) = C\ e^{-m(x)}$ where $m(x)\ m'(x) = a_1(x)/a_o(x)$ and u,v are two solutions of Lu=0. From this we know that:

Theorem 2: if $W(u,v:x=x_0) = 0$, then W(u,v:x) = 0 everywhere in the interval.

Theorem 3: solutions u and v are *dependent* $\Leftrightarrow$ $W(u,v:x_0) = 0$ for some point x_0 in interval
$\Leftrightarrow$ W(u,v:x) = 0 for all x in interval (Thm 2)

The Homogeneous Differential Equation Lu=0 (61)

Proof of the n independent solutions theorem ? Above I quoted a theorem in italics. I think here on page 61 is the closest that Stak comes to providing a proof. He does it for n = 2, but I see how it might go for general n. *However*, it depends critically on Theorem 1 above which he did not prove, so I guess we don't really have a proof yet.

The n=2 fundamental solution to Lu = δ (62)

Here on p 62 we just review what we already know for general n. This is a typical Stakgold repetition which is always a little confusing to the reader but OK. Once you pick your u(x), the other half function $v_\xi(x)$ is fully determined. This is so based on the initial-value conditions on v(x) at x = ξ. However, notice that we have not yet talked about overall BC's on Lu = δ .

The ∞ issue and the Volterra-form solution to Lu=f.

We are on page 63 and the logic is very tangled, but I think I now see the forest for the trees. Our usual Lz=f solution is $z(x) = \int_a^b d\xi\ g(x|\xi)f(\xi)$ + homo solutions, where we construct g from u(x) and $v_\xi(x)$ which are both solutions of Lu=0. Recall that each choice of u(x) results in a different form for g(x|ξ), but any form will work in our formula for z(x). Recall that the "causal" solution had u(x) = 0 and was as good as any other solution. Our *concern* is that if (a,b) = (-∞,∞), the integral might diverge. On the bottom of page 63 the *upshot* is that, *in this case*, we can write the solution as $z(x) = \int_\alpha^x d\xi\ v_\xi(x)f(\xi)$, where $v_\xi(x)$ satisfies the usual BC's of our "causal" fundamental solution which are these: $v_\xi(\xi) = 0$ and $v_\xi'(\xi) = 1/a_0(\xi)$. The constant α here means that z(x) satisfies the homo initial-value BC's at x = α (obvious that z(α) = 0, but also true that z'(α) = 0). So you can pick the constant point x=α as you see fit. Due to the Volterra type upper endpoint x, and picking a finite α, we have avoided any possible divergence of the integral, which alleviates the concern just mentioned.

In the derivation of this form of z(x), Stak refers to α as "a" which confused me on the (a,b) issue. He should have used some other symbol like α.

So here is how Stak comes up with the Volterra solution form. He writes the two obvious u=0 "causal" solutions and calls them t_1 and t_2. He picks a constant α (his a). For $x > \alpha$ he uses the t_1 form for g. For $x < \alpha$ he uses the t_2 form for g. In each case, the solution z(x) is the identical Volterra form $z(x) = \int_{\alpha}^{x} d\xi\ v_\xi(x) f(\xi)$ which of course does not mention t_1 or t_2. So why not use this solution for x anywhere in the range (-∞,∞)! For any finite x and α, there are no ∞ divergence issues. I now quote from the raw notes, using $a = \alpha$:

(4) Therefore, we may conclude that $z(x) = \int_{a}^{x} d\xi\ v_\xi(x) f(\xi)$ is a solution to Lz = f on the interval (-∞,∞) with the boundary conditions z(a) = 0 and z'(a) = 0, where $v_\xi(x)$ is the solution of Lv = 0 with the (different) boundary conditions that $v_\xi(\xi) = 0$ and $v_\xi'(\xi) = 1/a_o(\xi)$. There is only one such solution $v_\xi(x)$ to Lv = 0 with those BC's. We have found a particular solution of Lz = f. We know there are no solutions to Lz = 0 with BC's z(a) = 0 and z'(a) = 0 (our corollary earlier, ie, only solution is $z \equiv 0$), so there are no homo solutions to add, so our z(x) is then THE ONLY solution of Lz = f. This is the **Theorem p 63**.

1.5 Boundary Value Problems (64)

We are now going to formally add "boundary conditions" (BC's) to the soup, still n = 2. These then turn our ODE problems into "boundary value problems", the title of this book and section. For each of our two BC's we are going to say that a certain linear combination of u(a), u'(a), u(b),u'(b) = 0. I suppose we could have picked some other points inside the interval (a,b), but we don't. These two linear combinations are called $B_{1,2}(u)$ and the two equations $B_{1,2}(u) = 0$ are called "**homogeneous BC's**". Later we will set these to constants so $B_{1,2}(u) = \alpha,\beta$ and these would be "**inhomogeneous BC's**". Notice that we use the descriptors homo and inhomo *independently* to describe the BC's and to distinguish Lu=0 and Lu=f. This is a bit confusing. The homo word always means " = 0 ".

There are **two special cases** always of interest: (1) the **initial-value** case, where for example we only retain in $B_{1,2}$ the terms referring to a and not b (or the reverse, or even out in the middle at c). (2) the **unmixed** case where B_1 has terms only for a, and B_2 has terms only for b.

Fact: An unmixed BC has the form Au(a) + Bu'(a) = 0. Notice that if you go with the initial-value conditions at endpoint a, which is to say u(a) = u'(a) = 0, then you have satisfied the unmixed condition.

<u>definitions</u>:

- Complete Homo: Lu = 0 with $B_1(u) = 0$ and $B_2(u) = 0$ 1.54
- Complete Inhomo: Lu = f with $B_1(u) = \alpha$ and $B_2(u) = \beta$
- Green's Problem and Function: Lg = δ with $B_1(u) = 0$ and $B_2(u) = 0$ 1.55

Theorem: Suppose u_1 and u_2 are solutions of Lu = f with inhomo BC's. $B_i(u) = k_i$. Then the difference function $w = u_1 - u_2$ must be a solution of Lu=0 with $B_i(w) = 0$. That is to say, the difference between any two solutions of the complete inhomo problem is a solution of the complete homo problem. (the proof is obvious in 10 seconds).

Corollary: If the complete homo has no solutions, there is at most one solution to complete inhomo.

Fact: If complete homo Lu=0 has solutions, then operator L has a nullspace of some dimension N>0, and we know this will restrict the range of Lu=f (from our knowledge of later chapters in Stakgold). If f is not in the range, there will be *no* solutions to Lu=f. If f is in the range, there will be N+1 solutions given by a particular solution + the nullspace solutions. So "none or many" if N > 0.

Fact: Let $Lu_1=f_1$ and $Lu_2=f_2$. If $f = c_1f_1 + c_2f_2$, solution of Lu=f is $u = c_1u_1 + c_2u_2$.

Big Theorem: If complete homo has no solutions, then Lg = δ with $B_i(g) = 0$ has a unique solution. Notice that we are talking homo BC's here. This solution is called The Green's Function.

[In our string problem, complete homo $D^2u=0$ with pure u(a) = 0 and u(b) = 0 had no solutions.]

Stakgold proves this theorem first in two special cases, then claims it in general. He spends about 3 full pages on this, so I will at least comment. In each case, we already know the solution is unique due to the Corollary above, so all he has to do is construct a solution for g, then we know the solution *exists*.

1. The Case of Unmixed (Pure) BC's on g (p 66). His first step is to find two independent solutions to Lu=0 where u_1 only satisfies the a-end BC, and u_2 only satisfies the b-end BC. These exist due to the initial-value BC existence theorem stated earlier. He then shows that, if complete homo has no solutions, u_1 and u_2 are independent. Then using these two functions, in 66B he constructs $g(x|\xi)$ with unknown constants A,B. He applies the usual conditions g must have at $x=\xi$ and solves for A and B, and in this way he has constructed a $g(x|\xi)$ and therefore a solution must exist! Because u_1 and u_2 are independent, he knows that the denominators of A and B (Wronskian W) cannot be 0, so A and B are finite.

2. The Case of Initial-Value BC's on g (p 67). Here, he states the problem in 1.57, then proposes a "causal" type solution for g where now the $v_\xi(x)$ function is $A\,u_1(x) + Bu_2(x)$ where he implicitly claims he can find independent $u_i(x)$ for this problem as he did for Case 1 (but this is not shown). He then solves for A and B in similar fashion to Case 1 and the result for g is in 1.58 which really is just our causal solution. Again, the independence of the u_i makes A and B finite, and he has constructed a g.

3. The Case of arbitrary BC's. Stakgold does not handle this case! At the top of page 67 the heading suggests that he is going to do this, but then he really only does Case 2 above. We are referred to Exercise 1.53 for hints on doing Case 3, and I have some comments in the raw notes on this.

We now resume on page 67 at the bottom pencil line. **We continue to assume** that complete homo has no solutions, as assumed in Big Theorem above. Thus, from that theorem we know some $g(x|\xi)$ exists which satisfies Lg = δ with $B_i(g) = 0$. That is to say, regardless of the nature of the B_i -- fitting into one or more of our Cases above -- we know $g(x|\xi)$ exists and is unique. It is this $g(x|\xi)$ used in the following.

We then consider two general problems:

(a) Inhomo ODE with *homo* BC's: Lu=f $B_i = 0$

The solution to this problem is just the straight integral (1.61) that we always use, because there are no homo solutions.

(b) Inhomo ODE with *inhomo* BC's: $Lu=f$ $B_i = \alpha,\beta$

We try here a solution of the form $u(x) = \int g(x|\xi)\, f(\xi) + c_1u_1 + c_2u_2$. We have u_1 be a solution of $Lu=0$ with $B_1(u_1) = 0$ ONLY and u_2 be a solution of $Lu=0$ with $B_2(u_2) =0$ ONLY. You can then solve for the two constants c_1 and c_2. The result is 1.63 where α and β are the two inhomo BC numbers. Notice that the u_i are not solutions of complete homo (by assumption, there are none). They are special solutions of $Lu=0$ with oddball BC's. Also, since there are no complete homo solutions, we know that the two denominators in 1.63 (like $B_2(u_1)$) cannot be zero.

Adjoint, Symmetric, and Self-Adjoint Systems (69)

We continue in the context of n=2 ODE systems. The **adjoint L*** to operator L is defined in the usual way (parts) and I write out L and L* bottom of page 69. If $a_o' = a_1$, then L = L* and you can write L in the form $L = \partial_x(a_o\partial_x) + a_2$ which we will see a lot in Chap 4.

Now at this point we have in 1.65 that $\langle v,Lu\rangle = \langle L^*v,u\rangle + J(u,v)|_a^b$ where the J thing is the parts of the two parts integrations we did. Obviously we want this J thing to be 0 so we have our usual meaning for the adjoint of an operator. The domain D of L is restricted by BC's $B_i(u) = 0$. The domain of operator L* is defined such that $J(u,v)|_a^b = 0$ when $u \in D$, and this domain for L* is called D*. On the other hand, we presume that the domain of L* is controlled by some "adjoint BC's " we write as $B_i^*(v) = 0$ [* does not mean cc or any other operation on B_i.] . Here is the main point: we must have $J(u,v)|_a^b$ so we can say $\langle v,Lu\rangle = \langle L^*v,u\rangle$, although Stak does not stress this point. We know that $B_i(u) = 0$ defines D and we must have $u \in D$. Then the requirement that the J thing vanish produces for you a set of new "adjoint" BC's $B_i^*(v)= 0$ which define D* and then you need $v \in D^*$.

It turns out that, in the general case, it is a little hard to write out the $B_i^*(v) = 0$ if you are given the $B_i(u) = 0$. I did some raw notes work on this and proved to myself that the $B_i(u) = 0$ do in fact completely determine that $B_i^*(v)= 0$ which have the same "functional form" of a homo BC. In the language of Chapter 2. when you write $\langle v,Lu\rangle = \langle L^*v,u\rangle$, u and v are "admissible pairs".

For certain explicit $B_i(u) = 0$, it may turn out that $B_i^*(v)= 0$ are exactly the same equations as $B_i(u) = 0$. In this case we obviously have D = D*. If we have L=L* *and* D=D*, then the system is fully self-adjoint which Stak calls simply **self-adjoint**. If you only have L=L* and $D \neq D^*$, then L is **formally self-adjoint.**

For example, if the $B_i(u) = 0$ are unmixed BC's, then D=D* !

In a moment I will summarize my own raw notes regarding $B_i^*(v) = 0$, but first in order to stay in the Stakgold order of flow, we have some other facts to state:

(1) You can consider the problem $L^*h = \delta$ with $B_i^*(u) = 0$ as defining a certain "adjoint Green's" h. For a self-adjoint system, h = g, the regular Greens, that is, $h(x|\xi) = g(x|\xi)$. On the other hand, it is easy to show that $h(x|\xi) = g(\xi|x)$ in general. Here is a simple proof with matrix notation:

Lg=1 L*h=1

$0 = J\text{ thing} = \langle h, Lg\rangle - \langle L^*h, g\rangle = h^T.Lg - (L^*h)^T.g = h^T - g \quad \Rightarrow \quad g = h^T$

Therefore, we have this big result;

Fact: if the L system is fully self-adjoint, the Green's function is symmetric.

Fact: If L=L* formally, then Abel tells us $a_o(x)W(u,v;x) = C$ (easy, p 72 1.76)

The above fact is used a lot in proving things. And as already quoted above,

Fact: If $B_i(u)= 0$ are *unmixed*, then D=D* and any L=L* then becomes fully self-adjoint.

This is shown by me as a Theorem 1 in the raw notes, and it is easy to prove, 1 page or so of algebra.

[In contrast, as reported below, if $B_i(u)= 0$ are *initial-value* form, L can *never* be self-adjoint.]

Raw Notes General Case. I start with the general formula for J (not assuming L=L*), and the general formulas for the $B_i(u) = 0$. I then set $J(u,v)|_a^b = 0$ and try to find the resulting $B_i^*(v) = 0$. I do find these equations, and the result is my Theorem 2. For example, I define things like this:

$B_1(u) = Au(a) + A'u'(a) + Bu(b) + B'u'(b) = 0 \qquad B_1(u) = 0$
$B_2(u) = \alpha u(a) + \alpha' u'(a) + \beta u(b) + \beta' u'(b) = 0 \qquad B_2(u) = 0$

$$\begin{pmatrix} W & Y \\ w & y \end{pmatrix} \equiv - \left[\begin{pmatrix} A' & B' \\ \alpha' & \beta' \end{pmatrix} \right]^{-1} \begin{pmatrix} A & B \\ \alpha & \beta \end{pmatrix}$$

and my results are these

$[\ a_3(a) + W\, a_o(a)]\, v(a) - [a_o(a)]\, v'(a) - [w\, a_o(b)]\, v(b) = 0 \qquad B_1^*(v) = 0$
$[\ a_3(b) - y\, a_o(a)]\, v(b) - [a_o(b)]\, v'(b) - [Y\, a_o(a)]\, v(a) = 0 \qquad B_2^*(v) = 0$

which for L=L* simplify a bit.

$[\ W\, a_o(a)]\, v(a) - [a_o(a)]\, v'(a) - [w\, a_o(b)]\, v(b) = 0 \qquad B_1^*(v) = 0$
$[\ y\, a_o(a)]\, v(b) - [a_o(b)]\, v'(b) - [Y\, a_o(a)]\, v(a) = 0 \qquad B_2^*(v) = 0$

It seems odd that the resulting equations involve only 3 of the 4 "values" in each equation, whereas the original equations involve all four values in each equation. On the other hand, suppose I were to solve the second original equation $B_2(u) = 0$ for u'(b). If I were to insert that u'(b) into $B_1(u)$, then $B_1(u)$ would show only 3 of the 4 values. So maybe all OK. Still, it seems a little asymmetric.

Examples (starting p 72)

Preliminary Review for Example 1: Back on page 66, we construct the exact Green's solution for an arbitrary unmixed BC situation and the result was 1.56 where u_1 and u_2 are Lu = 0 solutions with initial-value BC's at the left and right ends. [If you look at the general unmixed BC situation 1.52 p 64, you see that one possible solution at each end is the initial-value BC's at each end.] So if you can find u_1 and u_2 and the Wronskian W, then you have your Green's g, since $W(u_1,u_2; x)\ a_0(x)$ appears in the bottom of 1.56. But for L = L*, Abel says this denominator is just a constant C, so the symmetric g is $u_1(x)u_2(\xi)/C$.

Example 1: We consider the *generic L=L* unmixed BC* situation (fully self-adjoint) and (a) write out g as just stated above in terms of u_1 and u_2. If Lu=0 has no solutions, then (b) the solution to Lu=f is "the usual integral" u = Gf, and we can just insert our form for g to get the result shown.

Example 2: (72) $L = D^2$ with u(a)=0 and u'(b) = 0, has no Lu=0 solutions. Since unmixed, and L=L*, system is self-adjoint and g is therefore symmetric. We get $u_1 = Ax$ and $u_2 = B$. Match g so $A\xi=B$. Match g-jump so that 0 - A = 1 so A = -1 and then $u_1= -x$ and $u_2 = -\xi$ and then Green's = $\{-x,-\xi\}$. There could hardly be a simpler example! Could compute $W(u_1,u_2; x)\ a_0(x)= C$ and you find C = -1.

Theorem Review: I show in the raw notes these two facts:

Theorem: Suppose we have initial-value BC's u(a) = u'(a) = 0 defining domain D. Then it is easy to show [by studying the J thing] that the equations defining D* are u(b) = u'(b) = 0; these are the $B_i^*(u) = 0$. But these BC's are NOT the same as $B_i(u) = 0$. That is u(a) = 0 differs from u(b) = 0, for example, since $a \neq b$. Therefore:

Fact: A system with initial-value BC's can never be self-adjoint since $D \neq D^*$, regardless of whether or not L=L* formally.

Example 3: (73) This long *generic initial-value BC example* has four things to show, which I just summarize here:
(a) a statement of the Fact just quoted above: such a system is never self-adjoint
(b) the solution to Lu=f with homo initial-value BC's can be written in my Volterra form of page 73B
(c) the solution to Lu=f with *inhomo* initial-value BC's (α,β) is stated in p 73 C with special u_1, u_2
(d) if L=L* in addition to (c). We then have a form for $v_\xi(x)$ so solution is p74 (1.79)

This nice **Appendix A.3** derives the general **heat flow equation** $\partial u/\partial t - 1/c\nabla\bullet(k\nabla u) = f/c$ where k is thermal conductivity, c is heat capacity, f is heat source distribution, and u is temperature. For constant k and for static situation, we get $\nabla^2 u = (1/k)\ f$ which determines temperature distribution. I quote the raw notes:

"So here we have three classical physics equations that are really the same: diffusion, electrostatics and heat conduction. I think they are all based on the continuity equation. The functions you solve for in these three cases are: matter density, electrostatic potential, and temperature. The sources in the three cases are matter source, charge density, and heat source. Coefficients are diffusion coefficient, "unity", and thermal conductivity. "

Example 4: (74) He does static **temperature distribution for a thin rod**, so $\nabla\bullet(k\nabla u) = f$ which in 1D becomes $-\partial_x[\, k(x)\partial_x] u = f$ which we see has the L=L* property and $a_o = k$. For BC's, we say u(a) = 0 and u(b) = 0 [ends are at 0 temperature], which are unmixed BC's so our system is fully self-adjoint. This problem falls under Example 1 as generic unmixed. The u_1 and u_2 are integrals of k(x) so we don't have explicit forms for them, but in terms of them we can write the Green's as usual.

So physically we have a thin rod with temperature reservoirs contacting each end. And we assume no conduction off the sides of the rod, there is no "radiative loss" in this problem.

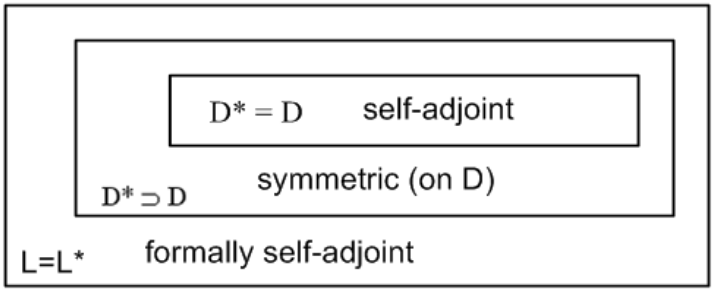
Suppose k = constant. Then the problem is just $D^2u = -f/k$ with u(a) = 0 and u(b) = 0. But this is just our string problem with T = k. For example, if you were to apply a uniform heat source f(x), the temperature would assume a symmetric quadratic distribution peaking at the center of the rod, and of course heat flows out through the ends of the rod at a rate equal to the rate the source supplies heat. Notice that in the heat static solution, something "flows", but in the string problem, nothing flows.

Example 5: (75) This example is the **Bessel ODE** with n=0 (order), so we get $L = \partial(x\partial) + x$ which is formally SA. We add u=0 at both ends (unmixed), so we are fully SA.

Appendix B.1 on p 329 discusses the solutions which are J_0 and N_0 with various details. Then **Appendix B.2** shows how you can easily compute various Wronskians as shown on page 331, all quite simple.

Stak writes $Z_{1,2}$ as obvious lincombs of J_0 and N_0 which meet the BC for left, right ends. Again we are in the Example 1 world so g is easy to write down as in p 75 C. Then we can compute the required Wronskian using the Appendix and we get C as shown in E.

Symmetric vs Self-Adjoint. This picture says it all:



In *both* inner rectangles we have $J(u,v)|_a^b = 0$ so that $\langle y,Ax\rangle^* = \langle x,Ay\rangle$ and A then has the matrix property that math people call symmetric: $A_{yx} = A_{yx}^*$ (Hermitian). However, only in the innermost rectangle do we have D* = D and then we are both symmetric and self-adjoint. In the intermediate rectangle we have $J(u,v)|_a^b = 0$ for all u and v in D.

Example 6: This gives a case where $D^* = R_L$ is larger than D, so we are symmetric but not SA. But I don't like the sample, so forget it.

Exercises (starting p 76)

Exercise 1.37. I did this in detail in the raw notes. Recall $B_i(u) = 0$ defines D, and $B_i^*(u) = 0$ defines D*. The **question** here is: if L=L*, what condition could put on the constants in the $B_i(u) = 0$ such that you end up with $B_i^*(u) = 0$ being the same BC's, so in fact L is fully self-adjoint, ie, D= D*. The answer is this:

$$a_0(b)\det(\alpha) = a_0(a)\det(\beta)$$

where

$$A = \begin{pmatrix} A & A' \\ \alpha & \alpha' \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \alpha \qquad B = \begin{pmatrix} B & B' \\ \beta & \beta' \end{pmatrix} = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \beta$$

Exercise 1.39. He notes that for real $a_i(x)$, you cannot have L=L* for n = **odd**, eg, L = D. The rest of the problem deals with L = D and also $\mathbf{L = D^4}$.

Exercise 1.40. Claims you **can always find** s(x) such that s(x)L = formally adjoint.

Exercise 1.50. A little **theorem** here that says: if Lu = f(x,θ) where θ is some parameter and where you have homo BC's, then if you want to solve $Lv = \partial_\theta f$ with the same BC's, the solution is $v = \partial_\theta u$. You could get this by starting with Lu(x,θ) = f(x,θ) and apply ∂_θ to both sides. This result is used in several of the later exercises.

Exercise 1.51. This is a **beam deflection problem** with its n=4 order ODE discussed in **beam.doc**. The beam end treatments are the BC's, and you could in theory find g . Recall this world deals with shear force V(x) and bending moment M(x). A delta M source is the same as a dipole V source by 1.50.

Exercise 1.52. When x = time t, people talk about g = **impulse response**, and z = **step response**.

1.6 Alternative Theorems and Modified Green's Functions (79)

Alternative Theorems

We are first given just the briefest possible review of rank and nullity in the **linear algebra world**. This will of course be a major subject in chapter 2. We are given Au = 0, Ay = f and A*v = 0 as equations 1,2,3 and some shades of an **Alternative Theorem**, but never really stated clearly, though I get the point.

Then we wander off to the **integral equation world** and look at (λK-1)u = 0 and (λK*-1)v = 0 as a regular and an adjoint homo equation , then (λK-1)u = f as the inhomo. We define operator B = (λK-1). Then we have simply Bu=0, Bu = f and B*v=0 as door number 1, 2 and 3. If 1 has only trivial so does 3 and then 2 always has a unique solution. If 1 has non-trivials, so does 3, and 2 has solution only if f is orthogonal to the v of 3. I would add that there are then many solutions u since you can add homos of 1. So we get a statement of the **Alternative Theorem** in this world.

Then we get on to the main act which is **the ODE world**.

We open by studying our **string in motion**. We assume the usual $e^{-i\omega t}$ time dependence for the driving term and F(x) for the driving spatial part (string pressure in this case), and this results in a spatial ODE LX(x) = F(x) where $L = -D^2 - \omega^2$ and where we assume $u(x,t) = X(x)\, e^{-i\omega t}$. If ω is not an

eigenvalue, we know that Lu=0 has no solutions and LX = F has a unique solution. But for any eigenvalue ω_k and eigenfunction $Z_k(x)$, we DO have a solution of LZ=0 so we have a 1D nullspace of L. In order that LX=F have a solution for $\omega = \omega_k$, F must be orthogonal to Z_k -- the consistency condition as shown in p82 (1.86) That is, we must have $F_k = 0$ so no action in this mode. If $F_k \neq 0$, there is no solution for X(x). Here we were trying out a solution $u(x,t) = X(x)\, e^{-i\omega t}$ and we find that if $F_k \neq 0$ there is no solution of this assumed form. We will soon find a solution of a different form.

Our sample problem here is self-adjoint since L = L* and we have unmixed BCs.

The explicit sine $Z_k(x)$ are stated in page 83 C. Then in mid page 83 we have an ODE version of our **Alternative Theorem specific to our problem** which I have already stated in the last paragraph.

An interesting aspect of this example is that our operator L has a "parameter" ω, and the nature of the alternative theorem's claim is dependent on whether ω is or is not an eigenvalue. If it is an eigenvalue, and if $F_k = (F,Z_k) \neq 0$, then there is no solution of the form $u(x,t) = X(x)\, e^{-i\omega t}$, but there is a solution of another form which now deduce.

Solving the driven string problem. This begins on page 83 and we assume driving $F(x)e^{-i\omega t}$. We have here a partial differential equation in x and t for solution function u(x,t). Stakgold takes the x-part of the problem and does a Fourier series sine-only transform to replace u(x,t) by $a_k(t)$, with the two transforms shown bottom of page 83. We end up then with an ODE in the t variable for $a_k(t)$ as in page 84 C which is an n=2 ODE with initial-value BC's. Stak then uses the "equation 1.79 method" which we already invested in to solve this equation to get D. We do the integrals to get E.

What does E tell us? It is written for ω not an eigenvalue. It says that each "mode" k runs at its natural frequency with some definite amplitude and phase (the last two terms), and on top of this we have some sine type motion at the driving frequency ω. The amplitudes and phases are determined by how far away ω is from ω_k of the mode.

If we let $\omega \to \omega_k$ in E, we get G which says that we get some sin(ωt) action but it gets added to action which is linear in t and thus "blows up" linearly over time. However, if $F_k = 0$ we there is no coupling into this mode, then we are OK and this particular $a_k(t)$ is then = 0.

So in either case we have our $a_k(t)$ from E or G and we stuff this back into (1.88) which adds the spatial part and we have our solution u(x,t) for string motion (not of the form $X(x)\, e^{-i\omega t}$ for $\omega = \omega_k$).

A subtle point quietly ignored by Stakgold is the BC's for the $a_k(t)$ ODE. These imply that our string was at rest for t<0 and is suddenly jerked into action at t=0 by the driving term, and this is why we have those defined-amplitude homogeneous terms in E. In a practical problem with damping, these terms would damp away and we would have only the $e^{-i\omega t}$ behavior of the solution, off resonance.

Stakgold fails to *stress* the notion of "diagonalization" here, where we can reduce the problem to just studying the "modes" which don't interact with each other and we just add the action in the modes to get a linear combination solution as in p 83 1.88.

Alternative Theorem Fourth time. Here on pages 84-85 he states it for our general n=2 ODE with operator L and BC's $B_i(u) = 0$ (and * for the adjoint). As before, we have door number 1,2 and 3.

Example 1. This is the **insulated rod heat problem** with $L = -D^2$ and $u'(0) = u'(\ell) = 0$. The homo Lu=0 *has* a non-trivial solution u = K. The consistency condition for Lu=f is $\int dx\, f(x) = 0$ and a physical

interpretation is given for this fact. For a certain specific f(x), he solves Lu=f and the solution is a particular one plus a constant, that being the homo add-on solution. I did a Maple plot.

This is exactly the problem which, we shall see, has an AWOL Green's function!

Example 2: This is an Lu = f problem where L has constant coefficients and is not self-adjoint. We find that Lu=0 has a single non-trivial solution, so we have a consistency condition and then particular + homo as the solution. The problem is not fully solved. The adjoint homo has also one solution, but it is not the same as the regular homo solution, since $L \neq L^*$.

The Modified Green's Function

This section is considered only for self-adjoint systems.

Whenever Lu=0 has non-trivial solutions (ie, a nullspace with n>0), the Green's equation $Lg = \delta$ has no solution! Why? If u_1 is a homo solution, we need $\int dx\ \delta(x-\xi)u_1(x) = u_1(\xi) = 0$ for all ξ, but that is a contradiction to saying u_1 is a non-trivial solution. Note that the existence of non-trivial solutions of Lu=0 is a function of both L *and* the boundary conditions used.

The Modified Green's Function is a "way out" of this problem. The book treatment assumes that L is fully self-adjoint. Instead of working with $Lg=\delta$, you work with

$$Lg_M = \delta(x-\xi) - \Sigma_{i=1}^{m}\ u_i(x)u_i(\xi) \qquad (*)$$

where the $u_i(x)$ are m orthonormal functions which span the nullspace of L. The RHS of the above equation then meets the required consistency conditions

$$\int dx\ [\ \delta(x-\xi) - \Sigma_{i=1}^{m}\ u_i(x)u_i(\xi)\ ]\ u_j(x) = 0 \qquad j = 1,2....m$$

and is thus in the perp space of L, so there can be a solution g_M which solves this equation and meets the BC's which I am not bothering to write.

You manually solve (*) for $g_M(x|\xi)$ and you will find there are m undetermined constants in line with the notion that solution = particular + linear combination of the m homo solutions. If you then impose these m conditions on g_M

$$\int dx\ g_M(x|\xi)\ u_i(x) = 0 \qquad i = 1,2..m$$

then this sets all m constants, and you find that the resulting g_M is symmetric (I proved this). Using this symmetric g_M, the solution to the problem Ly = f can then be written in normal Green's Function form, but it is the g_M that appears here instead of the usual g:

$$y(x) = \int d\xi\ g_M(x|\xi)\ f(\xi) + \Sigma_{i=1}^{m} K_i\ u_i(x) \qquad \text{// if we use the symmetric } g_M$$

which I also proved.

The text does a long but simple example. The ODE is just that for the string, but instead of the usual string BC's of being tied down at both ends, we have the derivative be 0 at both ends. This can be interpreted as the equation for heat flow in a simple thin insulated rod. The fact that there is no g jibes with the fact that there is no place for δ-injected heat to flow to get a static solution. Here is how the steps above are applied to this sample problem which has $u_1(x)$ = constant as the non-trivial Lu=0 solution:

$L = -D^2$ with unmixed, so self-adjoint // p 86 (1.89)

$u_1(x) = \sqrt{1/L}$ so that u_1 is normalized on (0,L) // p 87 A

$Lg_M = \delta(x-\xi) - 1/L$ // p 87 (1.91)

$\int dx\ f(x) = 0$ consistency // p 87 B

$\int dx\ g_M(x|\xi) = 0$ condition to make g_M be symmetric // p 88 A

$y(x) = \int d\xi\ g_M(x|\xi)\ f(\xi) + K$ // if we use the symmetric g_M // p 89 A

$g_M(x|\xi)$ is computed and shown explicitly in p 88 B (symmetric)

References for Chapter 1:

Classified into three areas: ODE, Green's and Distributions.