Meta Meta Review of Chapter 2 PhL date uncertain

This meta meta was originally at the front of the meta notes, but on 3.14.11 I put the meta meta notes into a separate doc (ie,. this doc) since that is done on all the other chapters. About 4.5 pages here.

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2.1 Functions and Transformations (Operators). (pp 92-96)

General notion of a **mapping** or "**transformation**" A:X→Y where X,Y are **sets** or **spaces**, but **domain** and **range** might be subsets of X and Y. If **1-to-1,** can talk about **inverse** A^{-1}. If spaces, can consider both **finite or infinite dimensional** spaces. A need not be linear.

2.2 Linear Spaces (= Vector Spaces). (pp 96-99)

Preview of **independent vs dependent** vectors, notions of **dimension** and of a **basis**.

2.3 Metric Spaces, Normed Linear Spaces, Inner Product Spaces (pp 99-116)

The **metric space** is discussed first, notion of **distance** and **metric** d(x,y) and **convergence** of a sequence of points in such a space which need not also be a vector space, ie, don't have to know what x+y or αx means to talk about **points** in a metric space. **Regular** and **Cauchy sequence** convergence ideas. Notion of space being **complete**, how to make it be complete. **Boundary** of an **open set**, notion of a set being **sequentially compact**. Notion of a set being **dense** in another set. Polynomials are dense in C(a,b) says the **Weierstrass Approximation Theorem**. Use overbar to show a completed or **closed set**.

The **normed vector space** adds a norm ||x|| to a vector space, totally unrelated to metric space a priori. Length of a vector need not be related to distance between two vectors. But if d(x,y) = ||x-y|| is added as the **natural metric** then a normed linear space (assuming complete) becomes a **Banach space**, 1932.

The **inner product space** is instead a vector space to which is added <x,y>. Usually one then uses the **natural norm** $||x||^2 = <x,x>$ and from that the **natural metric**, and the result is (if complete) a **Hilbert Space** (1900-1910). The famous **CSI** says $|<x,y>|^2 \leq <x,x><y,y>$. The prototype HS is R_n, and when acted on by linear operators, you have "linear algebra" aka **matrix algebra**. In **QM** the inner product order is reversed from this book. Graphic shows all the types of spaces and how related.

2.4 Properties of Separable Hilbert Spaces (HS). (pp 116-135)

Separable means a **countable spanning set** (like t^n), which is only a **basis** if the projected coefficients don't move as you improve a fit. In this case, they are generalized **Fourier Coefficients**. The **projection theorem** says a **subspace** (= **linear manifold**) can be treated as $M \oplus M^{\perp}$ and you can write any $x \in H$ as $x_1 + x_2$ etc. **Orthogonal** spanning set is a basis, many classical examples of such polynomials. The **Riesz-Fischer Theorem** (1907) makes a connection between convergence of a sequence of *vectors* in a HS and the convergence of a sequence of real *numbers*, but you have to do this with respect to a known basis in the space. The real numbers are the squares of the partial sums of the squared Fourier coefficients.

2.5 Functionals. (pp 135-139)

Notion of a **norm** ||T|| so T can be **bounded** over a set S. Opposite is **unbounded**. The definition of **continuity** of T in terms of sequences. Reminder of **uniform convergence** idea. Continuous at x=0 implies continuous in all of domain. Bounded = continuous for linear T. **Riesz Representation Theorem (1907)** makes isomorphism between bounded=continuous linear functionals T and elements t of H such that $T[x] = <x,t>$. [δ is not bounded.] The **nullspace** of a functional is defined, it has a **perp space**. Notion of **extending** or **restricting** T relative to its domain. All functionals linear in E^n are bounded and continuous. The **dual** or **reciprocal basis** idea.

2.6 Operators. (pp 139-146)

The **norm** ||A|| is introduced with various theorems which are just variations of the T theorems of the previous section. **Extension** and **restriction** as well. Notion of a **closable** and **closed operator** in terms of sequences. Close the domain as well. Theorems on closed operators. **Differentiation** is an unbounded but closed operator. Domain is **absolutely continuous** functions.

2.7 Operators in the Hilbert Space E_n(c) (pp 146-165) Finite Dimension Hilbert Spaces

This long section gets at basic operator ideas **but** within the limited confines of E_n . Later we come back and redo everything more generally to handle ∞ dim Hilbert Spaces. Operator here is a square **matrix**, either **regular or singular** based on det(A). Definition of the **adjoint operator** written as A*. **Self adjoint** means A = A* (same as complex **symmetric** or Hermitian). The **Alternative Theorem** says A has an inverse so Ax=f has a solution, or A has a nullspace and $R_A = (N_{A^*})^{\perp}$. So the **nullspaces** eats into the dimensionality of the range. Definition of **rank** (dim R_A) and **nullity** (dim N_A). Rule is that rank + nullity = n. Nullspace of $B = A-\lambda I$ is space of eigenvectors of A and eigenvalues come from detB = 0 which is the **secular equation**. Notions of **algebraic** k_i and **geometric** m_i **multiplicity** of an eigenvalue, and $m_i \leq k_i$. Every operator has at least one eigenvector. Every eigenvalue has at least one eigenvector which says $m_i \geq 1$. A full rank matrix A. If there are N eigenvectors all with different λ_i, they form a

basis. A **symmetric operator** is an important special case: eigenvalues are real, eigenvectors of different λ_i are orthogonal and form a basis. No missing eigenvectors, $k_i = m_i$ in each eigenmanifold. Can write $H = M_1 \oplus M_2 \oplus M_3 \ldots \oplus M_k$. **Spectral theorem** for symmetric operator lays it all out. The **spectral resolution** of $A = \Sigma_i \lambda_i P_i$. Review notions of **norms** I call ||A|| and |||A|||. For a symmetric operator they are equal and are equal to $|\lambda_{max}|$.

2.8 The inverse of an linear operator (pp 165-180) Infinite Dimension Hilbert Spaces

This section generalizes the notions of the previous section to arbitrary (∞ dim) Hilbert Spaces. This entire chapter is only about *linear* operators, note from title. The title seems a bit of a misnomer, this subsection really is about redoing all our previous E_n theory for ∞ dim HS, but of course the inverse A^{-1} is always of major interest if we want to solve Ax = f. **Inverse** exists if there is no **nullspace** (meaning only the trivial nullspace). More detailed definition of A being **regular**: empty nullspace, range is entire HS, A^{-1} must be bounded = continuous. **Essentially regular** if you can extend the range to be all of H. The notion of **closed operators** is introduced, to eliminate the possibility of a finite range limit point for the image of a **null sequence** in the domain. I drew a second **Visio picture** showing how linear operators are classified in various ways. There are **four distinct cases of closed operators**. The **adjoint operator** is defined as A*, and the new version of the **Alternative Theorem** is stated. Adjoint might exist only for some set of **admissible pairs** { y, A*y=g}. Facts about adjoints are stated. We are always interested in A* and N_{A*} because Alternative Theorem relates this to R_A and hence to the solvability of Ax = f. A **self-adjoint operator** has more requirements than it did in the R_n world. A **symmetric operator** is a less restrictive thing, and this is my third Visio picture. **Theorems** about self-adjoint and symmetric operators. Then a huge **EXAMPLES** section, including the mother of all examples A = d/dt with various BC's.

2.9 The spectrum of an operator (pp 180-184)

We review the classification of closed operators: one regular and three singular cases. We apply this to the operator B = (A - λI) and are then able to classify the possible values of λ into **four bins**. If B is regular, the bin is called the **resolvent set**. The other three bins are **singular bins** their union is the **spectrum**, that is, the range of λ values for which B is singular. Those bins are **point spectrum** (for eigenvalues), the **continuous spectrum** for when B^{-1} is unbounded, and the **residual spectrum** where $R_B < H$ so you have some $(R_B)^{\perp}$ there whose dimension is called the **deficiency of λ**, a strange name. Theorems on spectra are given. If A is symmetric, there is no residual spectrum. The spectrum of a self-adjoint operator lies entirely on the real axis (and there is no residual spectrum). EXAMPLES are given.

2.10 Completely Continuous Operators. (pp 184-187)

Review of the notion of **compactness**, and I wrote a separate document on this. We know that compact $\Rightarrow$ bounded, but the reverse is not true, and the **example** is given of the φ_n orthonormal sequence. The idea is that whereas a bounded = continuous operator always maps bounded sets into bounded sets, a completely **continuous operator** always maps bounded sets into compact sets, and some web sites call these compact operators. Interestingly, the identity is not completely continuous. Various theorems are given .

2.11 Extremal Properties for Bounded Operators (pp 187 - 190)

This section contains 6 theorems which are mostly pretty simple. Here are some of them:

Theorem 1: If A is bounded and $D_{\mathbf{A}} = H$, then $\| A \| = \| A^* \|$.

Theorem 2: If A is bounded and $D_{\mathbf{A}} = H$, then $|||A||| \leq ||A||$, for all $x \neq 0$.

Theorem 3: If λ is in any of the three kinds of spectra for A, then $| \lambda | \leq \| A \|$.

Theorem 4: If A is symmetric, $|||A||| = ||A||$.

Theorem 6: If A is symmetric *and* completely continuous, then the largest eigenvalue is $\pm \| A \|$.

PL Appendix: the resolvent.

So in our notation, we usually write $B = (A - \lambda I)$. We might then consider an equation of the form

$Bx = f$ $\qquad$ $Ax - \lambda x = f$ $\qquad$ $f = -\lambda x + Ax$ // integral equation form

$Lu - \lambda u = f$ // differential equation form

Our *formal* solution then is

$$x = B^{-1} f = (A - \lambda I)^{-1} f$$

and so B^{-1} is called the **resolvent**. I suppose we have to regard this inverse operator as a power series of some sort in A, and that is probably the famous series approach one takes to Fredholm approximation. For the differential equation above we use the more familiar L in place of A.
