Stakgold Chapter 2 Meta Notes PhL 1.17.09

These meta notes are excellent, if I do say so myself -- during a review 3.14.11.

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Meta Review of Chapter 2.

2.1 Functions and Transformations (Operators). (pp 92-96)

The notion of **1 to 1** and "inverse exists", notion of **domain** and **range** each within some space. The **A:X→Y** notation is strangely not used, I like to use the word "**mapping**". In general, X and Y are some kind of **spaces** or perhaps just **sets**. Things are interesting in the case that X and Y are both ***vector* (linear) spaces** so you can add and scale items (algebraic structure), which you cannot do with just sets. Then the objects in the spaces are vectors and not just boring points. Even if X and Y are linear spaces, the operator A need not be linear. Range and/or domain **can be less** than their full respective spaces Y and X. The dimensionality of the spaces could be **finite or infinite**. Another interesting case is A:X→X. So the subject in general is extremely **broad**. This book will deal with a subset of the broader subject, but even this subset will be broad relative to the simple matrix theory one gets when X = Y = E_n. This simple case however does provide a set of concepts which are then adapted and tuned for the more general spaces.

2.2 Linear Spaces (= Vector Spaces). (pp 96-99)

We learn here about the notion of **independent** vs **dependent** vectors, the notion of the **dimension** of a space, the fact that in a finite-N dimensional space, any set of N independent vectors provides a **complete basis** on which any vector in the space may be uniquely expanded. Orthogonality is not required for such a basis. The notation e_k indicates unit vectors like e_1 = (1,0,0. .)

2.3 Metric Spaces, Normed Linear Spaces, Inner Product Spaces (pp 99-116)

Metric Spaces

A metric space is a animal distinct from a vector space. It contains points with a rule for how you indicate the distance between any two points. This rule **d(x,y) ≥ 0** is called **the metric** of the space. In a vector space alone, although you can do things like add vectors, you don't in general know anything about the distance between two vectors because "distance" is not defined. So generally people add a metric to a vector space to get a space then which is both a vector space and a metric space.

Once we are talking distance, we can talk about **convergence** of a **sequence** of points. We learn the difference between a **regular sequence** $d(x_n,x)\to 0$ and a **Cauchy sequence** $d(x_n,x_m)\to 0$ where the actual limit we want to call x might not exist in the space X. If the limits exist, or if we somehow add them into our space, the resulting metric space is said to be **complete**. Thus, in a complete metric space, every convergent Cauchy sequence converges in the regular sense as well. Cauchy's main work was 1820-1828. Reals complete the rationals. An interesting space for X and Y is a certain space of functions called $\mathcal{L}_2$(a,b) so we have then a mapping f: $\mathcal{L}_2$(a,b)→ $\mathcal{L}_2$(a,b). Another space is that of continuous functions C(a,b). The metric you use tells you the distance between two functions defined on the interval (a,b).

Later in this section we consider a region S in a metric space that might not contain its **boundary**, sort of like an **open set.** In this case, sequences might converge to a point on the boundary not in S. You can **close** the region by adding all these missing sequence limits, and the closed S is called $\overline{S}$. The entire metric space might be closed to start with, or you might make it be **closed**. Stakgold mentions the notion of a set being **sequentially compact** on page 114, but does not use the idea much (yet). Every sequence contains a convergent subsequence if space is compact in this sense.

A related metric space concept is that of X being **dense** in Y. The idea is that for any $y \in Y$, you can get arbitrarily close to y with some $x \in X$ in the sense $d(x,y) < \varepsilon$. If you close X to be $\overline{X}$, then X is dense in $\overline{X}$. The rationals are dense in the reals. The **Weierstrass Approximation Theorem** says that polynomials are dense in the space of all continuous functions on some closed finite interval.

Normed Linear Spaces and Banach Spaces

Here we add some kind of **"length of a vector"** concept to a vector space, written $||x||$ for $x \in X$ and this length is called a **norm**. With the norm defined, a vector space becomes a **normed linear space**. It does not necessarily have a metric unless you add and use one. So, if you then define and make use of $d(x,y) = ||x-y||$ as "**the natural metric**" in your space, the space is called a **Banach Space**, where it is assumed that the space is complete as discussed a few paragraphs above. Of course you might reject this natural metric and use some other metric, but then X is not called a Banach Space. Banach's work was around 1932, a full century after friend Cauchy. Again: a normed linear space is a vector space to which you add a norm for the vectors. If you then use the natural metric this norm implies, your space becomes the combination of a vector space, a normed linear space, and a metric space. If space is complete, it is then Banach.

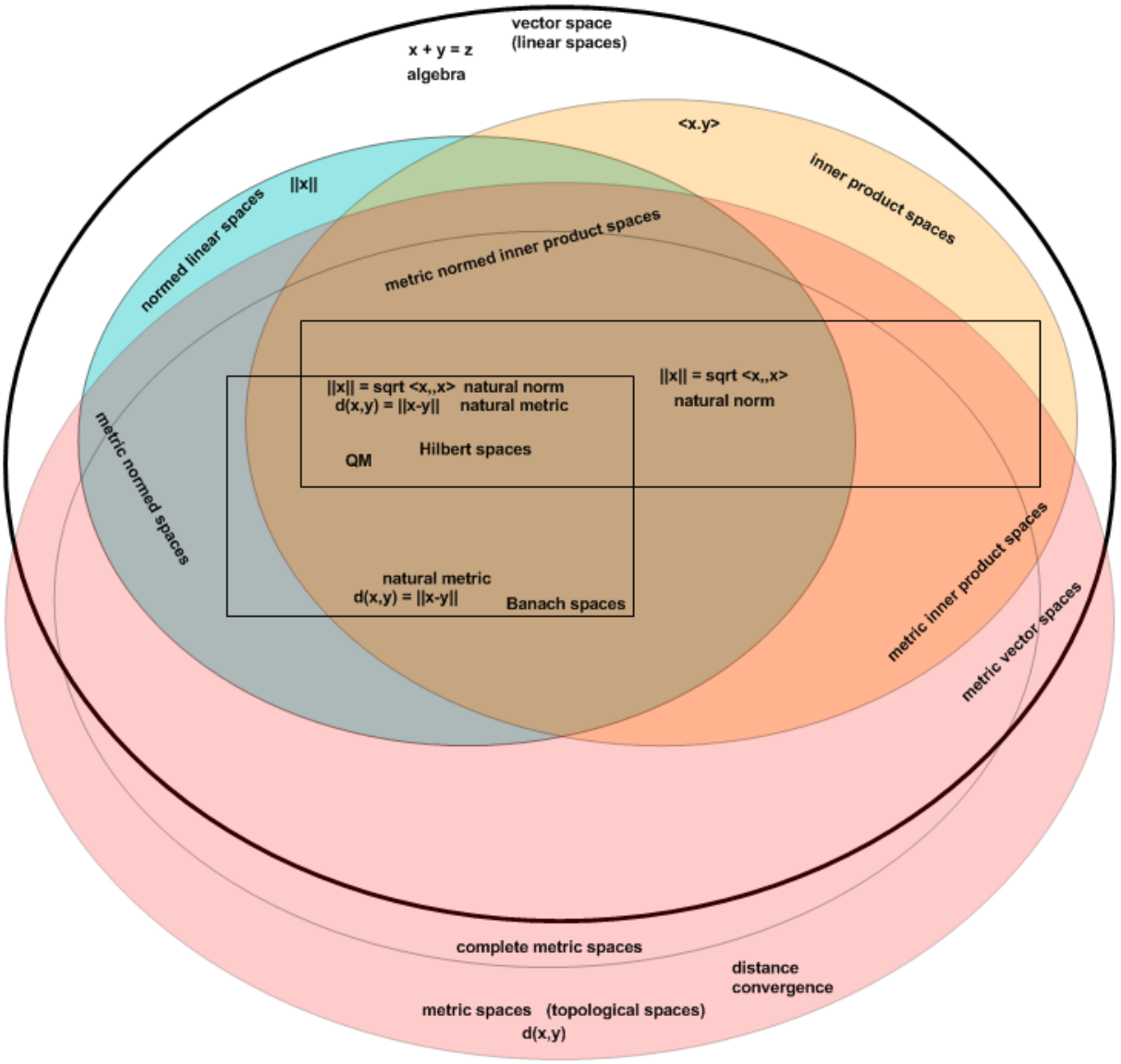
Inner Product Spaces and Hilbert Spaces

The metric is a mapping $d: (X,X) \rightarrow R$. An inner product is a similar mapping written $\langle x,y \rangle$, but we have (inner product): $(X,X) \rightarrow C$ (complex numbers). Whereas the norm is required to satisfy the triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$, the inner product has to satisfy some simple rules (addition and scaling etc) which lead to the **Cauchy-Schwarz Inequality** $|\langle x,y \rangle|^2 \leq \langle x,x \rangle \langle y,y \rangle$ where the thing on the right is a product, not a sum as with the metric. (CSI 1821, then 1888). An inner product can only exist inside a vector space since addition and scaling of vectors is required. A vector space with an inner product is called an **inner product space**. You could choose to use the natural norm $||x|| = \langle x,x \rangle^{1/2}$ to cause the inner product space to also be a normed linear space. Then you could use the natural metric $d(x,y) = ||x-y||$ to make this space also be a metric space. This combined X = vector space / inner product space / normed linear space / metric space object is called a **Hilbert Space** (1900-1910), provided the space is complete. You can always complete X (make it closed) by adding missing points.

There is a very famous **prototype** Hilbert Space we call R_n which is a generalization of R_3 , the 3D space in which we live. You can extend R_n to allow for complex components on the axes, then you have $E_n^{(c)}$. These Hilbert Spaces can be exactly modeled in terms of **matrix algebra** where a vector is a column of components, and an operator is a square matrix. This prototype Hilbert Space is usually referred to as Euclidian Space. Again, normally the components of a vector on each axis of Euclidean Space are real numbers as in our physical 3D space which Stakgold calls $E_n^{(r)} = R_n$, but you can then extend that to $E_n^{(c)}$.

In **Quantum Mechanics** we use the inner product $\langle a|b \rangle = \langle b,a \rangle$. This peculiar reversal of order arises because in QM we want operators to be linear on the kets $|b\rangle$ and antilinear on the bras.

Here is a picture I made that shows how all these types of spaces are related.



2.4 Properties of Separable Hilbert Spaces (HS). (pp 116-135)

The main thrust of this book is with ∞ **dimensional Hilbert Spaces** which go beyond the finite matrix idea. It proves useful to limit one's interest to HS's where the spanning vectors form a **countable** set, and so can be written f_1, f_2 .... . Such a Hilbert Space is called a **separable** Hilbert Space and this is all we will ever care about. The countable set of spanning vectors is called a **spanning set**. Using a spanning set f_i, you can get arbitrarily close to an element in X by doing a fit $x = \Sigma_{i=1}^{N} a_i f_i$ for large enough N (the word fit of course requires a metric, but HS has a metric.). In general, all the a_i

coefficients will "move" as you bump up N and crank down on your fit to achieve a smaller ε. If you can find an ∞ spanning set where the a_i *don't* move, then that spanning set is a **complete basis** and you can take N→∞ and all a_i stay put. In this case, basis vectors will turn out to be orthogonal, and the a_i are generalized **Fourier Coefficients**. A favorite counterexample is that in L_2, the powers t^k form a spanning set but not a basis.

The rest of this section seems to be a general grab bag of HS related facts. Any spanning set of *independent* vectors forms a basis, even if not orthogonal. Any *orthonormal* spanning set makes a basis. The **Projection Theorem** is the idea that if your HS H has a **subspace** M (= **linear manifold** in Stakgold), then $H = M \oplus M^{\perp}$ and you can write any $x \in H$ as $x_1 + x_2$ etc.

Each set of the famous **orthogonal polynomials** forms an orthogonal basis on some appropriate interval. Often they are not normalized, such as with the Legendres. They usually have a weight function included in the inner product relating to the concept of measure.

The **Riesz-Fischer Theorem** (1907) makes a connection between convergence of a sequence of *vectors* in a HS and the convergence of a sequence of real *numbers*, but you have to do this with respect to a known basis in the space. The real numbers are the squares of the partial sums of the squared Fourier coefficients. The **GSO process** for making a spanning set be orthogonal is described.

2.5 Functionals. (pp 135-139)

Up to this point we have just been talking about "spaces", and in particular Hilbert Spaces. Now for the first time we talk seriously about transformations *that map between our spaces* as was quickly outlined in Section 2.1. In this Section we talk about T: H → C where H is some Hilbert Space and C = the complex numbers. Such a transformation is called a **functional** (it need not be linear) and is written **T[x].** In Section 2.6 we will talk instead about transformations which map A: H→H so the Hilbert Space is mapped into itself. These transformations A are called **operators** and we write the transformation as Ax, (not as A[x]).

Since T(x) is in space C, || T[x] || = |T[x] |. And of course since the domain space H is a HS, it too has some norm, so ||x|| is well defined. We can consider the positive real number which is the value of the ratio || T[x] || / ||x|| as x moves around in some S ⊂ H. If this ratio has a maximum value < ∞, then the functional T is said to be **bounded on S**, and that max value is called the **norm** of T or || T ||. Conclusion: || T || ≡ max (|| T[x] || / ||x||) for x ∈ S. In the discussion above, we talked about the norm of a *vector* in a vector space, but || T || is the norm of a *transformation* between two vector spaces, so not the same animal, but it does fulfill the defining properties of a norm. If the norm is infinite, then the functional is **unbounded on S**. [In a sense, the norm is a measure of the maximum "amplification" of the functional T, where ||x|| is the input signal, and || T[x] || the output signal. Different signal inputs are amplified by different amounts.]

Regular convergence means $x_n \to x$. If this is in our domain space, we can consider whether or not we have $f_n = T[x_n] \to T[x] = f$ in the range space (call this **the image sequence**.) If we have $f_n \to f$ in the range space for every possible sequence $x_n \to x$ in some domain region S, then the functional T is said to be **continuous on S**.

This *reminds* us a little of the notion of **uniform convergence** of a function, but that is not the same concept. Suppose you consider a set of square-integrable functions $\mathcal{L}_2(a,b)$ for the interval (a,b). This thing $\mathcal{L}_2(a,b)$ is a Hilbert Space and its vectors are functions f(t). You could consider a sequence $f_n(t) \to f(t)$. If this convergence happens *for all t* in (a,b), then we say the sequence **converges uniformly** on

(a,b). Notice, however, that this range (a,b) is not the domain space of some transformation like T[x]. In uniform convergence, the parameter t in $f_n(t) \rightarrow f(t)$ which ranges through an interval is part of the definition of HS $\mathcal{L}_2(a,b)$. In contrast, when we talk in the previous paragraph about convergence $T[x_n] \rightarrow T[x]$, the transformation argument is a sequence of points in the domain space.

We are often interested in functionals which are *linear*. In this case, we can prove some useful theorems where S is some region in H: (both theorems are quickly proven in the text)

Theorem 1. If linear T[x] is continuous at x = 0 in S, it is continuous at all x in S.

Theorem 2: For linear functional T[x] , continuous on S $\Leftrightarrow$ bounded on S

Stakgold says he will only be interested in cases where S = a subspace of H, but I don't think that is a requirement for the above theorems to be true. He often refers to S as D_T.

Riesz Representation Theorem (1907).

Recall that T: H $\rightarrow$ C. We want to say that we can represent T[x] as T[x] = <x,f> for some f in H *for all x in H*. If this were true, it would say that we can completely characterize functional T by some fixed vector f in H, which seems pretty amazing to me. It turns out that this "representation" is true provided that the functional is linear and continuous on all of H, which Theorem 2 above says is the same a T being linear and bounded on H. This theorem therefore claims there is an isomorphism between two spaces: (1) the set of continuous linear functionals on H; (2) the set of vectors in H. Space (1) is called **the dual space**, but Stakgold does not get into that subject.

In the proof of this theorem, Stakgold defines the notion of the **nullspace** of T and calls it N. This space $N \subset H$ is a subspace of H (easy to show he says; a subspace is closed!). We can then write $H = N \oplus N^{\perp}$ where this latter is what I call the perp space. If $x \in N$, then T[x] = 0, that is what nullspace means. If there is no perp space, meaning N = H, then T[x] = 0 for all x in H, and we can "represent this" as <x,f> with f = 0. Otherwise, we can surely find and normalize some vector in the perp space $N^{\perp}$, call it f_o. He then shows that the required f to make T[x] = <x,f> is explicitly $f = \overline{T}[f_0]\, f_o$. The proof seems completely trivial as you read it. He also shows trivially that this f is unique, so that means whatever f_0 you select in $N^{\perp}$, you will find that $\overline{T}[f_0]\, f_o$ is always the same vector.

This site https://www.math.gatech.edu/~harrell/pde/Rieszapp.html shows trivially that the perp space of a linear functional is one dimensional, so there is really only one possible normalized f_o, which explains the mystery of the uniqueness of the $f = \overline{T}[f_0]\, f_o$.

The theorem requires that T be linear and continuous on all of H (which is same as being bounded on H), *but* as you read the simple proof, you see at once why linearity of T is required, but you do not see why continuity is required. I searched a little on the web and could not find an explanation of where continuity is used, and I see that most sites refer to concepts a little beyond Stakgold in dealing with this theorem. Perhaps this is it: if H were not bounded, then perhaps the solution we find which involves $T[f_0]$ would be infinite and therefore non-existent. If T is bounded, we know that $| T[f_0] | / ||f_0|| = | T[f_0] | \leq ||T||$ and must therefore exist. Thus, the solution f exists which makes T[x] = <x,f> exist. That must be it!

Now, once you can represent T[x] = <x,f>, I think T[x] is patently continuous because an **inner product is continuous in either argument.**

Extension of a functional from subspace to full space.

Page 137 talks about how you can **extend** a bounded linear functional from a subspace D_T to the entire space H. We know we can write $H = D_T \oplus D_T^{\perp}$, and we also can write $D_T = (N \oplus N^{\perp})$ because D_T being a subspace is also a Hilbert Space and our usual projection theorem applies with respect to T.

It is possible that the subspace D_T might not be closed. If that is the case, you close it, and then you can say that D_T is a Hilbert Space because it is then complete due to this closure. So close D_T to $\bar{D}_T$ if it is not closed up front. I will not use the overbar in what follows.

Notice that $D_T{}^{\perp}$ is therefore perp to both N and $N^{\perp}$. We know from Riesz that we can write T[x] = <x,f> for any x in D_T and we will have f be in $N^{\perp}$. We know that this formula applies for x in N (in which case it will happen that T[x] = <x,f> = 0), and it also applies for x in $N^{\perp}$, in which case T[x] = <x,f> ≠ 0. The idea is to *use this same formula* T[x] = <x,f> and extend it to x in $D_T{}^{\perp}$. Since $D_T{}^{\perp}$ is perp to $N^{\perp}$, we know that we will have <x,f> = 0. Thus, this method of extension just sets T[x] = 0 for x in $D_T{}^{\perp}$ and it does it all with the single formula T[x] = <x,f> now applying for any x in H ! This method of extension is thus making use of the Riesz Representation Theorem and thus you can only extend a functional in this manner if it is linear and bounded (= continuous).

Theorem 3: All linear functionals T[x] on $H = E_n$ are bounded (hence continuous).

On page 137-138 Stak shows how this theorem can fail for ∞ dim Hilbert Spaces and he concocts a specific example which is a sequence approaching a delta function.

Dual Basis Idea (p 138).

The claim is that if you start with a basis e_i that is non-orthonormal, you can find a set of "dual" (reciprocal) basis vectors $e_i{}^*$ which are orthonormal *to* the e_i (but not in general to the other $e_j{}^*$). Later on page 150 it is noted that if your e_i are themselves orthonormal, then we have $e_i{}^* = e_i$ so you don't have to think about the dual basis vectors. Thus subject arose in M&M where they wrote $e_i{}^*$ as e^i, and this was related to covariant and contravariant and metric tensors. Notice that this is unrelated to the dual space notion of certain functionals on H being isomorphic to H.

A chart on page 139 compares the world of functionals on $H = E_n$ to that on $H = \mathcal{L}_2(a,b)$, finite dim versus infinite dim. Here are the key points that are worth noting on the $\mathcal{L}_2(a,b)$ side:

(1) You can only express $T[x = \Sigma^{\infty} a_i e_i]$ as = $\Sigma^{\infty} a_i\, T[e_i]$ if T is linear *and bounded*, where e_i = some basis. Notice that this equality is an extension to infinity of the definition of T being "linear". Here you are interchanging the order of the operator T with the operator of infinite summation.

(2) Notice that T[x] = <x,f> for bounded T means $T[x] = \int dt\, f^*(t)x(t)$. Thus, any bounded T in $\mathcal{L}_2(a,b)$ can be expressed as a simple integral against some function f over (a,b).

2.6 Operators. (pp 139-146)

We have A:H→H now, and operator action is written as Ax = f. Concepts of nullspace, domain, range, linear, continuity, norm all carry over from the functional work done in the previous section. In particular we now have

norm = ||A|| = max(|| Ax || / ||x||) for x in D_A

which is the analog of the functional's norm. We get these three completely analogous theorems:

Theorem 1: Continuity of linear operator at x = 0 implies continuity over all of D_A.
Theorem 2: For a linear operator, boundedness ⇔ continuity (p140)
Theorem 3. In E_n, every linear operator is bounded (and continuous)

We can extend an operator A defined on D_A to operator B defined on all of H by something similar to what we did with the functionals. To make it work, as before we have to close D_A to $\bar{D}_A$ and then define Bx = 0 out in the $D_A^{\perp}$ area. In order to be able to close D_A, we must know that operator A is continuous, otherwise the limit points of sequences Ax_n might not be unique and we have closure problems. The discussion on page 141 also uses the boundedness of A. So if linear A is continuous (= bounded), then we can always extend an operator A from D_A to B defined on all of H, and we write **B ⊃ A**. Then B is an **extension** of A, and A is a **restriction** of B. In the functional world, the Riesz inner product gave us explicit continuity, but we still had to assume T[z] was bounded.

So in general things are under control for bounded = continuous operators A. But what can be said about unbounded linear operators? We are interested because we know that differentiation is unbounded.

Closed operators -- a somewhat painful topic. (starts below pencil line on page 141)

Stakgold introduces a new concept here, that of **closable and closed operators**. If an unbounded linear operator is a closed operator, it is not as "nice" as a bounded linear operator, *but* it is not as "ugly" as an unbounded linear operator which is not closed or closable. So closed unbounded operators are things we will be at least able to work with, but they are not as nice as bounded operators. The prototype example of a closed unbounded operator is d/dx, which will obviously be very important us.

So what does closable mean? For a *bounded* linear operator, we know that $Ax_n \to 0$ for a **null sequence** $x_n \to 0$, due to continuity. For an unbounded but **closable** operator, if this is not true, then at least we know that "Ax_n has no limit" will be the worst case for a null sequence x_n. So **closable means** either $Ax_n \to 0$ or $Ax_n \to$ does not exist (=∞ below). You cannot have the third bad option which is $Ax_n \to g \neq 0$ for x_n being a null sequence. So closable is thus defined. [For a closable operator, the image sequence in the range of any null sequence in the domain either converges to 0 or does not converge.]

Elaboration: Consider two sequences in the domain, $z_n \to z$ and $y_n \to z$. Then $x_n \equiv z_n - y_n$ is a null sequence and if A is closable, then $Ax_n \to 0$ or does not exist. If $Ax_n \to 0$, then we have $Az_n \to Az = f$ and $Ay_n \to Az = f$. All is well, the notation "Az" is well defined.

Suppose $Ax_n \to g$, some finite vector in the range. *This* is the case we want to avoid! We then somehow have the situation $z_n \to z$ => $Az_n \to$ "Az" = f_1 and $y_n \to z$ => $Ay_n \to$ "Az" = f_2 and $g = f_1 - f_2$ and $f_1 \neq f_2$. [Here, $(z_n - y_n) \to 0$ is our null sequence and $A(z_n - y_n) \to g$.] In this case, the notation "Az"

does not even make sense, it is not unique. Ugly! You might be able to deal with this, but special notation and methods would be required.

Suppose $Ax_n \to \infty_3$. We then have the situation $z_n \to z \Rightarrow Az_n \to \infty_1 = f_1$ and $y_n \to z \Rightarrow Ay_n \to \infty_2 = f_2$ and $g = f_1 - f_2$ or $\infty_3 = \infty_1 - \infty_2$. But we are happy to have $\infty = \infty - \infty$. For example consider the following: $\infty_1 = \lim(9n)$, $\infty_2 = \lim(2n)$ then $\infty_1 - \infty_2 = \lim(7n) = \infty_3$. Since we have f_1 and f_2 being ∞ in this case, we can thing of "Az" = ∞ for both cases, and we don't get the contradictory situation shown above when $Ax_n \to$ g finite. We can include in this case the meaning "limit does not exist" when we write the symbol ∞. Again, we get no contradictory notation.

The point is that we cannot have *different* limit points f in the range for different $q_n \to z$ in the domain. For a closable operator, either all those limit points like Az are the same (f) for all domain sequences like z_n and y_n, or all those limit points don't exist. If the limit points do exist, then we say $Az_n \to Az = f$.

Now, a closed operator could have some limit point z right inside D_A such that the limit Az_n does not exist for $z_n \to z$, and *that is OK*. It sounds sort of like a pole in a function is OK for meromorphic.

Closing a closeable operator to obtain a closed operator.

Notice that z (limit of z_n) might not be in D_A. In closing an operator, we to close D_A so it becomes $\overline{D}_A$ so that any such z in $z_n \to z$ gets included in the domain. This then is the idea of **closing an operator**. If A is closable, then the closed version of it is called $\overline{A}$. A closed operator $\overline{A}$ is different from a closed set $\overline{S}$, though the same word "closed" is used. The domain of A is D_A, and the domain of $\overline{A}$ is $\overline{D}_A$.

As the text points out, no interesting operator we ever encounter will by unclosed. So we will always deal either with (1) operators that are bounded = continuous, or which are (2) unbounded = not-continuous but are closable. These are only "mildly discontinuous" not "radically discontinuous". Notice that in case (1), if we take any null sequence $x_n \to 0$, we get $Ax_n = 0$, the case $Ax_n = \infty$ is ruled out. In this case, if $z_n \to z$, we have $Az_n \to Az$ = well defined. So any bounded =continuous operator is therefore certainly "closeable", since we have just removed one possibility of $Az_n \to \infty$. And if the domain is closed, then A is closed, the subject of the next theorem which we now see is quite obvious.

Theorem 10: A continuous (bounded) operator on a closed domain is a closed operator.
Corollary: A continuous (bounded) operator on all of H is a closed operator.

[This theorem seems pretty obvious. The image sequence of a null sequence for a continuous operator will be a null sequence in the range, and that is one of the two cases for a closable operator.]

Now here is the big new fact which will no doubt help us a lot:

Famous Theorem 11: A closed operator on a closed domain is continuous. (closed graph theorem)

[This seems to suggest that in this case, the image sequence of a null sequence cannot diverge. Somehow, having a closed domain "rescues" a closed operator and makes it be continuous.]

Note added 4.20.09. I have a big problem with Theorem 11. Suppose I start with a general unbounded operator A which is *closable*. I close the domain by including in it all possible missing limit points of domain sequences. A is then a *closed* operator and it acts on a closed domain. Why should A suddenly be

bounded, when I assumed from the start that it was unbounded? This seems to imply that there are no unbounded closeable operators. Web says Theorem 11 is the "closed graph theorem". I am misunderstanding some aspect of closed operators. The operator A = d/dt is unbounded but closeable. If I close the domain, does d/dt suddenly become bounded? Maybe the catch here is that for d/dt, you *can't* close the domain.

OK, here I think is a resolution of this issue: (Note: see "non-differentiable continuous functions.doc" in "calculus" which shows there are many functions such as the Weierstrass function which are continuous everywhere but differentiable nowhere!)

The Stakgold Issue (p 145 Example 7). The operator d/dt is unbounded and closeable on the domain of "differentiable functions" (Stak calls this set of functions "absolutely continuous" functions). Consider a sequence of functions $f_n(t)$ in the domain, where $f_n(t) \rightarrow f(t)$. He claims on page 145 that functions like f(t) are piecewise continuous (which he includes in his absolutely continuous class). He wants us just to include the limit functions like f(t) in the domain. The limits d/dt (f(t)) exist. Though they will be discontinuous functions, they are still fine L_2 functions. So here we are "closing the domain" in a certain sense so that that closable operator d/dt becomes a closed operator.

However (!), the domain of d/dt even including this closure is not a "closed set" within L_2 ! As noted above, the set of continuous functions which are differentiable *nowhere* is dense in L_2 ! Consider for example the Weierstrass or some other partial sum as $f_n(t) \rightarrow f(t)$. In this case, f(t) would not be included in our d/dt closure process mentioned above, since f(t) is differentiable nowhere. Thus, in the d/dt closure process we are NOT including the limits of ALL sequences of functions. We are only including the limits of sequences whose limits are differentiable to yield a function in L_2. The limit of the Weierstrass sequence does not make a function which is "differentiable to yield a function in L_2".

When you talk about a domain being "a closed set", you have to include the limits of ALL sequences.

So, you can take an open domain of d/dt, you can close it by adding the sequence limits discussed above, but even after this closing process, the domain is still open in the sense that not all sequence limits are in the domain. So the semantic confusion here for me was this: after you "close something" it is still "not closed". In our d/dt closing process, we are "partially closing" the domain, and "fully closing" the operator by so doing, by the definition of a closed operator.

Then we can get happy again about the "closed graph theorem" or Theorem 11 of my Stak Chapter 2 meta notes (text page 142) : an operator which is closed on a "closed domain" is in fact bounded and continuous. The operator d/dt is closed, but its domain of absolutely continuous functions (which includes PC) is not a closed set in L_2. Therefore we cannot apply our theorem to say d/dt is bounded.

Consider now an unbounded operator A acting on all of H, which is clearly a closed domain. If A were closed, it would have to be continuous by Theorem 11. But if it is continuous, it must be bounded by Theorem 2 (assuming A is linear, which we may have to assume everywhere here). Thus we have this fact:

Theorem 12: There can be no closed unbounded operators defined over a whole Hilbert Space H.
[for example, d/dt cannot be defined on all of H]

However, we *can* find such operators defined over a space which is dense in H! A dense subset of H is not a closed set, and thus Famous Theorem 11 does not apply.

Fact: We have seen how a closed operator might have an unclosed domain and unclosed range. The word "closed" is still used because the **graph of A**, which is the set of pairs (x,Ax) in graph space, forms a closed set. This is his remark 3 page 143.

Theorem 13: The nullspace of a closed operator is a closed set. (remark 4).

Fact: The **differential operator** acting on L_2 is the *prototype* of a closed unbounded operator. Perhaps the dense space contained in H which d/dx can act on is the space of "differentiable" functions in L_2. We know that d/dx cannot act on all functions in L_2 due to Theorem 12. I think this class of differentiable functions is what Stakgold later calls **absolutely continuous** functions, and this includes piecewise differentiable functions. This operator is discussed as Example 7 on page 145 and is the only example really that was unbounded.

Possible example of interest. Consider the sequence $x_n(t) = (1/n)\sin(n^2 t)$. I would argue this is a null sequence since $x_n(t) \to 0$. The integral of the square of this function on 0,1 is perhaps $1/(2n^2)$ which $\to$ 0. Now what is the image sequence for A = d/dt ? We get $f_n(t) = n\cos(n^2 t)$. This provides us then an example where the limit of the image sequence of a null sequence "does not exist", so the image sequence does not converge. Certainly the norm gets larger ~ n and so a limit if it existed would have infinite norm. This is really the idea of being unbounded. The fact is that the image function in this case is an ever larger amplitude faster moving cosine and I think it is fair to say that this does not approach any reasonable kind of limit.

2.7 Operators in the Hilbert Space $E_n^{(c)}$ (pp 146-165) Finite Dimension Hilbert Spaces

The opening sections are pretty easy, examples are given such as rotation and sheer transforms. Active and passive is mentioned. One good comment is that if $Ae_j = \Sigma a_{kj} e_k$ on some basis e_k that is not orthogonal, you can still invert to get $a_{nj} = \langle e^n, Ae_j \rangle$ using the *reciprocal* basis element e^j.

The following items all are the same for determining invertibility of A: A is invertible if: $\det A \neq 0$, A has full rank, A has no nullspace, Ax=f has exactly one solution, A is **regular**. If A is not invertible, then A is **singular**. Only two choices here.

Adjoint Operator.

Given linear operator A, you can define another linear operator A* (**the adjoint**) by this relation:

$$\langle Ax,y \rangle = \langle x,A^*y \rangle$$

If A = A*, then A is **self-adjoint**. This is the same as being **complex symmetric** = **Hermitian**.

Nullspace Facts.

The **Alternative Theorem** says that either Ax=f has a solution for all f (meaning the range is all of H), *or* the equation Ax = 0 has a non-trivial solution (there exists a nullspace, by which I always mean a non-trivial nullspace beyond the 0 vector). The more precise statement is that the following spaces are equal: $R_A = (N_{A^*})^\perp$ which is the same as $N_A = (R_{A^*})^\perp$. It turns out that A and A* have the same size (dimension) range and the same dimension nullspace, so for example we know that $\dim(N_A) = \dim(N_{A^*})$. The dimension of a nullspace of A is called the **nullity** of A or ν(A), while the dimension of the range is called the **rank** of A or ρ(A). So we just claimed that ρ(A*) = ρ(A) and ν(A*) = ν(A). Obviously $\dim(N_A)^\perp = n - \dim(N_A) = n - \nu(A)$, but

$$\dim (N_A)^\perp = \dim (N_{A^*})^\perp = \dim(R_A) = \rho(A)$$

and therefore **ρ(A) + ν(A) =** n. In the alternative theorem as stated above, if Ax = f always has a solution, then we are in the case ρ(A)= n and ν(A) = 0. The other alternative is that ρ(A) < n in which case we must have ν(A) > 0.

Connection between Nullspace and The Eigenvalue Problem.

We know we can solve the equation Bx = f for any f as long as detB ≠ 0. The reason is simple: in this case B^{-1} exists, and $x = B^{-1}f$ and there is your explicit solution.

Now suppose B = A - λI = A - λ. From the above, if detB ≠ 0, we can solve the equation Bx = f to get $x = B^{-1}f$ as the specific solution. Now detB = 0 is a polynomial of degree N in λ and it will have N roots which we call λ_i and these are called the eigenvalues. What is the significance of these eigenvalues λ_i ?

Eigenvalues, secular equation, geometric and algebraic multiplicities.

Definition: in this context, det[B(λ)] = 0 is called the **secular equation**.

Definition: An **eigenvalue** is a root of the secular equation.

(1) if $\lambda \neq \lambda_i$, then detB ≠ 0 and Bx=f has the solution $x = B^{-1}f$ and then $\rho(B) = \dim(R_B) = N$ which means that B has no nullspace, B has full rank, so Bx=0 has no solutions. But Bx=0 is the same as Ax = λx so when $\lambda \neq \lambda_i$, our eigenvalue equation Ax = λx has no solutions.

(2) if $\lambda = \lambda_i$, then $B_i x = f$ might still have some solutions, but they are not of the form $x = B_i^{-1}f$ because B_i^{-1}does not exist. Perhaps we will have $\dim(R_{Bi}) = \rho(B_i) = n < N$ and this will correspond go $\nu(B_i)= N-n > 0$. In this case, our eigenvalue equation $Ax = \lambda_i x$ will have some non-trivial solutions, and these are called the eigenfunctions of A (nullspace solutions of $B_i x = 0$). The number of independent solutions of $B_i x = 0$, or equivalently, the number of independent eigenfunctions for $\lambda = \lambda_i$, is $\nu(B_i)$ but has a fancier name: the **geometric multiplicity** of λ_i and we use the notation m_i for this multiplicity.

(2, refined). The refinement is that we can remove the qualifier "Perhaps" in the above paragraph.

If $\dim(R_B) = \rho(B) = N$ (B has full rank), we know that Bx = f has a solution for any f in E_n (meaning $E_n^{(c)}$). This means we can write that solution in the form x = Gf where G is some linear operator (you could show linearity easily I think). Therefore, we have f = Bx = BGf and this is true for any f in $H = E_n$.

Therefore it must be true that $G = B^{-1}$. This in turn implies that $\det B \neq 0$. Therefore, if we knew that $\det B = 0$, we would know that B must be less than full rank by at least 1. This means that the nullity of B must be at least 1. Thus, $\det B=0$ [which occurs when $\lambda = \lambda_i$, an eigenvalue] $\Rightarrow 1 \leq \nu(B) \leq N$. This means $Bx=0$ has at least one independent nullspace solution at this eigenvalue $\lambda=\lambda_i$, and it means that $Ax = \lambda_i x$ has at least one independent eigenfunction.

Theorem: Every operator A in E_n has at least one eigenvector.

Proof: We certainly know there is at least one eigenvalue because the secular equation has N roots and at worst they could all be the same, and there is your "at least one eigenvalue". It happens that this would only occur when A was a multiple of the identity, but in any case, there is at least one eigenvalue, so we just pick one. For this eigenvalue, we have $\det B = 0$ and we just showed above that $Ax = \lambda_i x$ then has at least one independent eigenfunction at that eigenvalue. We have really proven this stronger theorem:

Theorem: For every eigenvalue of A, there is at least one eigenvector. In other words, every eigenvalue of an arbitrary A has a geometric multiplicity of at least 1, so $m_i \geq 1$.

We know that λ_i is a root of $\det B = 0$ and that root will occur some number of times k_i where k_i ranges from 1 to N. This k_i number is the **algebraic multiplicity** of λ_i. Stakgold shows with an easy derivation that $k_i \geq m_i$ for any eigenvalue λ_i. Based on our theorem above, we really have $k_i \geq m_i \geq 1$.

Example 3: A simple "shear matrix" $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and secular says $k_i = 2$ with both eigenvalues $\lambda_i = 1$, but trivial solution for the eigenvectors finds that the only eigenvector is $x = k\,(1\ 0)$ so $m_i = 1$. So right here we have an explicit example of $k > m$, in case we were wondering if that were possible. Notice that this matrix is not symmetric.

Theorem: Suppose $Ax = \lambda x$ has $1 \leq \nu(B) \leq N$ eigenvectors, and suppose each eigenvector has a different eigenvalue, which means each eigenvalue has algebraic multiplicity = 1. In this case, those $\nu(B) \leq N$ eigenvectors will be linearly independent. We can also say $m_i = 1$ for each eigenvalue.

Stak proves this theorem by induction on page 159.

Corollary: If $Ax = \lambda x$ has $\nu(B) = N$ eigenvectors with distinct eigenvalues, then the eigenvectors are independent and form a basis for E_n.

Symmetric Operators.

After the voyage above we return now to this subject. In E_n we mean by the term "symmetric operator" one that is self-adjoint = Hermitian = complex symmetric. When we get to ∞ dimensional Hilbert Spaces, there will be a distinction between self-adjoint and symmetric, but in E_n they are the same. [C_n would be a better name for E_n, and R_n for the reals.]

For symmetric operators A, Stak shows on pages 159-160 these familiar facts:

(1) $\langle Ax,x \rangle$ = real for all x in C_n -- though components of x are complex numbers.

(2) eigenvalues are real
(3) eigenvectors of different eigenvalues are orthogonal
(4) eigenvectors form a basis (interesting proof uses perp spaces iteratively), so this means there must be a total of N eigenvectors forming this basis.

We know that $\Sigma_i k_i = N$ always. If there are N independent eigenvectors as in (4) above, then it must be true that $\Sigma_i m_i = N$ as well. This means that for each λ_i, we must have $m_i = \lambda_i$ and there are no "missing eigenvectors". A non-symmetric operator A very well might have some missing eigenvectors, like that shear matrix example above.

I think it is item (3) above that lets us treat each group of eigenvectors for a given λ_i as a subspace or **eigenmanifold**. These spaces are independent of each other. This is what lets us write:

$$H = M_1 \oplus M_2 \oplus M_3 \ldots\ldots \oplus M_k$$

using the direct sum notation (which Stak does not use). For each eigenmanifold M_i, we have $m_i = k_i$ and then we have $\Sigma_{i=1}^{k} k_i = N$ and same for m_i. At most there are k = N eigenmanifolds, and at least there is only 1 eigenmanifold.

Comment: If we don't orthonormalize the m_i eigenvectors in one or more eigenmanifolds M_i, then the total set of independent eigenvectors $\{ x_i\}$ still form a basis, but one which is not fully orthogonal. Here we reinforce our earlier idea that in a finite dimensional Hilbert Space like C_n, a basis does not have to be orthogonal, and this is just because some eigenmanifolds might have $m_i > 1$. Usually you would use GSO to orthonormalize within each M_i , and then $\{x_i\}$ becomes an orthonormal basis.

Spectral Theorem for Symmetric Operators in E_n

The above partitioning direct sum concept embodies what Stak calls The **Spectral Theorem for Symmetric Operators in E_n**. Here we just summarize it as he does:

(a) to each eigenvalue λ_i is associated an eigenmanifold M_i of some dimension $m_i=k_i$ (geo = alg)
(b) eigenmanifolds are pairwise orthogonal
(c) these eigenmanifolds partition the space, and you can decompose $x = x_1 + x_2 + \ldots$ where x_i in M_i.
(d) the eigenvalues λ_i are determined by $\det(A-\lambda I) = 0$
(e) due to (c), you can imagine a projection operator P_i for each eigenmanifold M_i.
(f) we can then write $A = \Sigma_i \lambda_i P_i$ known as the **spectral resolution of A** (easy to prove, remark 1)

Notice that we then have

$$Ax = A(\Sigma_i x_i) = \Sigma_i (Ax_i) \qquad \#1$$

$$= (\Sigma_j \lambda_j P_j)(\Sigma_i x_i) = \Sigma_j \Sigma_i \lambda_j P_j x_i = \Sigma_j \Sigma_i \lambda_j \delta_{ji} x_i = \Sigma_i (\lambda_i x_i) \qquad \#2$$

which one could derive in reverse starting with $Ax_i = \lambda_i x_i$ for the component of x which lies in the eigenmanifold M_i.

Remark #2. The eigenvalues of a symmetric operator are real, we know. So it is sometimes useful to list them in decreasing numerical order. It is also useful to have two different methods of listing them off, which I illustrate here:

$$\begin{array}{cccccccccccccc} \lambda_1 & \lambda_1 & \lambda_1 & \lambda_2 & \lambda_2 & \lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_5 & .. & \lambda_q \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 & \mu_{10} & \mu_{11} & \mu_{12} & .. & \mu_n \end{array}$$

Remark #3 Fact: A symmetric operator cannot have all eigenvalues be 0. If so, then $Ax_i = 0$ in every eigenmanifold, and you conclude that A = 0.

Extremal principles for Symmetric Operators (162)

Four functionals are defined bottom of page 162, here they are, where we have two kinds of norm type objects, one being the official norm.

$$\begin{array}{ll} \|A\| = \max I_2 \text{ over all } x & I_2 = \| Ax \| / \|x\| = [\langle Ax,Ax\rangle/\langle x,x\rangle]^{1/2} \\ \|A\| = \max I_1 \text{ over all } x \text{ such that } \|x\| = 1 & I_1 = \| Ax \| \\ & \\ |||A||| = \max I_4 \text{ over all } x & I_4 = |\langle Ax, x\rangle| / \|x\|^2 \\ |||A||| = \max I_3 \text{ over all } x \text{ such that } \|x\| = 1 & I_3 = |\langle Ax, x\rangle| \end{array}$$

The lower two objects are not directly related to ||A|| as far as I know

Page 153 very clearly proves that:

Theorem: for a symmetric operator, $\|A\| = |||A||| = |$ the largest eigenvalue$| = |\mu_1|$

You can see this pretty easily. In I_2 set $x = x_i$ and you get $I_2 = |\lambda_i|$, so if you max this, you get $|\mu_1|$. Similarly, you get $I_4 = |\lambda_i|$ and maxing it is the same thing. So for a symmetric operator, both these objects ||A|| and |||A||| are "the norm of A".

For a non-symmetric matrix, he claims *without proof* that this is all you can show

$$\begin{array}{l} \|A\| \geq |\mu_1| \\ |||A||| \geq |\mu_1| \end{array}$$

which seems an odd result to me, but he gives an example where this is born out. I could not find a proof of this in a quick web search due to having no good search handle.

2.8 The inverse of an linear operator (pp 165-180) Infinite Dimension Hilbert Spaces

Theorem: A is 1-to-1 $\Leftrightarrow$ A^{-1} exists $\Leftrightarrow$ [$Ax = 0 \Rightarrow x=0$, ie, there is no nullspace]

If there were a nullspace, that would imply multiple domain elements mapping to 0, so the mapping is not 1 to 1.

Now the term below "regular" really means an inverse exists for Ax=f for all f and the inverse is "nice". Thus, you can solve the equation Ax=f for x if A is regular.

Regular, essentially regular, and singular

Definition: **regular**. Recall that a E_n operator was either regular or singular, but now things are messier. In ∞ dimensions, to be regular we need 3 conditions to be met:

(a) $Ax=0 \Rightarrow x=0$, as with matrices; and this implies that A^{-1} exists;
(b) the range must be *the entire* Hilbert space, so Ax = f has a solution for any f in H;
(c) A^{-1} must be bounded (= continuous).

If your R_A does not include the boundary of space H, you can do the "usual" extension idea to get the boundary included to meet condition (b). Without the boundary, if all other conditions are met, you are **essentially regular**.

Notice that our old rule that $\rho(A) + \nu(A) = n$ now has problems since $n = \infty$. It seems from (b) that it is possible to have nullity $\nu(A) = 0$, but range is not the whole space. Condition (c) is a niceness condition that we just tack on so that "regular" means something reasonably nice.

Closed operator stuff

Now we consider once again those **closed operators**. Recall the various items of interest, operators being continuous, being bounded or unbounded, being closed, various sets being closed, etc.

Here is a densepack little review of our theorems from above: all operators are linear from now on, as far as I am concerned. [I think except for {..} that these are true for ∞ dimensional Hilbert Spaces.]

Theorem 1: Continuity of linear operator at x = 0 implies continuity over all of D_A.
Theorem 2: For a linear operator, boundedness ⇔ continuity (p140)
{ Theorem 3. In E_n, every linear operator is bounded (= continuous) }

Theorem 10: A continuous(=bounded) operator on a closed domain is a closed operator.
Corollary: A continuous(=bounded) operator on all of H is a closed operator.
Famous Theorem 11: A closed operator on a closed domain is continuous (=bounded).
Theorem 12: There can be no closed unbounded operators defined over a whole Hilbert Space H.
Theorem 13: The nullspace of a closed operator is a closed set.

Here then are some new facts concerning closed operators and the inverse A^{-1}:

(a) If A is closed and A^{-1} exists, then A^{-1} is also closed.

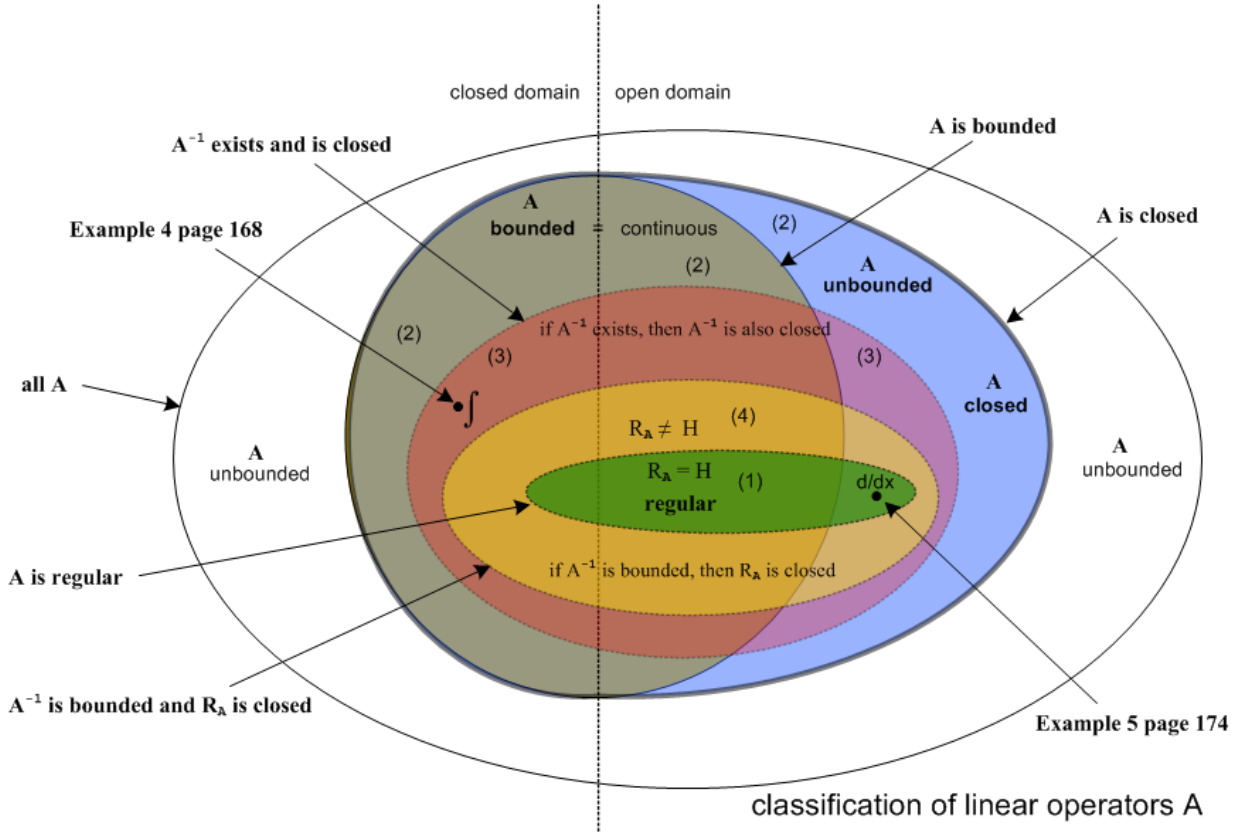
(b) If A is closed and A^{-1} is bounded, then R_A is closed.

Contrapositive: If A is closed and R_A is *not* closed, then A^{-1} is unbounded.
Apply to A^{-1}: If A^{-1} is closed and A is bounded, then D_A is closed

(c) If A is closed and D_A is closed (normal situation I think), then A is bounded.
Apply to A^{-1}: If A^{-1} is closed and R_A is closed, then A^{-1} is bounded.

A^{-1} is bounded and R_A is closed

I have tried to encapsulate all these "facts" in the following drawing"



The outer solid oval encompasses all linear operators A. Those on the left have D_A = closed, those on the right have open domains. The odd-shaped boundary contains the closed operators, and then the inner solid circle shows the location of bounded = continuous operators. You can see that on the left side, these two regions are the same. The dotted ovals have to be interpreted according to their location.. This is perhaps a tough chart to read, but it helps me to try at least to visualize the classification. I have tried to show where differentiation and integration fit in. The Hilbert Space H is meant to be arbitrary here. Now recall from just above:

Regular requires:

(a) $Ax=0 \Rightarrow x=0$, as with matrices; and this implies that A^{-1} exists;

(b) the range must be *the entire* Hilbert space, so Ax = f has a solution for any f in H;

(c) A^{-1} must be bounded (= continuous).

Classifying Closed Operators.

The picture shows that these four cases partition the space of closed operators.
The first case is "regular", the other three are singular:

(1) A is regular (meets the three criteria listed above) [green oval]
(2) A^{-1} does not exist, nontrivial Ax=0; [violates criterion (a)] [within blue but outside purple]
(3) A^{-1} exists *but* A^{-1} is unbounded [violates criterion (c) and (b)] [within purple but outside yellow]
In this case, we know from our drawing that $R_A \neq \overline{R}_A = H$
(4) A^{-1} exists but $\overline{R_A} \neq H$. [violates criterion (b)] [within yellow, but outside green]

I have tried to show the four cases as well on the chart above. Integration for example is singular in case 3 because its inverse, differentiation, is unbounded. It is really the *inverse* of an operator that determines its niceness regarding solving our Ax = f equation! We think if integration as nice, but it has the nasty inverse which is unbounded albeit closed. I think differentiation, in contrast, is regular! Unfortunately, this example was omitted on page 168-169 (but appears in spades starting page 174).

Richard Price Question comment added 3/15/11. Suppose A is the Love equation integration operator with its particular kernel. That integration has fixed endpoints and therefore does not seem to fit into the current Stakgold discussion. The question is: which of the four classes above does this operator A fit into? I suspect it is not invertible and that is why you won't be able to take Ax=f and get $x = A^{-1}f$ as some kind of differential equation. But this question is still open, maybe in Stak Chap 3 we can come back to it. I just want to note it as an "open question" here in the midst of the general operator discussion.

Adjoint operator stuff (A*)

Preliminary Factoid: $(S^{\perp})^{\perp} = \overline{S}$. From a "perp" point of view, we have $S^{\perp} \bullet \overline{S} = 0$, so that a point on the *boundary* of an open set S is still perp to $S^{\perp}$. Therefore, $S^{\perp} \bullet \overline{S} = 0 \Rightarrow (S^{\perp})^{\perp} = \overline{S}$. This factoid is used on the last line on page 171 with little comment and causes the bar we are about to mention.

First, a reminder as to *why* we care about the adjoint operator A*. In the ∞ dim HS world, the Alternative Theorem takes this form (p 171): (the new element here is the bar over R_A)

Alternative Theorem:
(similar to page 153 for E^n): $(R_A)^{\perp} = N_{A^*}$ so that by factoid, $\overline{R_A} = (N_{A^*})^{\perp}$

Corollary: If R_A is a closed set, the above says $R_A = (N_{A^*})^{\perp}$ which duplicates our matrix version.

So the reason we care about the adjoint is this fact just quoted: $\overline{R_A} = (N_{A^*})^{\perp}$. If we can find the nullspace of this adjoint operator A*, we learn something about the range of A, and thus we learn for which vectors y we might expect to have a solution of Ax = y.

Complication for unbounded operators regarding the adjoint.

If A is unbounded, it is not continuous, and Riesz does not apply to functionals like T[x] = <Ax,y> so y may not exist! We want to define the adjoint of A according to <Ax,y> = <x, A*y>, where A*y = g is some vector in our Hilbert Space H. However, for unbounded A we can only state this equation (intended to be true for all $x \in D_A$) for certain vectors y, and not for all vectors y in H. In other words, for certain vectors y, we won't be able to find $g \in H$ such that <Ax,y> = <x, g> is true. Such a value of y is said to be **inadmissible**. If g exists such that <Ax,y> = <x,g>, then y is **admissible** and we write A*y = g. Therefore you can say that the space spanned by these admissible vectors y is the domain of A*, or D_{A^*}. One says that { y,g } form an **admissible pair** of y is admissible.

There is then an added technical detail. If D_A is not dense in H, then g might not be unique. Suppose $g_1 \neq g_2$ were both solutions, then we have to say $A^*y = g_1$ and $A^*y = g_2$ but we don't want to have an operator A* which maps a vector into multiple vectors in the range, so we cannot allow this. Stak shows that if D_A is dense in H, then g will be unique. So we have to limit our interest to domains of A which have the property $\bar{D}_A = H$.

So our conclusion is this about the adjoint for an *unbounded* operator A:

$\langle Ax,y\rangle = \langle x, A^*y\rangle$ when $x \in D_A$ where $\bar{D}_A = H$, and $y \in D_{A^*}$

STak makes a point that D_{A^*} includes 0, so that y=0 is admissible.

So in order to even be able to *talk about* an A* operator for unbounded A, you have to assume that $\bar{D}_A = H$ and x must be in D_A and y must be in D_{A^*} and you just hope that D_{A^*} is not an empty space.

Now let's jump ahead for a moment. Suppose we define

A is **symmetric** ≡ : $\langle Ax,y\rangle = \langle x, Ay\rangle$ for all x,y in D_A

If we assume that $D_A \subset D_{A^*}$, then for all y in D_A we can have an A* object and we can write

$$\langle Ax,y\rangle = \langle x, A^*y\rangle$$

So if A is symmetric, and if $D_A \subset D_{A^*}$, we can combine these two results and say

$\langle x, Ay\rangle = \langle x, A^*y\rangle$ for all x and y in D_A (*)

If we don't assume that $D_A \subset D_{A^*}$, then A* won't exist for certain y in D_A and we cannot write the above. We are trying to work toward a concept of A = A* as we had in the E^n world. Being able to write (*) is a major step in this direction.

Stak comments (page 172, Th2) that you can interpret the above discussion as saying

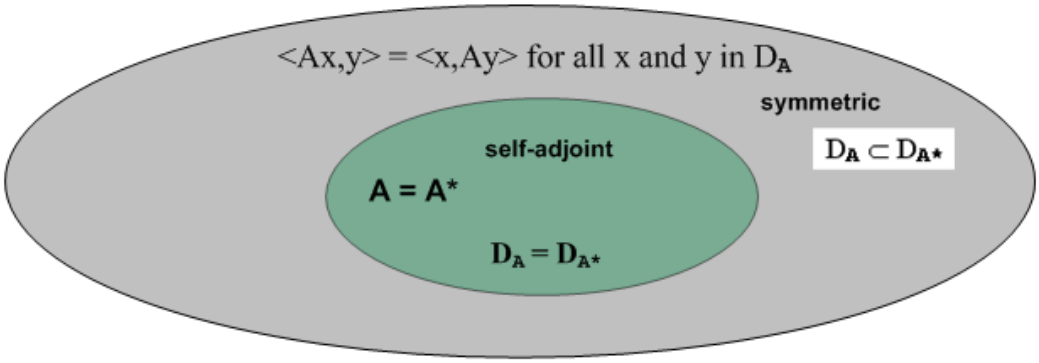
A is symmetric ⇔ $D_A \subset D_{A^*}$

That is to say, based on our definition of "symmetric", an operator can only possibly be symmetric if $D_A \subset D_{A^*}$, for only then do we know that a candidate y will an admissible value of y.

If you just add the extra condition that $D_A = D_{A^*}$, then you get A being self-adjoint. But we still have that extra required fact that $\overline{D}_A = H$, or we cannot even be talking about A*.

Now Stak says it is possible that you could have $D_A \subset D_{A^*}$ and not have $D_A = D_{A^*}$. In this case, A symmetric does not imply A self-adjoint.

Now suppose it happens that $D_A = H$. Then $D_A \subset D_{A^*}$ says $H \subset D_{A^*}$ and of course that implies that $D_{A^*} = H$, no other choice. So in this case, we have $D_A = D_{A^*}$ (= H), and then symmetric A is also self-adjoint. So any symmetric A whose domain is the whole Hilbert space is self-adjoint. Here then is my picture:



At this point, we get a very heavy dose of unproven adjoint-related claims that I don't want to get sidetracked on unless one of these claims is later needed. Here they are (A is assumed linear I presume)

(1) If $\overline{D}_A = H$, then A* exists with some D_{A^*} containing 0 (just showed this above)
(2) A* is linear
(3) A* is closed (regardless of whether A is closed), thus if A=A* self adjoint, A is closed.
(4) $B \supset A$ (operator B is an extension of A) $\Rightarrow A^* \supset B^*$
(5) if A is bounded on all of H, then so is A*
(6) If (A*)* exists [requires that $\overline{D}_{A^*} = H$] , then $A \subset (A^*)^*$
(7) If A is closable, then $A^* = (\overline{A})^*$. [The overbar means closure of operator A, not c.c.]
(8) If A is closed, then [$\overline{D}_A = H \Rightarrow \overline{D}_{A^*} = H$]] , so then A** exists.

Self-Adjoint Operators

An operator A is self-adjoint iff: (page 172)
(a) $D_A = D_{A^*}$ (which implies $\overline{D}_A = \overline{D}_{A^*} = H$)
(b) $Ax = A^*x \; \forall \; x \in D_A = D_{A^*}$.

Back in our E_n world, "self adjoint" was a matrix equality $A = A^*$ and $D_A = E_n$ = the domain of all operators in E_n, and two identical matrices of course implies (b).

$\langle Ax,y \rangle = \langle x,Ay \rangle = \langle x,A^*y \rangle$ for all x and y in D_A

Symmetric Operator Facts

We then get a cluster of theoremlets (page 172 just in the text)

Theorem 1: A is self-adjoint $\Rightarrow$ A is symmetric
Theorem 2: $A^* \supset A$ (A* is an extension of A) $\Leftrightarrow$ A is symmetric
Theorem 3: ($D_A = D_{A^*}$) + A is symmetric $\Leftrightarrow$ A is self-adjoint => (3): $Ax = A^*x \ \forall\ x \in D_A = D_{A^*}$
Theorem 4: If A symmetric on *all* of H, then A is self-adjoint. (this is the case in E_n)(see para above)

The first and third follow from my figure. I just accept the second. The third suggests that when the domains are the same, you don't really have to add the requirement that Ax = A*x on the domain. Four was already mentioned above.

EXAMPLES: This section is very long and should probably only appear in the raw notes, but I wrote these notes here because things were confusing so I leave them here. These examples are required if one is to have even the dimmest understanding how you find A* from A, and how you find R_A, uses of the Alternative Theorem, restrictions and extensions, and so on . These are very tedious because each 5 word sentence requires brain work on the reader's part with concepts that have just been loaded into the reader's data banks. The concepts are very foreign from those a physics person usually encounters.

In each case, we want to know: (1) given the A shown, what is the adjoint operator A* ? (2) what are the domains and nullspaces of A and of A* ? (3) what does the Alternative Theorem tell us about R_A given what we know about N_{A^*}? It gives the answer if the range is closed.

Example 1: [Ax](t) = tx(t) where H = $L_2(0,1)$. We find A* = A, self adjoint. <tx,y> = <x,ty> makes things pretty clear, since t is real.

Example 2: A = k1. If k is real, A is self-adjoint. Otherwise, get $A^* = \bar{k}\ 1 \neq A$. In this case, we are again saying <kx,y> = <x,ky> if k is real.

Example 3: $[Ax](t) = \int_0^t ds\ x(s)$. We find $[A^*x](t) = \int_t^1 ds\ \bar{x}(s)$ so clearly $A \neq A^*$. If Ax=0, x = 0 because only the function 0 has indefinite integral 0! Thus, A has no nullspace. Similarly, A* also has no nullspace. What is R_A ? It must be a function f(t) such that f '(t) = x(t). Thus, the range is not all of L_2, but is the subset of L_2 which is differentiable functions which Stakgold calls absolutely continuous functions. Range is thus not closed but is dense in L_2. [Since R_A is not closed, we *cannot* use the alternative theorem to say $R_A = (N_{A^*})^{\perp}$ = all of L_2, which is consistent with what we just found.] This operator is singular as case (3) since the inverse we know is unbounded.

Example 4: $[Bx](t) = \int_0^t ds\ x(s) - \lambda\ x(t) = (A - \lambda I)(t)$, $\lambda \neq 0$. In this case, B still has no nullspace, but now the R_B is all of L_2. In fact, this operator meets *all* the requirements for B to be **regular**: no nullspace so inverse exists; range is all of H; B^{-1} is bounded. I think this last item he proves explicitly by constructing the solution of Bx = f explicitly.

Example 5: [Ax](t) = dx/dt. This is **the Mother of all Examples** (p174-179), filling 5 full pages of the text! This is the one we are going to be VERY interested in for the rest of Stakgold's two volumes! A new idea to me (it shouldn't be!) is that we are going to regard boundary conditions like x(0) = 0 as restrictions on the domain, beyond the usual restriction to absolutely continuous functions. He is going to study 7 different boundary conditions, which each get a label like IV in the text! Each different BC along with A = d/dt really describes a different operator A!

(a) If there are no BC's, we have only condition I (x is absolute continuous), and he calls the operator A_1. He first shows that A_1 is closed. He then asks what are the admissible pairs {y, g} such that $\langle A_1x,y\rangle = \langle x,g=A_1^*y\rangle$. He shows that in order for y to be admissible, it must satisfy condition II which is that y(0) = y(1) = 0. In this case, the adjoint operator turns out to be $A_1^*y = g = -dy/dt$ and of course this minus sign arises from a parts integration and the condition II makes the parts go away. Obviously $A_1 \neq A_1^*$. So the domain of A_1^* is conditions I + II. A_1 has a nullspace since x(t) = C is in it. But A_1^* has no nullspace because y(t) = C is ruled out by the boundary conditions. Range of A_1 is all of L_2 hence closed. The alternative theorem says in this case that $R_{A1} = (N_{A1^*})^{\perp}$, and since $(N_{A1^*}) = 0$ only, we conclude that the R_{A1} = all of L_2. If we find a solution of Ax = f for some f, we can always add to it a solution of Ax = 0 which in this case is x = C. Thus we get $x(t) = \int_0^t ds\, f(s) + C$

(b) Add a domain restriction **x(0) = 0, condition III**, call the operator A_2 . We find A_2 is closed, as was A_1. In looking for A_2^*, we again ask what are the admissible pairs. This time due to III, we only need y to satisfy y(1) = 0 (condition IV) to make the parts vanish, and again $A_2^*y = g = -dy/dt$. With conditions III and IV, this time we find that the nullspaces of both A_2 and A_2^* are empty, R_{A2} is again closed, and we again conclude that R_{A2} = all of L_2 by the same argument as in (a) above. In this case, for given f we get the unique solution $x(t) = \int_0^t ds\, f(s)$; there is no C to add since A_2 has no nullspace. [The only possibility for having a nullspace is if x = C or y=C can survive the boundary conditions.]

(c) This time start again with A_1 but add the domain restriction **x(0) = x(1)** [**condition V**], call the operator A_3 which yet again is closed. In order to make those parts vanish this time, we need the condition y(0) = y(1) which is the same as condition V, so $D_{A3} = D_{A3}^*$. Both A_3 and A_3^* have some nullspace since x=C is legal. Given this nullspace, what is $(N_{A^*})^{\perp}$? Well, we need $\langle x(t), C\rangle = 0$, and that means that $\int_0^1 dt\, x(t) = 0$. So functions with zero integral are in $(N_{A^*})^{\perp}$. The range of A is closed, so we get to use the alternative theorem without the bar, $R_A = (N_{A^*})^{\perp}$, and we thus learn that for A_3, the range is functions whose integral is zero! $x(t) = \int_0^t ds\, f(s) + C$ is the solution here.

(d) Start again with A_1 but add the domain restriction **x(0) = 0 and x(1) = 0 [condition VI]** and call the operator A_4. This condition is really a special case of condition V, but it alone now makes the parts vanish, so D_{A^*} has no conditions on it other than condition I and $A_4^* = -d/dt$ as in all previous cases. Now the equation $A_4^*y = 0$ has y= C as its non-trivial solution while $N_{A4} = 0$ due to VI. As in example

(c), we find $(N_{A*})^{\perp}$ consists of functions such that $\int_0^1 dt\ y(t) = 0$ [the "consistency condition"] And since $N_{A4} = 0$, we don't add a constant, so solution is $x(t) = \int_0^t ds\ f(s)$.

(e) This time our condition is **x(0) = 0 and x'(0) = 0** [**condition VII**] and call it A_5 . When Stak writes this sequence $A_4 \subset A_2 \subset A_1$ he is saying that we have been restricting A_1 more and more with our boundary conditions. A_1 had none, A_2 had x(0) = 0, and A_4 had both x(0) = 0 and x(1) = 0. That is, we are restricting the domain more and more. And the domain of the A*'s gets more extended at the same time. Clearly $A_5 \subset A_2$ because we added x'(0) = 0. But in this case, it will turn out that instead of the expected $A_2^* \subset A_5^*$, we will get $A_2^* = A_5^*$ (he proves this at length). This fact tells us A_2^* has no nullspace, and we can see that A_2 also has no nullspace. He claims that the R_{A5} is not all of L_2 so it must be that this range is not closed (else alt theorem would say range = all of L_2 since nullspace N_{A2*} empty, as usual in many of the above examples). It turns out that the range condition is f(0) = 0.

2.9 The spectrum of an operator (pp 180-184)

The first step is to review the "classification" of closed operators into one regular and three singular cases, we copy from above, replacing A with B:

(1) B is regular, meaning B^{-1} exists, $\overline{R_B} = H$, and B^{-1} is bounded
(2) B^{-1} does not exist, nontrivial Bx=0;
(3) B^{-1} exists *and* B^{-1} is unbounded and $R_B \neq H$ but $\overline{R_B} = H$.
(4) B^{-1} exists *and* R_B does not close to H so there are points in H which lie outside our range.

Next, we focus now specifically on the operator

$B \equiv (A - \lambda I)$

where A is a closed linear operator on a domain D_A that is dense in H. We imagine that as we let λ vary over the complex plane, we might "hit on" any of the above cases. The set of λ values that gives each of the above cases has a name. Although we might be thinking about operator B, these names refer to the operator A. Notice that the word "spectrum" applies only to the three singular cases here. So you can say that { set of all values of λ } = { resolvent set | the spectrum }.

Point, continuous and residual spectra; the resolvent set.

(1) the **resolvent set of A** (B is regular)
(2) the **point spectrum of A** (λ_i are the eigenvalues, B has some nullspace, B^{-1} does not exist.)
(3) the **continuous spectrum of A** (B^{-1} unbounded, R_B dense in H)
(4) the **residual spectrum of A** ($H - R_B = (R_B)^{\perp}$ has dimension called **the deficiency of λ.**)

In E_n, either an operator B is regular, or it is singular and B^{-1} does not exist (det B = 0). These are the first two cases above. The latter two cases arise only in ∞ dim HS problems.

We now get four heavy duty theorems:

Theorem 1: If λ is in the residual spectrum of A, then $\bar{\lambda}$ is an eigenvalue of A* which has geometric multiplicity m $\equiv$ the deficiency of λ.

The proof is stated in 3 lines, I follow it, but it omits this logic which I now add: B* has a nullspace of dimension m (= number of indep eigenvectors of A*), so nullity $\nu(B^*) = m$. The Alt Theorem says $(R_B)^{\perp} = N_{B^*}$ so the dimension of $(R_B)^{\perp}$ is also m. But this is, by definition, the deficiency of this point λ. Note that the number m as first defined here is the geometric multiplicity of A*, and that for our ∞ dim HS stuff, there is no meaning to the term algebraic multiplicity, so we just say "multiplicity".

Theorem 2: If A is symmetric, then the eigenvalues of A are real, and <x,Ax> is real for x in D_A. (simple proof, same as in matrix world).

Corollary 12A: From Theorem 1, if λ is in the residual spectrum of A, then $\bar{\lambda}$ is an eigenvalue of A*. For a self-adjoint case A = A*, eigenvalues of each operator are the same and real, so if λ is in the residual spectrum, it must be an eigenvalue of A. But obviously such a point is in the point spectrum of A, so we have a contradiction. Therefore, for a self-adjoint A, the residual spectrum is empty.

At this point, we get two interesting lemmas which are used to prove theorem 3.

Lemma 1 says: If B^{-1} is unbounded, you can construct a sequence x_n in D_B of unit norm whose image sequence in R_B is a null sequence, and in particular, such that $\| Bx_n\| < 1/n$. If B^{-1} were bounded, we know that such a range null sequence would have to be null in the domain, so in that case this lemma could not apply. It seems very reasonable.

Lemma 2 says $\|Bx\|^2 \geq \text{Im}(\lambda)^2 \|x\|^2$ which was easy to show using the CSI.

Theorem 3: The *continuous* spectrum of a *symmetric* operator A is limited to the real axis.

The proof is readable and quite peculiar. If $\eta \neq 0$, then for all norm-1 sequences x_n we have $\|Bx_n\| \geq |\eta|$ according to Lemma 2. This means there can be no range null sequence as postulated in Lemma 1 (unless $|\eta| = 0$). So if $\eta \neq 0$, then B^{-1} must be bounded, but in this case λ is not in the continuous spectrum by definition of same! So if λ is in the continuous spectrum, we must have $\eta = 0$ meaning λ = real.

Theorem 4: The entire spectrum of a *self-adjoint* operator lies on the real axis, and there is no residual spectrum, meaning that case 4 never occurs.

This is just Theorem 3 plus our Corollary 12A above.

EXAMPLES.

Example 1: A = integral operator. In our Example 4 above, we showed that in this case B = regular unless $\lambda = 0$, so $\lambda = 0$ is the only point in the spectrum of this A. In the raw notes, we show that λ is not an

eigenvalue because nullspace of A is empty: Ie, Ax = 0x (λ=0) only has solution x = 0, no eigenvector here. And since A* also has no nullspace, we know $(R_{A^*})^{\perp}$ = null so λ cannot be in the residual spectrum. Therefore, it must be in the continuous spectrum.

We are now going to have a reprise of the various versions $A_{1,2,3,4}$ of the d/dt operator, as we studied above in Example 5.

Example 2(a). A_1 (no other BC's). Since dx/dt = λx has solution $e^{\lambda t}$ for every λ, then every complex or real λ is an eigenvalue! Fascinating. Here the *point* spectrum is the entire complex λ plane, and other two spectra sets are empty, and so is the resolvent set. So here the point spectrum is actually a continuous set.

Example 2(b). A_2 = d/dt with x(0) = 0. For this problem, since x(t) = C $e^{\lambda t}$ there are *no* eigenvalues, so no point spectrum. Using the usual Green's method (remember that the Green's function respects the BC's) , equation (A - λI)x = f has a solution for any f, so there can be *no residual spectrum* (full range). And it turns out that $(A - \lambda I)^{-1}$ is bounded which means *no continuous spectrum*. [Integral operator is bounded.] Conclusion: the *entire spectrum is empty*! That means that every λ is regular.

Example 2(c), A_3 = d/dt with x(0) = x(1). Point spectrum is $\lambda = 2\pi i n$ n = 0,1,2,... (or negative). Since different BC, the Green's is now different, but we still get the full range result (so no residual spectrum) and $(A - \lambda I)^{-1}$ is bounded (so no continuous spectrum) , so A_3 has ONLY this point spectrum. Other λ are regular and so are in the resolvent set.

Example 2(d). A = d/dt with x(0) = 0 = x(1). As in (b), no eigenvalues so no point spectrum. New situation is that now not all f of Ax = f are allowed, so we can have a residual spectrum. The adjoint operator A* allows y = C in it's domain (see example 5.d), so the range of A is functions of 0 integral. But also the adjoint of B which is B* = A* - $\bar{\lambda}$I, which has nullspace solutions exp(-$\bar{\lambda}$t), so that the range of B is then functions which are orthogonal to this expo which eliminates a lot of functions from the range of B! Since B^{-1} is bounded, there is no continuous spectrum. We don't have $\bar{R}_B$ = H I guess, so there is no resolvent set of regular λ values. Turns out all λ are in residual spectrum.

2.10 Completely Continuous Operators. (pp 184-187)

definition: A **set** S of elements in a HS is **bounded** if ||x|| < c for all x in S.

Theorem: Set S is finite $\Rightarrow$ S is compact (since any sequence must have a convergent subsequence)

Theorem: set is compact $\Rightarrow$ set is bounded.

This is not obvious to me from the above compact definition, and Stak does not prove it. Obviously true in a finite-dim space where compact = closed + bounded. A contrapositive proof is no doubt best. I would have to show that unbounded implies there is a sequence without a convergent subsequence. If a set is unbounded, I just make a sequence that marches off forever in one of the directions in which it is unbounded. Think for example of the positive integers. Find a convergent subsequence! There is none, it just keeps on going forever without converging. Therefore not bounded => not compact, so compact must imply bounded.

In some of Stakgold's statements on page 185 top, I think he assumes that sets are **closed**. For example, the BW theorem says that a subset of R_n is sequentially compact iff it is bounded and closed. This caused me much confusion for a while.

Theorem: set is bounded $\not\Rightarrow$ set is compact.

There is a very standard example always given here. The φ_n are regarded as some orthonormal functions in a sequence, such as the normed Legendres on (-1,1). Since orthonormal, $\| \varphi_n \| = 1$, so the set of φ_n is clearly bounded. Riesz-Nagy point out that $\|\varphi_n - \varphi_m\| = 2$ for $n \neq m$ since orthonormal, so pretty clear that there can be no convergent subsequence here! So this set of φ_n is bounded but not compact. This does seem to require an infinite dimensional Hilbert Space. In a finite dimensional space, we know that bounded + closed $\Rightarrow$ compact. In a sense you might argue that the set of φ_n here is not "closed", but that is not really the right language. It is not a problem with the limit of a convergent sequence being missing from our set (being on the boundary), it is that there IS no limit.

Theorem: Operator A is bounded $\Leftrightarrow$ A transforms bounded sets into bounded sets
Theorem: Operator A is continuous $\Leftrightarrow$ A transforms bounded sets into bounded sets

Same theorem, and I proved it both ways in the raw notes.

Definition: Operator A is **completely continuous** $\Leftrightarrow$ A transforms bounded sets into compact sets.

Example: The identity operator is not completely continuous. It maps our φ_n set mentioned above into itself. The φ_n set was bounded but not compact.

Example : We will see that Hilbert Schmidt integral operators are completely continuous, which is why we care.

Theorem 1. If A is completely continuous and ϕ_n is infinite orthonormal sequence, then $A\phi_n \rightarrow 0$.

So here is our old friend the φ_n sequence again. We know from the definition that the image sequence has to be compact so has to have I think an accumulation point. But this theorem which appeals to Riemann-Lebesgue, says that the image has such an accumulation point at 0 and converges there.

Implication of theorem 1: certain non-convergent sequences (like φ_n above) get mapped into convergent ones. But that is not good for the inverse operator! It now maps a convergent sequence into a non-convergent one!

Theorem 2: If A is completely continuous and A^{-1} exists, then A^{-1} is unbounded if in ∞ dim space.

The proof of this one is easy. We just consider our usual sequence φ_n and we end up with $A\phi_n \rightarrow 0$, so with respect to A^{-1} we have boundedness set by $\| \varphi_n \| / \| A\varphi_n \| = 1/\| A\varphi_n \| \rightarrow \infty$, so A^{-1} is unbounded.

Theorem 3: Consider $A_n \rightarrow A$. If the A_n are completely continuous, so is A (some restriction).

This has a long 1/2 page proof which I will skip for now (1.18.09) . $A_n \rightarrow A$ means we are approximating in the norm so that $\| A_n - A \| \rightarrow 0$. The R-N book also discusses this theorem. I think Stakgold (1967) just got all this stuff from their book (1955 English).

Comment: This subject (completely continuous) is mentioned in my Riesz-Nagy book on page 177. Reisz says that he invented the idea in 1917 (reference 9)

Comment: This section on completely continuous operators was very painful for me, largely due to the word "closed" being omitted in several places. Also, we don't have a single example of a c.c. operator yet, and even the identity operator is not one! I am sure he will reference this section later in the integral equations chapter.

2.11 Extremal Properties for Bounded Operators (pp 187 - 190)

This section is a set of six theorems most of which have simple proofs that I review in the raw notes.

Theorem 1: If A is bounded and $D_A = H$, then $\| A \| = \| A^* \|$.

Theorem 2: If A is bounded and $D_A = H$, then the ratio $|\langle Ax,x\rangle|/\|x\|^2 \leq \|A\|$, for all $x \neq 0$.

This ratio I called I_4 in the E_n earlier section of these notes. And I said $|||A||| = \max I_4$ over all x. So this theorem says, in my language: $|||A||| \leq \|A\|$. So for a bounded operator on the full domain, this triple bar norm is in general ≤ the official double bar norm.

Corollary: Obviously, if M_A is the largest that $|\langle Ax,x\rangle|/\|x\|^2$ can go as you vary x, then $M_A \leq \|A\|$. So M_A is a different notation for my $|||A|||$ "norm" thing.

Theorem 3: If λ is in any of the three kinds of spectra for A, then $| \lambda | \leq \| A \|$.

This generalizes a result we had in the E_n section with a similar conclusion.

Theorem 4: If A is symmetric, $M_A = \|A\|$. { that is to say, $|||A||| = \|A\|$ }

Theorem 5: If A is symmetric, there exists a domain sequence x_k with $\|x_k\| = 1$ which, in the limit, would become an eigenvector of $Ax = \lambda_1 x$ where $\lambda_1 = \pm \|A\|$. BUT, this theorem allows that the limit itself might not exist, so theorem is stated as a $\lim Ax_k \rightarrow \lim \lambda_1 x_k$.

Theorem 6: If A is symmetric *and* completely continuous, then: (p 190)
(1) At least one of these numbers is an eigenvalue: $\|A\|$ or $-\|A\|$.
(2) There is no eigenvalue μ for which $|\mu| > \|A\|$.

What exactly is this saying? Suppose μ_1 is the eigenvalue having the largest absolute value of all the eigenvalues. We don't know the sign of μ_1. But we know that $|\mu_1| = \|A\|$.

the largest eigenvalue is $\pm \| A \|$.

The proof is simple. By adding the completely continuous condition we are then able to show that the limit in the previous theorem exists, and then λ_1 really is an eigenvalue and $x_k \rightarrow x$ converges.