Stakgold Chapter 4 Meta Notes PhL 5.2.09

The book runs 259-322 = 63 pages. My raw notes are 38 pages, meta notes are 16 pages, and these meta meta notes are 5 pages.

The Spectral Theory of Second Order Differential Operators

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4.1 Introduction to the Regular Boundary Value Problem (259)

Stak opens this chapter with a long example of a "regular BV problem", the string problem,

$L_\lambda = (L - \lambda)$ where $L = -D^2$, interval is $(0,\ell)$, BCs are $u(0)=0$ and $u(\ell)=0$

He finds the EV's λ_k and EF's φ_k of $L_\lambda u=0$, and the φ_k form a complete set. He then solves the inhomo $L_\lambda u=f$ to get u(x) below, and then some Big Gun results are developed: [s(x) = 1 in this example]

$u(x) = \Sigma_k\, u_k\, \varphi_k(x) = \Sigma_k\, f_k\, \varphi_k(x)/\,(\lambda_k-\lambda)$ (00) $= \int d\xi\; g(x|\xi;\lambda)\, f(\xi)$

$-(1/2\pi i) \oint d\lambda\; u(x;\lambda) = f$ (0) $f_k \equiv \int d\xi\; \varphi_k(\xi)\, f(\xi)$

$-(1/2\pi i) \oint d\lambda\; g(x|\xi;\lambda) = \delta(x-\xi)$ (1) // special case of (0)

$g(x|\xi;\lambda) = \Sigma_k\, \varphi_k(\xi)\, \varphi_k(x)/\,(\lambda_k-\lambda)$ (2) // the bilinear expansion

$\delta(x-\xi) = \Sigma_k\, \varphi_k(x)\varphi_k(\xi)$ (3) // insert (2) into (1)

$g(x|\xi;\lambda) = [\, 1/\sqrt{\lambda}\, \sin(\sqrt{\lambda}\,\ell)\,]\, \sin(\sqrt{\lambda}\, x_<)\, \sin(\sqrt{\lambda}\,(\ell - x_>)$ // poles at λ_k

$g(x|\xi;0) = [\, 1/\ell]\, x_<\, (\ell - x_>)$ // $\lambda\to 0$ limit of above

He shows the "integral equation" corresponding to our ODE which contains $g(x|\xi;0)$. If $\lambda=\lambda_i$ he shows that the solution of $L_\lambda u=f$ is $u(x) = \Sigma_{k\neq i}\, u_k\, \varphi_k + A\, \varphi_i$ and consistency condition $f_i = 0$ on range. He then

does a very strange thing: he expresses the solution of $L_\lambda v = 0$ with α,β inhomo BC's in terms of the φ_k which has 0,0 homo BC's. There are problems at the endpoints so don't get uniform convergence. This example ODE system associates with the Fourier Sine Series Transform, discrete spectrum, one way is sum, the projection is an integral. There are 3 exercises and one is to repeat the above with u'=0 BC's so you get the Fourier Cosine Series Transform.

4.2 The General Regular Boundary Value Problem (268)

The first step here is to "generalize" the example of the previous section, So here is "regular":
- Real functions p(x) and q(x) are added, such that L is always formally self-adjoint, L = L*
- The interval (a,b) must be finite.
- Function p(x) must be positive in the interval
- A new real "measure" function s(x) is added, and it too must be positive in the interval
- The HS is generalized from L_2(real) to L_2(complex) with change in the inner product
- The BC's at each end are unmixed with real coefficients, causing L to be fully self-adjoint

The unmixed BC situation causes the system to be fully self-adjoint and $\langle Lu,v\rangle = \langle u,Lv\rangle$ for any two functions u and v in the domain of L. The BC's force $W(u,\bar{v}; a) = W(u,\bar{v}; b) = 0$.

Facts: For an L satisfying our "regular" requirements:
- EV's are real
- EF's of different EV's are orthogonal when weighted by s(x) $\quad L\varphi_n(x) = \lambda_n s(x)\varphi_n(x)$
- EV's are denumerable, which means we have a discrete spectrum. (HS is separable).
- Only possible accumulation point is $\lambda = +\infty$.

We construct our Green's in a very standard manner

$$g(x|\xi; \lambda) = [1/C(\lambda)]\, w(x_<, \lambda)\, z(x_>,\lambda) \qquad C(\lambda) = p(x)\, W[\, w(x,\lambda), z(x,\lambda)\, ; x\,]$$

and then the zeros of C(λ) are the poles of g(x|ξ; λ) which are the EV's. Stak goes through the usual integral equation form to show that EV's form a complete set with weight s(x). The Big Gun results are:

$$u(x) = \Sigma_n\, \varphi_n(x) \langle f,\varphi_n\rangle/(\lambda_n - \lambda) \qquad \text{// notice that inner product does not contain s}$$

$$g(x|\xi; \lambda) = \Sigma_n\, \varphi_n(x)\, \bar{\varphi}_n(\xi)/(\lambda_n - \lambda)$$

$$-(1/2\pi i) \oint d\lambda\, g(x|\xi; \lambda) = \delta(x-\xi)/s(x) = \Sigma_n\, \varphi_n(x)\, \bar{\varphi}_n(\xi) \qquad \text{and corresponding orthogonality with s}$$

We then get some exercises. Initial value BC causes Volterra form. Number of negative eigenvalues is finite due to the accumulation rule quoted above. Normal form, with no linear D term. Heat in a rod with end point parameterized by β, EV equation is transcendental. Discussion of Bessel over order ν = 0.

Clarification: How the Major Results are adjusted when weight s(x) ≠ 1.

I have given names like "inhomo2" to mimic "inhomo Fred 2", but of course these are ODE's, not integral equations, so the Fred word does not apply. The L_λ operator is used because it generates every kind of equation we are interested in, to wit:

operator:	$L_\lambda = L - \lambda s$	$L = -\partial_x(p\partial_x) + q$	
Fred general:	$L_\lambda u = f$	$Lu - \lambda su = f$	// inhomo 2
Set f = 0:	$L_\lambda \varphi = 0$	$L\varphi = \lambda s\varphi$	// homo 2 = the EV problem
Set λ = 0:	$L_0 u = f$	$Lu = f$	// inhomo1
Set f = δ:	$L_\lambda g = \delta$	$Lg - \lambda sg = \delta$	// Green's function problem with λ

Orthogonality:	$\langle \sqrt{s}\,\varphi_n, \sqrt{s}\,\varphi_m \rangle = K_n \delta_{n,m}$	
Jump condition:	jump $(\partial_x g) = -1/p$	
Green's solution:	$g(x\|\xi;\lambda) = [1/C(\lambda)]\, w(x_<,\lambda)\, z(x_>,\lambda)$	$C(\lambda) = p(x)\, W[\, w(x,\lambda), z(x,\lambda)\, ; x\,]$
Integral equation:	$\mu u = \int d\xi\, s(\xi)\, g(x,\xi)\, u(\xi)$	$\mu = 1/(\lambda-\theta)$
Green's as separable:	$g(x,y) = \Sigma_n \varphi_n(x)\varphi_n^*(y)/(\lambda_n-\lambda)$	
Completeness:	$-(1/2\pi i) \oint d\lambda\, g(x\|\xi;\lambda) = \delta(x-\xi)/s(x) = \Sigma_n\, \varphi_n(x)\, \bar{\varphi}_n(\xi)$	
inhomo 2 solution:	$u_n = F_n/(\lambda_n-\lambda)$ $\quad u(x) = \Sigma_n\, u_n\varphi_n(x)$	$F(x) = f(x)/s(x) = \Sigma_n\, F_n\varphi_n(x)$

So you see that s(x) appears in these places: (1) the EV equation and the Green's equation; (2) integrals of orthogonality and of the integral equation. (3) completeness. ν

Comment on Consistency Conditions.

Your first task is to find the eigenfunctions of L, call them φ_n. Then to solve inhomo2 you just expand everything which appears in $Lu - \lambda su = f$ and if λ is not an EV, you get $u_n = f_n/(\lambda_n-\lambda)$, all done. But suppose $\lambda = \lambda_1$, say, one of the EV's. Then look what happens if f has a φ_1 component. Look just at that component. You have then $Lu - \lambda_1 su = \varphi_1$. The solution u cannot have any φ_1 component because we know that $L\varphi_1 - \lambda_1 s\,\varphi_1 = 0$, but we need to get φ_1, not 0. Could solution u have a φ_2 component? We would then need to get $L\varphi_2 - \lambda_1 s\,\varphi_2 \sim \varphi_1$. But in fact $L\varphi_2 - \lambda_1 s\,\varphi_2 = L\varphi_2 - \lambda_2 s\,\varphi_2 + (\lambda_2-\lambda_1)\varphi_2 = (\lambda_2-\lambda_1)\varphi_2$, so not $\sim\varphi_1$. Similarly for $\varphi_{3,4\ldots}$. Thus, if f has a φ_1 component, there can be no solution because nothing on the LHS can generate φ_1! The consistency condition is that $f_1 = 0$.

4.3 Introduction to the Singular Boundary Value Problem: Examples (283)

If you allow the interval (a,b) to become infinite, for example with $b\to\infty$, the discrete pole spectrum can become a branch cut on the real axis. Here he looks at our *same* Section 4.1 example but on $(0,\infty)$. It has a branch cut, and gives rise to the Fourier Sine Transform, integrals both directions. If u(0) = 0, there are no normalizable EF's of $L_\lambda u=0$! He redoes all the main results as a limit $\ell \to \infty$. Here are some results:

$\varphi_\nu(x) = \sqrt{2/\pi}\,\sin(\nu x)$ with $\nu^2 = \lambda$ as a "non-normalizable" EF.

$g = (1/\nu)\sin(\nu x_<)\exp(i\nu x_>)$ and $\quad [g] = (2i/\nu)\sin(\nu x)\sin(\nu\xi)$

$\delta(x-\xi) = \int d\nu\ \varphi_\nu(x)\varphi_\nu(\xi)$ $\qquad$ $g(x|\xi;\lambda) = \int d\nu\ \varphi_\nu(x)\varphi_\nu(\xi)/(\nu^2 - \lambda)$

He shows that if $\mathrm{Im}\lambda \neq 0$, the inhomo has usual solution $u = Gf$. Alternatively, he solves the inhomo by a diagonalization method, similar to doing a Laplace Transform idea. He then dives into 5 pages of exercises. He talks a little about our L and L_λ as being the A and B = A-λ of Chapter 2. [Notice that the "regular boundary value problem" has "singular points" λ_i which are in the point spectrum of L.] In general, when λ is off the real axis, the operator L_λ is "regular" and such λ is in the "resolvent" set of L. This has nothing to do with some z being an "ordinary point" of an ODE !! One of his examples produces the Fourier Cosine Transform, and yet another the full Fourier Transform. He does a heat problem that has a branch cut plus one pole on the negative λ axis.

Comment: Notice that when an endpoint goes to ∞ or $-\infty$, we no longer worry about that endpoint having a BC. Just requiring normalizability for the Green's at such an endpoint is condition enough.

4.4 The General Singular Boundary Value Problem (295)

This is a very long 33 page chapter section and contains the chapter's most complicated stuff. The notion of a "singular point" is just a little vaguely presented. One example is an endpoint which goes to ∞, but another example is a finite endpoint at which $p(x) = 0$ or $s(x)$ is singular. The issue really is the convergence of an integral at a problem endpoint, whether finite or infinite. Lack of convergence for a candidate solution function $u(x)$ means it is not "finite s norm" and thus not a solution.

Weyl's Theorem. First we have the theorem, then second we have implications. The theorem says two things: Part I: If $L_\lambda u=0$ has two s-norm solutions at any one value of λ, then it has two s-norm solutions at all λ. Part II: For any λ off the real axis, $L_\lambda u=0$ has at least one s-norm solution. In limit-point that is all there is, and in limit-circle there are two s-norm solutions. For λ = real, there might be zero s-norm solutions.

The Part II world involves lots of tricky steps which I outlined in the meta notes. We assume one singular endpoint "b" and one regular endpoint "a". At the regular endpoint we impose correlated BC's on two generic solutions ψ and φ to make them independent, then we consider $u = \varphi + m\psi$. We take the limit $b_0 \to b$ while maintaining the BC $p(b_o)W(u,\bar{u};b_0) = 0$ and we find that $m = m(\lambda)$ is either a circle or a point in the m-plane, based on its radius. The radius is 0, causing the limit point case, if the ψ s-norm integral diverges, causing u to be the only s-norm solution. In limit circle, both ψ and u are solutions for any point on the limiting circle.

Part I tells us this: if we are limit-circle at some particular λ, then we are limit-circle at all λ. So either you are limit-circle at all λ, or you are not. Stak fails to state this clearly. The usual idea is to pick some simple test value of λ such as $\lambda = 0$ and ask: "for this λ, are there, or are there not, two s-norm solutions?". If there is only 0 or 1 s-norm solution, then we are not limit-circle. In this case, we are limit-point which means for any λ off the real axis, there is exactly one s-norm solution. In this limit point case, for λ *on* the real axis, there may be either 0 or 1 solution depending on the value of λ -- but we know there cannot be two solutions for any such real λ.

Stak then considers various special function ODE examples. Remember that usually λ is something added to a special function ODE, and you have to set λ to some special value to get the official special

function ODE, but you can set it to some other convenient value to do the circle vs point test. And of course you have to do this test at both endpoints if they are both singular.

1. $L = -D^2$ on $(0,\infty)$ is limit point at the ∞ endpoint, only one s-norm solution (decaying expo).

2a. **Bessel ν=0** on (0,b) is limit circle at the 0 endpoint, solutions are $J_0(\sqrt{\lambda}\, x)$ and $N_0(\sqrt{\lambda}\, x)$.
2b. **Bessel ν=0** on (a,∞) is limit point at the ∞ endpoint, solution is $H_0^{(1)}(\sqrt{\lambda}\, x)$.

3. **Hermite** on $(-\infty,\infty)$ is limit point at both endpoints, so only one s-norm solution for $\mathrm{Im}\lambda \neq 0$. In this example, the test value used is $\lambda = -1$. We later learn that there are some s-norm solutions for certain λ on the real axis, namely, $\lambda = 2k+1$ with k = integer (ie, these are the EV's)

4. **2D Laplace Radial** on $(0,\infty)$ is limit point at both endpoints, $s(x) = 1/x$, $\lambda=0$ is the test value.

5. **Legendre** on (-1,1) is limit circle at both endpoints, two s-norm solutions are $P_\ell(x)$ and $Q_\ell(x)$ where $\lambda = 2\ell + 1$ with ℓ being any complex value you want. Test value here is $\lambda = 0$.

Limit Point Analysis

The plan here for Green's is to use ψ and the regular end and u at the singular end, so

$$g(x|\xi;\lambda) = -[1/(\alpha_1^2 + \alpha_2^2)]\,\psi(x_<,\lambda)\,\{\varphi(x_>,\lambda) + m(\lambda)\,\psi(x_>,\lambda)\} \qquad \text{// 4.93 p 303}$$

Here, $m(\lambda)$ could have discrete or continuous stuff, so can get discrete or continuous spectra! For limit point, the solution to $L_\lambda u = f$ is usual answer $u = G(sf)$.

Then we have two examples showing limit point situations:

1. **Bessel general order ν≥1 on (0,1) .** Is Bessel equation when $\lambda = 1$, limit circle at 0 endpoint $\nu < 0$, but is limit point when $\nu \geq 1$. In this case he builds the Green's for $\lambda = 0$, does integral equation, φ_i set is complete as usual. Here $\varphi_i(x) = k\ J_\nu(\sqrt{\lambda_i}\, x)$ are the only s-norm solutions, zeros of J_ν determine the λ_i. So this limit-point problem is entirely discrete, get the **Fourier Bessel Transform**.

2. **2D Laplace Radial** on $(0,\infty)$ is limit point at both endpoints, $s(x) = 1/x$, $\lambda=0$ is the test value. This example has only a continuous spectrum and gives the **Mellin Transform**. Variable change takes you to the normal Fourier Transform.

Limit-Circle Analysis

Let b be the singular end, and impose $p(b)\, W(u, \bar{u}; b) = 0$ there. Here I show how the two end points are treated from the BC point of view:

Green's problem $L_{\lambda 0} g = \delta_\xi$ with $L_{\lambda 0} = L - s\lambda_0$. Here are the two BC's:

$$B_a(g) = 0 \text{ at } x=a \qquad \text{same as} \qquad p(a)\, W[g,\bar{g}; a] = 0 = p(a)\, W[\psi, \bar{\psi}; a]$$

----- at x=b (limit $b_0 \to b$): p(b) W[g,$\bar{g}$; b] = 0 = p(b) W[u, $\bar{u}$; b]
= p(b) W[g, $\bar{u}$; b]

Inhomo problem $L_{\lambda 0} v = sf$. Here are the two BC's:

$B_a(v) = 0$ at x=a same as p(a) W[v,$\bar{v}$; a] = 0
----- at x=b (limit $b_0 \to b$): p(b) W[v,$\bar{u}$; b] = 0

When the homo problem is cast into an integral equation, kernel is symmetric and H-S so we know we can only have a discrete spectrum. Thus, limit circle always goes with a purely discrete spectrum.

Example: **Bessel with ν=0 on (0,1).** Here 0 is limit-circle. In this problem he refers to u as φ. When λ = 0 he gets ψ = log(x) and φ = -1 + A log(x) as the two limit-circle s-norm solutions. The EF's are a complete set. Our BC at the singular a endpoint is p(a) W[w,$\bar{\varphi}$; a] = 0 involves arbitrary parameter A and can be written as in p 313 4.125a. For A = 0 it is just $\lim_{x \to 0}$ xw'(x) = 0. Only the $J_o(\sqrt{\lambda_i}\, x)$ meet this requirement, ie, these are the EF's, the N_0 cannot meet BC's at both ends, see 4.126. Same φ_i really as we got in our earlier treatment above.

And now comes the final gasp of exercises.

1. **Bessel ν=0 on (0,∞)** is limit circle at 0, but limit point at ∞. Use A = 0. Get J_0 on the left and Hankel on the right of Green's, this leads to the **Hankel Transform** of order ν=0, which is all continuous. Thus, the limit point "won out" in the sense that only it allows a continuous spectrum component.

2. **Bessel ν=ν on (0,∞)** and this gives the **Hankel Transform** of order ν, same idea.

3. **2D Laplace Radial** on (0,1) is limit point at endpoint 0. We do the Green's and end up here with the **Mellin Sine Transform** which is all continuous. Can convert it to Fourier Sine Transform.

4. Change one BC and get **Mellin Cosine Transform**.

5. **Kantorovich-Lebedev Transform** . Is purely continuous, involves K functions.

6. **Modified Green's** Approach. Do this when you need to work at $\lambda = \lambda_i$. Use g_M instead of g.

7. **Legendre in (-1,1)** is limit circle at both ends. Obtains P_ℓ from truncation requirement. Then does an example here of a modified g_M approach for λ = 0.

8. **Hermite on (-∞,∞)** is limit point at each end. Finds complete set of EF's $u_k(x) = \exp(-x^2/2)H_k(x)$ and derives various oddball Hermite properties.

9. **Bessel order ν=0 on (a,∞).** We know from above this is limit point at ∞. Our exercise is to "do everything" for this problem.

10. **Schrodinger Equation for 1D particle in square well.** This problem has a mixed spectrum with a finite number of bound states on the negative λ = E axis. He only assigns this as a problem to study.

Comment: In case the main point was missed: For each ODE and for each interval and for each set of (unmixed?) BC's at the regular end(s), you have "a problem" (a Sturm-Liouville Problem). You can do the following things:

1. Construct the Green's function $g(x|\xi;\lambda)$ using an appropriate homo solution at "each end" and evaluating the jump to get the constant. This thing will have poles, a cut, or both. These poles and cuts are the "spectrum" of our ODE system, ie, of our operator L_{λ} with BC's. They are the "eigenvalues" of our problem $L_{\lambda}u = 0$.

2. Throw the Green's thing into the LHS of :

$$(1/2\pi i) \oint d\lambda\, g(x|\xi;\lambda) = -\delta(x-\xi)/s(x) = -\Sigma_{n} \varphi_{n}(x)\, \overline{\varphi}_{n}(\xi) - \int d\nu\, \varphi_{\nu}(x)\, \overline{\varphi}_{\nu}(\xi) \qquad // (4.95)$$

Since this yields an expansion for $\delta(x-\xi)$, it always implies some kind of "transform". One direction of the transform is always an integral as the projection, the other side might be an integral or a series or both! Using this method, you can actually find the normalized eigenfunctions φ_{n} and/or φ_{ν}. In the φ_{ν} case things are generalized functions which might not be officially normalizable.

3. In the regular boundary value problem, the spectrum is always discrete and you always get a complete set of eigenfunctions $\varphi_{n}(x)$ of $L_{\lambda}u = 0$ with eigenvalues λ_{n}.

4. In the singular boundary value problem, situation will be either limit-circle or limit-point and you can determine which using Weyl's Theorem at a convenient value of λ.

If limit-circle, the spectrum is discrete. If limit-point, spectrum could be discrete or continuous or both. You might have limit-point and limit-circle in the same problem for two endpoints. Regardless, you go ahead and construct the Green's $g(x|\xi;\lambda)$ using "appropriate" finite s-norm homo solutions at each "end", and then you apply the above contour integral and you come up with some transform.

5. We can explicitly display the singularities of the Green's this way:

$$g(x|\xi;\lambda) = \Sigma_{n} \varphi_{n}(x)\, \overline{\varphi}_{n}(\xi)/(\lambda_{n} - \lambda) + \int d\nu\, \varphi_{\nu}(x)\varphi_{\nu}(\xi)/(\nu^{2} - \lambda) \quad = \text{poles + branch cut}$$

Comment added 8.12.09: Sturm and Liouville.

Stak has neglected to mention the names **Sturm and Liouville** which are strongly associated with the subject matter of this chapter, which seems very odd. Here is a great Wiki summary of the connection:

Sturm–Liouville theory

From Wikipedia, the free encyclopedia

In mathematics and its applications, a classical **Sturm–Liouville equation**, named after Jacques Charles François Sturm (1803–1855) and Joseph Liouville (1809–1882), is a real second-order linear differential equation of the form

$$-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = \lambda w(x)y. \qquad (1)$$

where y is a function of the free variable x. Here the functions $p(x) > 0$ has a *continuous derivative*, $q(x)$, and $w(x) > 0$ are specified at the outset, and in the simplest of cases are continuous on the finite closed interval $[a,b]$. In addition, the function y is typically required to satisfy some boundary conditions at a and b. The function $w(x)$, which is sometimes called $r(x)$, is called the "weight" or "density" function.

The value of λ is not specified in the equation; finding the values of λ for which there exists a non-trivial solution of **(1)** satisfying the boundary conditions is part of the problem called the **Sturm–Liouville problem** (S L).

Such values of λ when they exist are called the eigenvalues of the boundary value problem defined by **(1)** and the prescribed set of boundary conditions. The corresponding solutions (for such a λ) are the eigenfunctions of this problem. Under normal assumptions on the coefficient functions $p(x)$, $q(x)$, and $w(x)$ above, they induce a Hermitian differential operator in some function space defined by boundary conditions. The resulting theory of the existence and asymptotic behavior of the eigenvalues, the corresponding qualitative theory of the eigenfunctions and their completeness in a suitable function space became known as **Sturm–Liouville theory**. This theory is important in applied mathematics, where S–L problems occur very commonly, particularly when dealing with linear partial differential equations that are separable.

So first you have to have the right "**equation**" which is set up so that L is Hermitian (self adjoint). The endpoints (a,b) could be finite or infinite. Then we have the "**problem**" which is finding the EV's λ_i and the corresponding EF's. A Hilbert Space is defined by the problem, L^2 on the interval, with the weight function w (wiki) or s (Stak). The study of the solutions and general related analysis is the "**theory**", and this would include Weyl's theorem.

In my world, this S-L business is of course associated with the subject of "**special functions**" and the nature of their singular points. Each one has an S-L equation of the form (1), and each has solutions for certain λ values. And don't forget the associated transform.

You see the general 1850 time frame on this subject.

Whittaker and Watson talk a lot about specific special functions, and they have a chapter on ODE's which is sort of Frobenius-oriented. But they make no mention of Sturm or Liouville! Bateman also says nothing, has no "general theory" at all. M&M say nothing as well. So Stakgold is the only book I own that talks about Sturm-Liouville Theory!!! [but now I have others....]