Stakgold Chapter 4 Meta Notes PhL 4.29.09

The Spectral Theory of Second Order Differential Operators

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4.1 Introduction to the Regular Boundary Value Problem (259)

The operator here is $L_{\lambda} = (L - \lambda)$ where $L = -D^2$, interval is $(0,\ell)$, BCs are $u(0)=0$ and $u(\ell)=0$. Stakgold does many things in this section, I will try to list them concisely. His idea is to introduce some pretty fancy notions with a rock solid specific example always in full view.

1. The homo is $L_{\lambda}\varphi = 0$ which has EV's λ_k and EF's φ_k. We know from Chapter 3 that the φ_k form a complete set. [This always requires converting to an integral equation and showing k is sym and H-S.]

2. We then consider $L_{\lambda}u = f$ and expand u and f on the complete set φ_k. This produces a diagonalized equation which is simply $(\lambda_k-\lambda)u_k = f_k$, where u_k and f_k are projections onto the φ_k. As long as $\lambda \neq \lambda_k$, our problem is completely solved as $u(x) = \Sigma_k\, u_k\, \varphi_k = \Sigma_k\, f_k\, \varphi_k/\, (\lambda_k-\lambda)$.

3. This last result directly tells us that $-(1/2\pi i) \oint d\lambda\, u(x;\lambda) = \Sigma_k\, f_k\, \varphi_k = f$.

4. If we apply 3 for $L_{\lambda}u = f$ to $L_{\lambda}g = \delta_{\xi}$ we get $-(1/2\pi i) \oint d\lambda\, g(x|\xi;\lambda) = \delta(x-\xi)$. We also get $f_k = \varphi_k(\xi)$, and then $g_k = f_k/(\lambda_k-\lambda)$ so $g = \Sigma_k\, g_k\, \varphi_k = \Sigma_k\, \varphi_k(\xi)\, \varphi_k(x)/\, (\lambda_k-\lambda) = g(x|\xi;\lambda)$. So with very little effort we quickly obtain these Big Gun results: [which we shall slightly generalize in the next section]

$$-(1/2\pi i) \oint d\lambda\, g(x|\xi;\lambda) = \delta(x-\xi) \qquad (1)$$

$g(x|\xi;\lambda) = \Sigma_{\mathbf{k}}\, \varphi_{\mathbf{k}}(\xi)\, \varphi_{\mathbf{k}}(x)/\, (\lambda_{\mathbf{k}}-\lambda)$ (2)

$\delta(x-\xi) = \Sigma_{\mathbf{k}}\, \varphi_{\mathbf{k}}(x)\varphi_{\mathbf{k}}(\xi)$ (3) // insert (2) into (1)

5. Now we do for the first time what we will do about 20 times later in this chapter. We "manually" construct the Green's function as our "first step", then see where it leads us. For this problem we get

$$g(x|\xi;\lambda) = \left[\, 1/\sqrt{\lambda}\, \sin(\sqrt{\lambda}\, \boldsymbol{\ell}) \,\right] \quad \sin(\sqrt{\lambda}\, x_<) \quad \sin(\sqrt{\lambda}\, (\boldsymbol{\ell} - x_>)$$

which is the prototype for the very many cases we get later. The left factor meets the left BC, the right factor meets the right BC, the constant comes from the jump condition, each factor is a homo solution. We examine the λ-plane structure of this $g(\lambda)$ and see the poles arising from the constant. We then use our all purpose result $-(1/2\pi i) \oint d\lambda\, g(x|\xi;\lambda) = \delta(x-\xi)$ and observe its "standard form" (3), then we know at once the normalized $\varphi_{\mathbf{n}}$ as in p 263 4.12. When we talk about the "spectrum" of L, we are really talking about the singularities in λ of $g(x|\xi;\lambda)$. Each singularity (pole or later cut) contributes to the "completeness" condition for $\delta(x-\xi)$ and is thus part of the "spectrum".

Remarks : At this point, we encounter a set of 5 remarks which I will call R1 through R5

R1. Another way to derive our solution u(x) given above in item 2 above is to use $u = G_{\boldsymbol{\lambda}} f$ with (2) above:

$$u(x) = \int d\xi\, g(x|\xi;\lambda)\, f(\xi) = \int d\xi\, \Sigma_{\mathbf{n}}\, \varphi_{\mathbf{n}}(x)\varphi_{\mathbf{n}}(\xi)/\, (\lambda_{\mathbf{n}}-\lambda)\, f(\xi) = \Sigma_{\mathbf{n}}\, \varphi_{\mathbf{n}}(x) \,/\, (\lambda_{\mathbf{n}}-\lambda) \int d\xi\, \varphi_{\mathbf{n}}(\xi)\, f(\xi)$$

$$= \Sigma_{\mathbf{n}}\, f_{\mathbf{n}}\, \varphi_{\mathbf{n}}(x) \,/\, (\lambda_{\mathbf{n}}-\lambda)$$

That is to say, we have $L_{\boldsymbol{\lambda}} u = f$ so $u = G_{\boldsymbol{\lambda}} f$.

R2. Here he writes out our homo and inhomo ODEs in "standard integral equation form". He notes that our current solution to our current problem replicates a solution we found back in Chapter 3. Notice that it is always $g(x|\xi;\lambda=0)$ that appears. Here is why: [G is the inverse of L]

$L_{\boldsymbol{\lambda}} u = f \quad \Rightarrow \quad Lu = f + \lambda u \quad \Rightarrow \quad u = G\,(\, f + \lambda u) = F + \lambda G u$

R3. What happens if $\lambda = \lambda_{\mathbf{i}}$? The solution of $L_{\boldsymbol{\lambda}} u = f$ is $u(x) = \Sigma_{\mathbf{k}\neq\mathbf{i}}\, u_{\mathbf{k}}\, \varphi_{\mathbf{k}} + A\, \varphi_{\mathbf{i}}$. In this situation, operator $L_{\boldsymbol{\lambda}\mathbf{i}}$ has a null space with $L_{\boldsymbol{\lambda}\mathbf{i}}\varphi_{\mathbf{i}} = 0$ so we get a consistency condition $f_{\mathbf{i}} = 0$ and that is why that specific i term is knocked out of the $\Sigma_{\mathbf{k}\neq\mathbf{i}}$ and replaced with $A\varphi_{\mathbf{i}}$: $u_{\mathbf{i}}$ is not determined by $(\lambda_{\mathbf{i}}-\lambda)u_{\mathbf{i}} = f_{\mathbf{i}}$. Another way to interpret this is we are adding a homo solution to a particular solution. Later we shall see this method formalized in terms of modified Green's functions.

R4: This is a strange section. He changes the BC's for the homo and inhomo equations to α and β, but he continues to "work with" the $\alpha=\beta=0$ basis functions $\varphi_{\mathbf{n}}(x)$. I would have found the new homo basis functions I think for the α,β BC's, but I think he is trying to make a point here. You CAN use the $\varphi_{\mathbf{n}}(x)$

and you CAN diagonalize the problem as in p 265 C, and you CAN get a homo solution as in 4.18. But this solution cannot possibly work at the endpoints of the interval since $\varphi_n(x) = 0$ there, so we do not get *uniform* convergence of our $\varphi_n(x)$ series solution on our interval. We get convergence in the mean only. This is a subject that got lots of discussion earlier in the book.

R5: Here he uses our last example to show that term by term differentiation is not legal in a situation where the endpoints are "screwed up". The shift in BC's causes trouble, and maybe it is exactly this non-uniformity of convergence that is the problem.

P6: This problem yields a Fourier Sine Series Transform as per page 260 B. It is tuned to functions on the interval $(0,\ell)$. It is usually expressed on $(0,\pi)$ and is the full Fourier Series Transform for periodic functions, but limited to functions f(x) which are odd!

Exercises (267)

This is a small set of only 3 items. The main idea is that we could alter our BC's from u(a)= 0 to u'(a) = 0 and then the whole ball of wax involves cos instead of sin for the $\varphi_n(x)$. And if we go do a general unmixed BC, we would get a combination of sin and cos.

4.2 The General Regular Boundary Value Problem (268)

The first step here is to "generalize" the example of the previous section, So here is "regular":
- Real functions p(x) and q(x) are added, such that L is always formally self-adjoint, L = L*
- The interval (a,b) must be finite.
- Function p(x) must be positive in the interval
- A new real "measure" function s(x) is added, and it too must be positive in the interval
- The HS is generalized from L_2(real) to L_2(complex) with change in the inner product
- The BC's at each end are unmixed with real coefficients, causing L to be fully self-adjoint

Fact: If two functions u and v satisfy the unmixed BC at end a, then $W(u,\bar{v}; a) = 0$. This is not quite obvious, and I prove it in the raw notes where it says Repair. Same at end b.

Fact: If two functions u and v are in the "domain D of L" , they must satisfy the BC's at both ends, and therefore $W(u,\bar{v}; a) = W(u,\bar{v}; b) = 0$. Perhaps $Lu = f_1$ and $Lv = f_2$. Notice that u and v need not be solutions say of Lu=0 or of any other ODE. They are just in the domain of L.

Fact: For u and v in the domain of L, we know that $\langle Lu,v\rangle = \langle u,Lv\rangle$ because these differ only by the Wronskians shown in 4.29b p 269. That is why unmixed BC's cause L to be fully self adjoint on D.

Facts: For an L satisfying our "regular" requirements:
- EV's are real
- EF's of different EV's are orthogonal when weighted by s(x)
- EV's are denumerable, which means we have a discrete spectrum. (HS is separable).

On page 271 we are given the outline of the method we use over and over again. We construct a Green's function solution to the Green's ODE problem as $g(x|\xi;\lambda) = A(\lambda)\, w(x_<,\lambda)\, z(x_>,\lambda)$ where w satisfies the left BC and z the right BC, and both w and z are EF's of the homo equation $L_\lambda u = 0$ ($L_\lambda = L - \lambda s$). In order to find $A(\lambda)$, we apply the usual jump condition which says $\Delta(dg/dx) = -1/p(\xi)$. If we write $A = 1/C$ we get

$$g(x|\xi;\lambda) = [1/C(\lambda)]\, w(x_<,\lambda)\, z(x_>,\lambda) \qquad C(\lambda) = p(x)\, W[\, w(x,\lambda), z(x,\lambda)\, ;\, x\,]$$

where we replace ξ with general x in the $C(\lambda)$ formula according to the self-adjoint-L Abel rule we learned on page 72. So for the regular problem, w and z are analytic in λ and $C(\lambda)$ can have zeros λ_i, and those are the poles of $g(x|\xi;\lambda)$ which we know (general form later) are the EV's of $L_\lambda u = 0$. In our many examples and exercises to come, we almost always start out exactly as outlined here! Notice from the above:

$$0 = C(\lambda_i) = p(x)\, W[\, w(x,\lambda_i), z(x,\lambda_i)\, ;\, x\,] \quad \Rightarrow \quad w(x,\lambda_i) = k_i\, z(x,\lambda_i) \qquad // \; 4.45$$

On page 272 Stak carefully converts our ODE system to an integral equation and then invokes Chapter 3 Theorem 6 on page 220 which says the EF's of a self-adjoint ODE form a **complete set** (weight s(x)). Somewhere along this path, the fact that L is "regular" is required. For example, finite a,b make the HS requirement relatively simple. Also, regular lets you find a non-EV real λ he always calls θ. I don't think in Chapter 3 that Stak consider "singular" integral equations in the sense we will soon be investigating singular ODEs.

At this point Stak goes on to write our generalized Big Gun results including s(x), then he applies everything to the example where the BC's have primes on them so we get cos instead of sin everywhere. This just lets us apply all our of "rules" to an example.

I will now insert a summary section directly from my raw notes.

My summary of the Regular Boundary Value Problem.

1. The **general form** of the ODE system must be as shown bottom page 269 with all the "**conditions**" there listed. $L = L^*$ always, and $L_\lambda = L_\lambda^*$ when λ real.

2. The **BC's** must be the **unmixed** ones, one for x=a and one for x=b, as shown 4.26, again with the "conditions" as listed there. These determine the domain D_L.

3. The difference between $\langle Lu,v\rangle$ and $\langle u,Lv\rangle$ can be expressed in terms of Wronskians at the endpoints of the interval, such as $W(u,\bar{v}; a)$. If u and v are in the space of functions D_L which satisfy the BC's just stated, it is easy to show that both Wronskians vanish, and therefore that $\langle Lu,v\rangle = \langle u,Lv\rangle$, which says that **L is "symmetric" on D_L.**

4. The EV's of $L\varphi = \lambda s\varphi$ are **real**, the EF's are **orthogonal with weight s(x),** and the number of EV's is **denumerable**. The spectrum of EV's therefore cannot fill the entire real axis. EV problem is also $L_\lambda\varphi = 0$.

5. The "symmetric" **Green's** for $L_\lambda g = \delta_\xi$ can be constructed as (1/C) $w(x_<)\, z(x_>)$ where, as usual, w satisfies just the left side BC, and z satisfies just the right side B, and C = p W[w,z; x] . Everything here is a function of λ.

6. One can recast the EV problem $L\varphi = \lambda s\varphi$ into an **integral equation**, and then use the theory of integral equations to claim that the set of EF's $\psi_n(x) = \sqrt{s(x)}\, \varphi_n(x)$ form a **complete orthonormal set**, and that the only possibly accumulation point for λ is at $\lambda = +\infty$. That is, $L\varphi_n(x) = \lambda_n s(x)\varphi_n(x)$. We don't of course know exactly what the EV's λ_n are at this point, but they are denumerable and accumulate at $+\infty$ if they are infinite in number. [the spectrum is discrete!]

7. To say that the $\psi_n(x)$ form a complete orthonormal set is also to say that

$$\delta(x-\xi) = \Sigma_n\, \psi_n(x)\, \bar{\psi}_n(\xi) = s(x) \quad \Sigma_n\, \varphi_n(x)\, \bar{\varphi}_n(\xi) \qquad // \ 4.43$$

8. If $\lambda \neq$ EV, one can solve the inhomo $L_\lambda u = f$ to get result shown in 4.41:

$$u(x) = \Sigma_n\, \varphi_n(x) \langle f,\varphi_n\rangle/(\lambda_n - \lambda) \qquad // \text{ notice that inner product does not contain s}$$

As usual, if $\lambda = \lambda_n$, we need $\langle f,\varphi_n\rangle = 0$ as our consistency condition.
If we apply this result to $L_\lambda g = \delta_\xi$, we get 4.42,

$$g(x|\xi;\lambda) = \Sigma_n\, \varphi_n(x)\, \bar{\varphi}_n(\xi)/(\lambda_n - \lambda) \qquad // \text{ here are your EV's and normalized EF's !}$$

9. If we apply great circle $\oint d\lambda$ to the above formula (and use our δ result above) we get

$$-(1/2\pi i) \oint d\lambda\, g(x|\xi;\lambda) = \delta(x-\xi)/s(x) = \Sigma_n\, \varphi_n(x)\, \bar{\varphi}_n(\xi)$$

10. Now that we see from 8 that $g(x|\xi;\lambda)$ has a pole at each eigenvalue, when we write
$g(x|\xi;\lambda) = w(x_<;\lambda)z(x_>;\lambda)/C(\lambda)$ it is not hard to conclude that C(λ) must have a simple zero at each eigenvalue, and that near such an eigenvalue $\lambda \approx \lambda_n$ we will have $C(\lambda) \approx C'(\lambda_n)(\lambda-\lambda_n)$. Since we are then saying $C(\lambda_n) = 0$, we must from p 272A have $W[w(x_<;\lambda_n)z(x_>;\lambda_n); x] = 0$ which says the functions w and z are linearly dependent when $\lambda = \lambda_n$, and we write $w(x,\lambda_n) = k_n\, z(x,\lambda_n)$ as in 4.45. It is then only a small step to relate the functions w and z to our functions φ_n:

$$\varphi_n(x) = \pm\, [k_n C'(\lambda_n)]^{-1/2}\, w(x,\lambda_n) = \pm\, [k_n/C'(\lambda_n)]^{1/2}\, z(x,\lambda_n)$$

Degeneracy Question: From our integral equation theory we knew that the multiplicity of an eigenvalue μ had to be finite, except for the case μ = 0 which could have an infinite multiplicity. In our ODE work to this point, he seems to be assuming that the λ_n all have multiplicity 1. We saw this to be true in our prototype long example, and it seems to be true in our various examples. It seems odd that he has never commented on this fact. I am sure degeneracy will arise when we go to > 1D problems. So I will just state this question and then let it ride for the timing being.

Exercises: (276-283) // selected comments only

4.5 If you use **initial-value** type BC's, the ODE casts into a **Volterra** integral equation

4.7. In this very detailed exercise, the conclusion is that there can only be a **finite number of negative eigenvalues** and that $\lambda = +\infty$ is the only accumulation point. If we think of the 1D **Schrodinger Equation** where $\lambda = E$, this tells us that we can only have a finite number of bound states. For $E > 0$ we get a branch cut but this is because we allow a, b = $(-\infty,\infty)$ which removes us from the "regular" ODE situation. But for the H atom, we get an infinite number of bound states. The radial equation there must be singular to allows us to have an infinite number of bound states. Saxon p 286 71 shows the ODE where r = ky and it appears to me that p = 1 and s = 1 but $q(y) = -2Z/y + \ell(\ell+1)/y^2$ and interval $(0,\infty)$. So here we have "irregularity" possibly with the sign of q (which is supposed to be positive) and with q(y) blowing up at the singular endpoint y=0. The solutions are related to associated Laguerre functions. This would be a good ODE to study from the Stakgold point of view in the singular realm.

4.10 Relates to casting an ODE into "**normal form**" which has $L = D_t^2 + Q(t) - \lambda S(t)$.

4.11 This is a very long example of **heat in a rod** $(0,\ell)$ where the left end is held at zero temperature, but the right end BC is parameterized by **variable β**. We first form the Green's as usual. The problem divides into three cases depending on the sign of λ. If $\lambda < 0$, we get a single negative eigenvalue. If $\lambda > 0$, we have the usual infinite number of positive EV's. We can then write the $\delta(x-\xi)$ expansion for different ranges of parameter β. In this problem, all EV's are determined by **transcendental** equations. This situation arises also in the 1D square well SE problem.

4.12 The ODE here is **Bessel order 0** and solutions are $J_0(kx)$ and $N_0(kx)$ where $k^2 = \lambda$. If we force simple 0 at each endpoint (a,b), we get a transcendental equation for the eigenvalues which includes these special functions!

4.3 Introduction to the Singular Boundary Value Problem: Examples (283)

We now redo our $L = -D^2$ example on some more dangerous intervals.

On the full $(-\infty,\infty)$ there are no normalizable EF's. We might be able to think of the non-normalizable EF's as generalized functions, but Stak does not wander off in that direction here.

On the interval $(0,\infty)$ we know there is one expo EF that normalizes at the high end. However, if we want u(0) = 0 at the low end, this expo is not a true EF, so same problem as with $(-\infty,\infty)$. But we can go ahead and make a Green's function using this expo at the high end. And if we want u(0) = 0 at the low end, we can use a sin there, so we can construct a Green's in the usual manner as shown in 4.53 p 285. But now we have $C(\lambda) = \sqrt{\lambda}$ so for the first time in this book, g has a **branch cut** on the positive real axis 0-- the coalescence of all the poles as $\ell \to \infty$. If we throw this g into our usual g contour integral, out pops our first "continuous spectrum" result for $\delta(x-\xi)$, see below. This is the basis for the **Fourier Sine Transform**, where both directions are integrals.

Stakgold redoes everything at this point as a **limiting process** $\ell \to \infty$ and we get the same results.

I will again insert a summary section directly from my raw notes.

My summary of the Introductory Singular Boundary Value Problem.

1. The ODE is just $L_\lambda = -D^2 - \lambda$, but the interval is taken as $[0,\infty)$ and the only BC is $u(0)=0$. With this BC, the only possibly EF (expo decay) is ruled out, so there are *no L_2 eigenfunctions at all*! However, we might in some sense think of $\varphi_\nu(x) = \sqrt{2/\pi}\sin(\nu x)$ with $\nu^2 = \lambda$ as a "non-normalizable" EF. There is surely some connection to "distribution theory" here, but Stakgold has chosen not to shine any light on it at this time. Note that this $\varphi_\nu(x)$ meets the BC requirement that $u(0) = 0$.

2. A good Green's Function $g = (1/\nu)\sin(\nu x_<)\exp(i\nu x_>)$ can be constructed. First, note that

$$\nu = \sqrt{\lambda} = |\lambda|^{1/2} e^{i\theta/2} \quad \text{where } \theta = \arg(\lambda) \text{ runs 0 to } 2\pi.$$

Then

$$\exp(i\nu x) = \exp(ix\,|\lambda|^{1/2}\{\cos\theta/2 + i\sin\theta/2\}) = \exp(ix\,|\lambda|^{1/2}\cos\theta/2)\exp(-x\,|\lambda|^{1/2}\sin\theta/2).$$

The only place this $\exp(i\nu x)$ will be non-integrable is when $\theta = 0$ which is the positive real λ axis. So, this Green's is square integrable for all other λ.

3. The above Green's has a branch cut on the positive real λ axis with this discontinuity:

$$[g] = (2i/\nu)\sin(\nu x)\sin(\nu\xi) \qquad \text{where here } \nu \text{ just means } |\lambda|^{1/2} \text{ since } \theta = 0.$$

4. The great circle integral theorem is still valid:

$$(1/2\pi i)\oint d\lambda\, g(x|\xi;\lambda) = -\delta(x-\xi)$$

5. We distort the contour to pick up the discontinuity along real λ and we get this "completeness",

$$\delta(x-\xi) = \int d\nu\, \varphi_\nu(x)\varphi_\nu(\xi) \qquad (4.58a) \qquad \text{where} \quad \varphi_\nu(x) = \sqrt{2/\pi}\sin(\nu x)$$

which we can compare to our previous result (and $s = 1$ in our introductory singular problem)

$$\delta(x-\xi) = \Sigma_n\, \psi_n(x)\,\bar{\psi}_n(\xi) = s(x)\,\Sigma_n\, \varphi_n(x)\,\bar{\varphi}_n(\xi) \qquad (4.43)$$

6. The Green's function can be written in this form [derived in 10 below]

$$g(x|\xi;\lambda) = \int d\nu\, \varphi_\nu(x)\varphi_\nu(\xi)/(\nu^2 - \lambda) \qquad (4.67)$$

which we compare to our previous result

$$g(x|\xi;\lambda) = \Sigma_n\, \varphi_n(x)\,\bar{\varphi}_n(\xi)/(\lambda_n - \lambda) \qquad (4.42)$$

7. The above completeness relation serves as the basis of the Fourier Sine Transform:

$$F_s[\nu] = \int_0^\infty dx\, f(x)\, \varphi_\nu(x) \qquad (4.60) \qquad \text{projection}$$

$$f(x) = \int_0^\infty d\nu\, F_s[\nu]\, \varphi_\nu(x) \qquad (4.61) \qquad \text{expansion}$$

$$\int_0^\infty dx\, |f(x)|^2 = \int_0^\infty d\nu\, |F_s[\nu]|^2 \qquad (4.63) \qquad \text{Parseval's equality}$$

8. All of the above results can be obtained using $\ell \to \infty$.

9. The inhomo equation $L_\lambda u = f$ has the unique solution $u = G_\lambda f$ with the above Green's. This is true for λ off the positive real axis which then means g is square integrable so the integral $G_\lambda f$ will converge. This is **Theorem 1**, and $u = G_\lambda f$ is our "normal" solution to the inhomo problem which was $L_\lambda u = f$. So in some sense it seems that we can write $G_\lambda = L_\lambda^{-1}$ for λ off the real axis.

10. Alternatively, the inhomo $L_\lambda u(x) = (-D^2 - \lambda)\, u(x) = f(x)$ can be diagonalized by the Fourier Sine Transform to give: [special case of Fourier Transform which is group diag wrt group T(1)]

$$\nu^2 U_s[\nu] - \lambda\, U_s[\nu] = F_s[\nu]$$

which can be solved

$$U_s[\nu] = F_s[\nu] / (\nu^2 - \lambda) \qquad \text{where } F_s[\nu] = \int_0^\infty dx\, f(x)\, \varphi_\nu(x)$$

then

$$u(x) = \int_0^\infty d\nu\, U_s[\nu]\, \varphi_\nu(x)$$

There results all look just like the discrete results we had before, such as $u_n = f_n/(\lambda_n - \lambda)$! This last is similar to our normal use of the Laplace Transform. The ODE is converted into a simple polynomial thing as derivatives become powers of the conjugate variable. (This is all **Theorem 2**)

11. If we apply the above to $L_\lambda g = \delta_\xi$, we get $\delta_s[\nu] = \varphi_\nu(\xi)$ and then $g = \int_0^\infty d\nu\, \varphi_\nu(x)\, \varphi_\nu(\xi) / (\nu^2 - \lambda)$ which is the result we quoted above in item 6.

12. One gets the impression that every ODE of this type with $b = \infty$ will give some kind of "transform" similar to the Fourier Sine Transform associated with the above example singular problem.

Exercises: (290-295, 5 pages)

4.14 Checking single integral of $k(x,\xi)$ to see if K is bounded.

4.15 Apply this to our big sin example, G is not H-S but G is bounded. HS relates to the double integral, being bounded need not. And boundedness is what counts below in operator classification (but H-S with symmetric is what counts for completeness of EF's).

4.16 $L_\lambda u = 0$ has nullspace symbolic function solution $\varphi_\nu(x) = \sqrt{2/\pi}\sin(\nu x)$, $\nu^2 = \lambda$. Thus, $L_\lambda u = f$ only has a solution if $\langle f,\varphi_\nu\rangle = 0$ as consistency condition. Solution will be u = particular + $A\varphi_\nu(x)$. Exercise 4.17 sets $f = e^{-x}$ and follows through. These two problems are strange, ignore them. This really needs to be done inside the distribution theory umbrella.

4.20 Here we try to tie our ODE work to our **general operator theory** of chapter 2.

(1) *for λ not on the real axis* : In 4.15 above we showed that $G = L_\lambda^{-1}$ is bounded in our $(0,\infty)$ $L = -D^2$ problem. We know that $L_\lambda u = 0$ with $u(0) = 0$ has no (normalizable, genuine function) solution, only the trivial solution. Therefore the range of $L_\lambda u = f$ is the entire L_2 HS. Thus, operator L meets the three criteria for such λ to be "**regular**" points for L: ($B = L_\lambda = L - \lambda$, L = A) and such λ are in the **resolvent set** of L.

B Regular requires:

(a) $Bx=0 \Rightarrow x=0$, as with matrices; and this implies that B^{-1} exists;

(b) the range must be *the entire* Hilbert space, so Bx = f has a solution for any f in H;

(c) B^{-1} must be bounded (= continuous).

(2) *for λ on the positive real axis*, $G = L_\lambda^{-1}$ becomes unbounded, and this puts us in the **continuous spectrum** classification for L. So yes, branch cut on positive λ axis puts those λ into continuous spectrum, officially!

(3) the **continuous spectrum of A** (B^{-1} unbounded, R_B dense in H)

The issue of R_B dense in H is mentioned in the exercises, it is met.

4.21 The **Fourier Cosine Transform** Example. $(0,\infty)$ u'(0) = 0

4.22 The full **Fourier Transform** presented two ways. $(-\infty,\infty)$

4.24 This is a heat problem which has a **single pole on the negative λ axis** + continuous piece.

Comment: Notice that when an endpoint goes to ∞ or -∞, we no longer worry about that endpoint having a BC. Just requiring normalizability for the Green's at such an endpoint is condition enough.

4.4 The General Singular Boundary Value Problem (295)

Several ideas here. First is the notion of a Hilbert Space H_s with $\langle u,v\rangle_s$ and $\|u\|_s$. The domain of L is the set of L_2 functions which are of **finite s-norm**. The second idea is that we should take limits to approach singular points $(a_0, b_0) \rightarrow (a,b)$ (separately). We want to maintain the idea, as we do these limits, that pW = 0, as we had in the regular BV problem case, and then as usual $\langle u,Lv\rangle_s = \langle Lu, v\rangle_s$ = symmetric. Thirdly, the ideas A,B,C on page 296 apply to any u,v in the domain D of L. These three ideas

become A',B',C' on page 297 when we apply them to a solution of $L_\lambda\varphi = 0$. These items will be used in the proof of Weyl's Theorem.

Weyl's Theorem. This is a little slippery and was not stated very clearly. First we have a theorem with two parts, but then we have some facts that immediately derive from this theorem.

Weyl's Theorem: [1910] Let "x=b be a singular point" (this can have several meanings). Two claims are made:

Part I: Suppose for *some* λ (call it $\lambda=\lambda$) our ODE (with no BC's) has two independent solutions which are both of finite s-norm. Then for *any other* λ (call it $\lambda=\mu$), there are also two independent solutions of finite s-norm. This encourages you, when studying a case, to pick a λ which makes the equation have simple solutions.

Part II: For any λ off the real axis, our ODE has at least one finite s-norm solution. In the limit-point case there is only one finite s-norm solution, while in the limit-circle case there are two finite s norm solutions. The cases limit-point and limit-circle are mutually exclusive, and the meaning of these word pairs is not known to the reader until he reads through all the steps below in the proof of part II.

The proof of Part I though not trivial is conceptually simple, you just do it.

The proof of Part II brings many things into play and needs much comment just to explain the general lay of the land.

(1) The main idea is to start with b_0 and then take limit $b_0 \rightarrow b$. Endpoint a is assumed regular. We assume λ is some fixed value off the real axis.

(2) We imagine two homo solutions of $L_\lambda u=0$ called ψ and φ. We force (require) them to have "correlated" BC's at the non-singular endpoint "a" (bottom page 298). Either of these BC's can be regarded as a general unmixed BC at a, constants are α_1 and α_2 and p(a) is snuck in, see A and B on page 299. Notice that φ and ψ do *not* have the same inhomo BC at endpoint a, hence the term "correlated" I used above. Things have been set up very carefully so that the following facts are true: The BC for ψ implies that $p(a)W(\psi,\overline{\psi}; a) = 0$, and the BC for φ implies that $p(a)W(\varphi,\overline{\varphi}; a) = 0$. The correlation forces the two solutions ψ and φ to be independent because we find that $p(a)W(\varphi,\psi; a) = -(\alpha_1^2 + \alpha_2^2) \neq 0$. [See Theorem about Wronskians and BC's in raw notes, Repair.] So the upshot is that we manage to have φ and ψ be independent, and to both satisfy the same familiar fact $p(a)W(f,\overline{f}; a) = 0$ which, you will recall, is the object you get when you do parts integration and you try to do a "swing around" of operator L (or A) in your Hilbert Space (but with both functions the same) . I said above that φ and ψ do not have the same inhomo BC, but when we get to $p(a)W(f,\overline{f}; a) = 0$ form they are both the same in that regard.

(3) We attempt to construct a candidate finite-s-norm solution called u(x) which will somehow be well defined as we later take our limit $b_0 \rightarrow b$. This u(x) must be a lincomb of φ and ψ, so write $u = \varphi + m\psi$.

(4) We impose a general unmixed BC at b_0 on $u = \varphi + m\psi$, constants β_1, β_2. This BC forces the quantity $p(b_o)\, W(u, \overline{u}; b_0) = 0$ as in 4.82a (Theorem "Repair" in raw notes). Later we shall see that the details of

the β_i values don't matter much. The condition we really care about is that $p(b_o)\, W(u, \bar{u}; b_0) = 0$ where the β_i don't appear.

(5) We can insert $u = \varphi + m\psi$ into $p(b_o)\, W(u, \bar{u}; b_0) = 0$ and solve for $m(\lambda)$ as in 4.83.

(6) It is not obvious from 4.83, but if we were to vary $h = \beta_1/\beta_2$ I guess over $-\infty$ to $+\infty$, the point m describes a **circle** in the m-plane centered at a point A shown in 299 D. The circle fact becomes more obvious if we first rewrite 4.82a as 4.83a, which in turn can be written as $|m-A|^2 = r^2$. The radius r of the circle is given in p 300 4.84 after some computational steps. So we can think of the points on this circle as representing variation of the β_1/β_2 ratio in our b_0 end imposed BC.

(7) Now we take the limit $b_0 \rightarrow b$. The formula 4.84 for r shows there are **two possibilities**.

If the integral of $s|\psi|^2$ diverges as $b_0 \rightarrow b$, which says ψ is not finite s norm, then r=0 and circle has contracted to a point at the location m = A(b), see p 299 D. But according to p 301B, the function u(x) is finite s norm. So we can think of u(x) and $\psi(x)$ as independent functions but only u(x) is finite s norm. We conclude that in this "**limit-point**" situation, we have only one finite s norm solution to our ODE, because the other candidate one diverges at the b endpoint.

If the integral of $s|\psi|^2$ is finite as $b_0 \rightarrow b$, which says ψ is finite s norm, then r>0 and circle has reached a **limit-circle** centered at A(b). In this case, both u(x) and $\psi(x)$ are finite s norm solutions. This was all for our particular selected off-axis value of λ, but we then use Weyl Part I to conclude that there are in the limit-circle case two finite s normal solutions for ANY value of λ.

(8) In the limit-circle case, notice that for each m on the limit-circle, you get a different u(x), but the u(x) is always finite s norm.

(9) As we take the limit $b_0 \rightarrow b$, the m-circle gets smaller in diameter, and the center shifts a bit, but you get a sequence of circles in which each circle lies inside the previous circle. The circles are not concentric in general. just nested.

Examples (301) // they are all excellent!

Example 1. For our usual $\mathbf{L = -D^2}$ problem on $(0,\infty)$, we know if $\mathrm{Im}\lambda \neq 0$ there is only one finite s norm solution which is the decaying expo, so this must be a limit-point case.

Example 2. Bessel n=0. (a) Here s = x and p = x and interval is (0,b). Since p(0) = 0, a=0 is a singular endpoint. For λ=0, however, it is easy to find two finite s-norm solutions. We can then apply Part (1) of the Weyl theorem to conclude that for any λ there will be two finite s-norm solutions, and these are the $J_o(\sqrt{\lambda}\, x)$ and $N_o(\sqrt{\lambda}\, x)$ functions. So this is limit-circle case for a = 0.

(b) If you change the interval to (a,∞), then b is a singular point. For $\lambda = 0$, both our previous solutions are non finite s norm. To be in limit-circle, both would have to in fact be finite s norm, so we must be in limit-point. Remember that one solution must exist for $\mathrm{Im}\lambda \neq 0$, so none need exist for $\lambda = 0$ or on the real axis. In this case it is one of the Hankel's that is the single finite s norm solution.

Example 3. Hermite. Here s=1 p=1 q=x^2 and (-∞,∞). The trick is always to pick some λ for which you can find simple solutions. Here, λ = -1 yields two solutions, but at least one of them is non-finite s norm at both end points. Each endpoint is separately a singular point. This has to then be limit-point at each end.

Example 4: 2D Laplace Radial. Here s = 1/x and p=x and (0,∞). So here we have two singular points to think about. We take λ = 0 and identify two solutions. They both diverge at both endpoints, so here we are in limit-point at both endpoints, independently. Here it is the s = 1/x that causes divergence at the lower endpoint for the u_1 = 1 solution (for example), and you get ln(∞) divergence at the upper end as well. For Imλ ≠ 0, the single solution is $x^{-isqrt(\lambda)}$.

Example 5. Legendre. Here p = 1-x^2 s = 1 and (-1,1), Both endpoints are thus singular. Pick λ = 0, there are two simple solutions. At each endpoint separately, you have finite-s norm for each solution, so both endpoints are the limit-circle case. Here λ = -***ℓ***(***ℓ***+1) see Schaum p 146, so λ=0 means ***ℓ*** = 0 and one solution is then going to be $P_o(x)$ = 1 as noted here. The other solution is $Q_o(x)$ = (1/2)ln[(1+x)/(1-x)] just as he claims. So, we know that for any ***ℓ*** (any λ), there will always be two normalizable solutions on this interval! I guess that is something I did not know -- the Q_0 is always normalizable. If is not of course a polynomial. We don't seem to use these much in practice. [just wait till you do other coordinate systems]

The Green's Function when endpoint b is singular (303)

We take as our Green's g = A ψ($x_<$) u($x_>$), write as g(x|ξ; λ) = A(λ)ψ($x_<$,λ) u($x_>$,λ). At the non-singular a end (the ψ end) we apply our generic unmixed BC p 299B. For each m on the limit circle, or for the one m at the limit point, we know u($x_>$,λ) is finite s norm at the upper end. The jump condition tells us that A(λ) = - 1/ ($\alpha_1^2 + \alpha_2^2$) and in this case "A" does not depend on λ. But u = φ + m(λ) ψ. So

$$g(x|\xi;\lambda) = -[1/(\alpha_1^2+\alpha_2^2)]\,\psi(x_<,\lambda)\,\{\varphi(x_>,\lambda) + m(\lambda)\,\psi(x_>,\lambda)\} \qquad // 4.93 \text{ p } 303$$

When we are at b_0 before going to b, we know m(λ) is meromorphic as in p 299 4.83, so our spectrum is just the poles of m(λ), and of course spectrum is discrete. But what happens when $b_0 \to$ b? We consider the two cases separately. In each case we will have some theory followed by examples.

Limit-Point Analysis

In the limit $b_0 \to$ b, the resulting m(λ) shown above might have poles and/or a cut. If both are present, we have this general result showing a "mixed spectrum".

$$(1/2\pi i)\oint d\lambda\, g(x|\xi;\lambda) = -\delta(x-\xi)/s(x) = -\sum_n \varphi_n(x)\,\bar{\varphi}_n(\xi) - \int d\nu\, \varphi_\nu(x)\,\bar{\varphi}_\nu(\xi) \qquad // (4.95)$$

Examples of Limit-Point Cases (p 305)

Example 1. (p 305) **Bessel's Equation of general order ν (the Fourier-Bessel Transform)**

We have to set λ = 1 to get the actual Bessel equation, and ν is ≥ 0 as Schaum confirms. We have s = x and p = x and interval is (0,1) at least for this example. Since p(0) = 0, *left* endpoint 0 is singular. But set

λ=0 to get a simple situation where solutions are $\varphi = x^{\pm\nu}$. For $0 \le \nu < 1$ both solutions are finite s norm so we are limit-circle, but $\nu \ge 1$ has only one s-norm solution so is limit-point, and this is our interest in this section.

The Green's for λ=0 is shown in D with C = 1/2ν. Using this λ=0 Green's, in the usual way he writes the corresponding integral equation showing general λ and where s is "absorbed" into kernel k. Things are symmetric and HS, so get complete set of $\varphi_i(x)$.

By doing a variable change, you can show $\varphi_i(x) = k\ J_\nu(\sqrt{\lambda_i}\, x)$ are the proper EF's, they are finite s norm at the x=0 end (the N_ν are not). Applying $\varphi_i(1) = 0$ at the regular end lets us find the $\sqrt{\lambda_i}$ values from the simple equation $J_\nu(\sqrt{\lambda_i}\ ) = 0$, so $\sqrt{\lambda_i}$ are just the zeros of these functions.

He next constructs the general-λ Green's using these functions, as shown in p 306 I = 4.100. This thing it turns out has nothing but poles, at our λ_i. By computing the residues of these poles, on page 307 he comes up with the correct normalization factor for the $\varphi_i(x)$ which I just called k above, p 307 J. The δ rule is shown in I, and this leads to the Fourier-Bessel Transform which is a series expansion with an integral as the projection.

So this example produces just a discrete spectrum for the limit-point example, not mixed.

Example 2. (p 308) **The Mellin Transform**

This involves that "2D Laplace radial ODE" on (0,∞). It turns out both ends are singular, and in fact are both limit-point singular. He builds the general-λ Green's as shown in p 308 F and it has no poles but has a cut. As usual, we use our λ contour to develop a δ expansion and of course this problem then has only a continuous spectrum, and this is in fact the Mellin Transform, integral both ways. It is easy to make a variable change and convert this to the regular Fourier Transform.

So this double limit-point example produces a purely continuous spectrum.

After this second example, he considers the inhomo $L_\lambda u = f$ and the usual answer is u = G(sf).

Limit-Circle Analysis

As usual, interval is (a,b) with b the singular end. The general λ Green's has ψ at the regular end and our limit-circle u at the singular end. We normalize things so Green's is very simple as in 4.113. Remember that singularities are going to be in m(λ) within u(x), not in a leading constant (which here is 1). In our Weyl analysis, we imposed $p(b_o)\ W(u, \bar{u}; b_0) = 0$ at the b end and that is where our m-space circle came from. Here the plan is to maintain this constraint in the limit, so want $p(b)\ W(u, \bar{u}; b) = 0$ and this should correspond with the limiting circle. We don't mess with any β_1 and β_2 BC constants, this IS the BC at the singular b end.

Stak separately considers the homo and the inhomo systems and here are the conclusions:

Green's problem $L_{\lambda 0} g = \delta_\xi$ with $L_{\lambda 0} = L - s\lambda_0$. Here are the two BC's:

$B_a(g) = 0$ at x=a	same as	$p(a)\ W[g,\bar{g}; a] = 0$	$= p(a)\ W[\psi, \bar{\psi}; a]$
----- at x=b	(limit $b_0 \to b$):	$p(b)\ W[g,\bar{g}; b] = 0$	$= p(b)\ W[u, \bar{u}\ ; b]$
			$= p(b)\ W[g, \bar{u}\ ; b\ ]$

Inhomo problem $L_{\lambda 0} v = sf$. Here are the two BC's:

$B_a(v) = 0$	at x=a	same as	p(a) W[v,$\bar{v}$; a] = 0
-----	at x=b	(limit $b_0 \to b$):	p(b) W[v,$\bar{u}$; b] = 0

In both cases we put ---- to indicate we have no formal BC of the form B_b(g or v) = 0. Our BC is the fact that p(b) W[g, $\bar{u}$; b] = 0 in the Green's case, and p(b) W[v,$\bar{u}$; b] = 0 in the inhomo case. This last result is perhaps not "obvious" but he proves it is the right thing to have at the bottom of page 311.

Homo problem $L_{\lambda 0} w = 0$. Same BC's as inhomo problem with $v \to w$. He then casts this as an integral equation, shows the kernel is symmetric and H-S and thus has a discrete spectrum. This then shows that in the limit-circle case, you can only get a discrete spectrum, no continuous piece.

Example: Bessel with ν= 0 and with parameter λ (limit-circle example, p 313)

We recall from above that ν = 0 is a limit-circle situation for the Bessel system, interval (0,1) with 0 being the singular end. He constructs the λ = 0 Green's as in p 313 4.123. We use ψ = log(x) at the regular end "b" and φ = -1 + A log(x) at the singular end "a". This φ is what we have always been calling u, but here he changes it to φ which I found a bit confusing since φ had another Weyl meaning. We have just w(1) = 0 at the regular end. He constructs the λ=0 Green's in 4.123. As usual, he uses this λ=0 Green's to convert the ODE to an integral equation with general λ, and he shows that the kernel is symmetric and HS, so our EF's here will form a complete set. So what are the eigenfunctions?

Now, the condition at the singular end is here p(a) W[w,$\bar{\varphi}$; a] = 0, again u is called φ here. But we have an explicit function for φ, so our singular BC is the limit p313 4.125a and we are free to pick any real A we want. So in 4.126 he writes out our ODE system and shows this A thing as one of the BC's, all in line with the above except for the confusion between φ and u. Then on page 314 he returns to that integral equation just mentioned and claims the EF's are complete as I said at the end of the last paragraph, I have things a little out of presentation order. If A = 0, the BC becomes extremely simple: $\lim_{x \to 0} xw'(x) = 0$. It turns out that only the $J_o(\sqrt{\lambda_i}\, x)$ functions meet this requirement, not the N_0, so the $J_o(\sqrt{\lambda_i}\, x)$ are the EF's, and the λ_i are determined (as in our limit-point treatment) as the zeros of J_0 by the BC at the regular end. But this is a special case of the EF's we found in our $\nu \geq 1$ limit-point analysis of this same ODE system, with same normalization, I guess that is no great surprise.

He ends this little Example section writing down the problem for general A instead of A = 0. The equation for the EV's is a real mess, and the EF's are a mix of J_0 and N_0 . I am not sure why he did not just attack the $\nu \leq 1$ range of ν all at once here since it is all limit-circle, but OK, he used just ν = 0.

Exercises (315-322) // These involve both LC, LP and combination LP/LC !

Exercise 4.25 Bessel for $0 \leq \nu \leq 1$. OK, here is where we are supposed to on our own do the entire Bessel limit circle range $\nu \leq 1$. Pretty much things don't change at the boundary ν = 1! It's just that you have limit point on one side and limit circle on the other, and in both cases we have a discrete spectrum.

Exercise 4.26. Hankel Transform (of order ν = 0). Here we continue with the Bessel ν = 0 example, but now on (0,∞) instead of on (0,1). He shows that at 0 we are limit circle as in our example, but at ∞ we are limit point. At the 0 end with A=0 we still have the BC 4.128 so we still want a J_0 at that end. We know that only the Hankel is a solution for the right half of the Green's (recall previous work p 302 top), so the

Green's seems right as shown in p315 B. Since these Bessel functions are of known normalization, we cannot just set the Green's constant to 1, we have to compute it, and he claims the result shown there. In this problem there is a branch cut from $J_0(\sqrt{\lambda}\, x)$ that is not cancelled by the constant in g (as happened earlier), and I suspect the Hankel has no cut. He must then compute [g] and then do the g contour integral to get 4.130 which is then all continuous. The corresponding integral (both ways) transform is called the Hankel Transform (of order $\nu = 0$ here).

This example has both endpoints singular, and one is limit-point and the other limit-circle. The spectrum comes out completely continuous, so the limit-point "won".

Exercise 4.27. Hankel Transform for order ν . Just redo the previous for $\nu \leq 1$. Conclusions are all the same.

Exercise 4.28. Mellin Sine Transform On page 308 Example 2 we did Mellin on $(0,\infty)$. Here we repeat for (0,1) instead. The endpoint 1 is regular and endpoint 0 is limit point, as was shown back on page 302 Example 4. Our instructions are to find the Green's (it is shown). We insist now that $g(1) = 0$ so the Green's is a little different from what it was in the $(0,\infty)$ case. We have the usual branch cut, compute [g], and we end up with a strange transform as shown which he makes up the name **Mellin Sine Transform**. As in the previous case, a change of variable changes this into something more familiar: the Fourier Sine Transform.

This example is purely limit-point and gives a purely continuous spectrum.

Exercise 4.29. Mellin Cosine Transform Repeat the above but with $g'(1) = 0$.

Exercise 4.30. Kantorovich-Lebedev Transform. In this problem on $(0,\infty)$ with $s = 1/x$ and $p = x$ it turns out that both endpoints are limit point. We end up getting a purely continuous spectrum. The Green's is a mix of the I and K "modified" Bessel functions mentioned in the Appendix. The eigenfunctions here are combinations of sqrt sinh and K functions. Strange Russian name for the thing.

Exercise 4.31. The Modified Green's Function . Sometimes, he says, we are interested in studying a problem where $\lambda = \lambda_1$, some eigenvalue. The g_M in this problem is obtained from g by subtracting off a term corresponding to that EV. If we want or need to do this, then we can use the g_M modified Green's function to answer all our questions. This could be any generic problem where we have some L_λ as shown in 4.140. We have a homo, inhomo, and Green's equation to think about, as always. That is to say, homo = 4.140, Green's = p 318 A, inhomo not stated but we know what it is. We are always going to want to know if the φ_n of the homo form a complete set. For everything regular, we know this will be the case from earlier work, but for singular cases, we have to do an *ad hoc* check in each case. If in such a case we need to use some $\lambda = \lambda_1$ which is an EV, we can still do that "check" as follows: construct the modified Green's g_M as instructed in this exercise, then test to see if the corresponding symmetric kernel as shown in p 319 A is Hilbert-Schmidt! If it is, then our homo φ_n are a complete set. I suspect we shall see this concept used in one or more upcoming exercises.

Exercise 4.32 The Legendre Functions (319). This is on (-1,1) and both endpoints are limit-circle. He constructs the usual power series solutions (about $x = 0$) and shows that we only get bounded solutions if the even and odd solutions truncate. For any λ, only one of the two candidate solutions (even and odd)

truncates. So this is how you can discover the $P_\ell(x)$ functions. The Q_ℓ functions of course exist as well but are not analytic at x = 0, but he does not mention Q_ℓ functions.

He then uses this as a test bed for the modified Green's idea. We decide to work at $\lambda = 0$, but this is in fact an EV. So we make g_M by doing the "subtract off" trick. He finds the explicit g_M and then shows that the corresponding integral equation kernel is symmetric and H-S so the $P_\ell(x)$ are a complete set. You don't need the Q_ℓ !

Exercise 4.33 The Hermite Equation (320). Earlier on page 302 we studied this ODE a little and showed it was limit-point at each end of $(-\infty,\infty)$. For $\lambda = -1$ we know some independent solutions (involving I think the error integral), and he then comes up with two solutions which are "OK" at one or the other end (ie, integrable there) so the Greens for $\lambda = -1$ is as shown in p 320C.

We are supposed to show that the corresponding integral equation is HS, so our solutions are a complete set. Those solutions are $u_k(x) = \exp(-x^2/2)H_k(x)$. Eigenvalues are $\lambda_k = 2k+1$, fully discrete spectrum (we get this from the recursion relation truncation business again). We are then supposed to derive various Hermite properties that will no doubt get used in Volume II.

Exercise 4.34 Bessel's of order 0, with parameter λ, and on (a,∞) (321). We are asked to find "the spectral decomposition" for this "operator" (meaning ODE system). We would need to do the following: (1) construct a Green's. (2) examine its λ analytic structure to see if poles or cuts or both; (3) maybe along the way think about whether the endpoint is LC or LP (I would guess LP) ; (4) write a $\delta(x-\xi)$ formula which then shows the "spectral structure" ; (5) present this as a Transform formula. We are given no hints now because we are supposed to be Big Boys now.

Exercise 4.35 Particle in a finite square well potential. We are supposed to cast the SE into our Stakgold world with $\lambda = E$. We know there will be a finite number of poles on the left λ axis, and a branch cut on the positive λ axis. The interval is $(-\infty,\infty)$ and I guess it is limit-point at each end, and this is consistent with the fact that this example has a mixed spectrum. He just states the problem, we are supposed to solve it. I comment on solution approaches in the raw notes.

End of Stakgold Volume I, today is May 1, 2009. All meta notes are done.

Comment (put in singular section): although I have finished Stak Volume I, he has not mentioned "**regular singular points**". I wonder why this is missing from his book. It seems that this is an important part of ODE analysis. // See regular singular points etc.doc for an explanation!