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Chapter Title: The Spectral Theory of Second Order Differential Operators

4.1 Introduction to the Regular Boundary Value Problem

[p259] What we have here is an 8-page "example" which probably shows most major ideas of the section to follow. The operator is $L_\lambda = (L - \lambda)$ where $L = -D^2$ and where λ is the $1/\mu$ for the μ that appeared earlier in Fredholm's second kind inhomo2 integral equation. This λ is a "free parameter" that turns out to be useful. The BC's first considered are 0 at both ends, so unmixed. Since $L=L^*$, the system is fully self-adjoint. He states both the inhomo and homo equations with this operator. The homo for L_λ is the EV problem for L. And the inhomo for L_λ is just $L_\lambda u = f$ which is that inhomo2 thing for L. The interval is taken to be $(0,\ell)$.

[p260] The homo $L_\lambda \varphi_n$ is easily solved and we find EV's λ_n and EF's φ_n as in A. The φ_n form a complete set because we studied this case earlier, so in considering the inhomo $L_\lambda u = f$, we expand both u and f on the φ_n with coefficients u_n and f_n, as shown in B. From the theory of Fourier Transforms "I think", we know that both series converge "in the mean", as shown in C. Take $\int dx\ \varphi_n \{L_\lambda u = f\}$ and use orthonormality of the φ_n on both sides, to get D (which contains u") and then after parts E.

Comment: The ODE is now what I like to call "**diagonalized**". Just as a partial wave analysis puts things in terms of the representation label ℓ of SO(3), I think here our group is the translation group and it is a Laplace Transform type diagonalization. Of course Stak says nothing about this, and no need to digress. But it is true that result E is a very simple equation with label n, there are no differential operators and no integral operators, just trivial algebra.

Stak is going to first consider that $\lambda \neq$ some λ_k , so $\lambda \neq$ EV. In this case, you solve E as in F and you have your u_n and therefore you have your u(x) solution in terms of the fn as shown in (4.4). This is the solution to our inhomo problem!

[p261] We can see after the first = sign in (4.4) that $u(x;\lambda)$ has only poles at the λ_k with obvious residues, and we thus by inspection obtain (4.5). This is the first time a λ-space contour integral has appeared in this book. The integral is an infinite great circle that picks up "all" the poles. The parameter λ appeared in the form $1/\mu$ in Chap 3, but we never did much with it. It did enter into the convergence requirement for the Neumann series involving $\|K\|$. Now λ did appear on page 81 in Chap 1, but again nothing was done with it, this being the ODE version of the inhomo2 equation. So:

$$L_\lambda u = f = -(1/2\pi i) \oint d\lambda\ u(x;\lambda)$$

and we have this strange new integral representation of the ODE operator L_λ.

New topic: what is our Green's Function? We write the $L_\lambda g_\lambda = \delta$ problem in 4.6 and we simply treat it as a special case of $L_\lambda u = f$ and just read off the solution and its "supporting" forms. That is to say, we have $f = \delta$, and so $f_n = \langle f,\varphi_n\rangle = \langle \delta_\xi,\varphi_n\rangle = \varphi_n(\xi)$ from our usual distribution theory of Chapter 1, as shown A. The inverse of this equation is the top left of p 260 B which now says:

$\delta(x-\xi) = \Sigma_n\ \varphi_n(x)\varphi_n(\xi)$ // as in 4.7 or $\langle x|\xi\rangle = \Sigma_n \langle x|n\rangle\langle n|\xi\rangle$

and we have the completeness of closure form. I think we have seen this earlier in the book, but not sure where, but here it is now. The eigenfunctions of this particular self-adjoint ODE form a "complete set". Then finally, we get the result for g_λ itself, which is another famous "form"

$$g(x|\xi;\lambda) = g_\lambda(x|\xi) = \Sigma_n\ \varphi_n(x)\varphi_n(\xi)/(\lambda_n-\lambda)$$

And the last "form" is then to do the λ integral as shown above on u to get

$L_\lambda\ g_\lambda = \delta_\xi = -(1/2\pi i) \oint d\lambda\ g(x|\xi;\lambda) = \delta(x-\xi)$ which is 4.9

[p262] In the above, we found our Green's function g_λ by treating $L_\lambda g_\lambda = \delta_\xi$ as just a special case of the equation $L_\lambda u = f$ and we used the solution (4.4) for u as our result for g_λ . But here, we do a Chapter 1 style direct solution for g_λ. In A we have it as a homo solution on each side where we pick constants that make it meet the BC's at x=0 and x=ℓ . At this point, Stak is very careful with the quantity $\sqrt{\lambda}$ and he defines it with the branch cut taken to the right with the main sheet on the top at θ=0. He then does the usual jump condition at x=ξ to get result 4.10 which is manifestly symmetric as it must be, and in 4.11 he writes the Green's when λ = 0, which I might call g_o.

[p263] Here, he points out that, although g(x|ξ; λ) is clearly a function of $\sqrt{\lambda}$, it does NOT have a branch cut on the right half real axis in the λ plane. In fact the form 4.10 for g_λ only has simple poles at the eigenvalues λ_n due to the denominator $\sin(\sqrt{\lambda}\,\ell)$ factor. [These must be the same poles you see in 4.8]

Where are we at this point? We have manually constructed g_λ on page 262, and we have the result and we see that it has poles. Now we want to see if we can find the $\varphi_n(x)$ as if we did not know them. He first computes the residue of 4.10 at each pole as in B, and then with help from C we get D and then 4.12. But we know that $\delta(x-\xi) = \Sigma_n\, \varphi_n(x)\varphi_n(\xi)$ for a complete set, so we can then just "read off" the $\varphi_n(x)$ from 4.12.

[264] On this page we start with a series of 5 long "remarks, I will comment on each of these.

Remarks (1) through (5)

(1) Suppose we wrote $u = G_\lambda f$ as our solution to $L_\lambda u = f$. And suppose we throw in 4.8 for g in G. We get

$$u(x) = \int d\xi\, g(x|\xi;\lambda)\, f(\xi) = \int d\xi\, \Sigma_n\, \varphi_n(x)\varphi_n(\xi)/\,(\lambda_n-\lambda)\, f(\xi) = \Sigma_n\, \varphi_n(x)\,/\,(\lambda_n-\lambda) \int d\xi\, \varphi_n(\xi)\, f(\xi)$$

$$= \Sigma_n\, f_n\, \varphi_n(x)\,/\,(\lambda_n-\lambda) \;= \text{the result in 4.4}$$

So this would be another way to obtain the solution 4.4 without using the u_n expansion.

(2) My scribble here is on page 259. Start with $L_\lambda u = f$ and write out as $(L - \lambda)u=f$ and apply G_0 from the left and you get $G_0\,(L - \lambda)u = G_0\, f$. But $G_0 L = 1$ so we have $u - \lambda\, G_0\, u = G_0\, f$, so $u = \lambda\, G_0\, u + G_0\, f$, and this is result A on p 264 which he then rewrites in conventional inhomo2 integral equation form in 4.13. So he is just making the usual observation that you can rewrite the inhomo2 ODE as an IE.

And then he writes the homo ODE problem also as an IE EV problem in 4.14. We solved this IE back in Chapter 3, result is quoted as B which he then rewrites as C which then replicates our solution of this chapter.

(3) Now for the first time in this chapter he asks: suppose λ in our ODE is exactly one of the EV's? We then have situation p 264 D where we took $\lambda = \lambda_k$. D is our diagonalized ODE. If $n \neq k$, everything is as before, but if n=k, we must have $f_k = 0$ in which case u_k is undetermined.

[p265] The right part of 4.15 is then our "particular solution" of the ODE where we have set $f_k = 0$, which means we have assumed the consistency condition shown there. Of course now our homo equation is just $L_{\lambda k}\varphi_k = 0$ and since λ is this EV, this has the solution φ_k. Thus, we add this ONE solution of the homo equation to our particular solution and that is what 4.15 is saying. When $\lambda = \lambda_k$ (specific k), the nullspace of L_λ is 1D and the consistency condition is exactly what we expect to see, saying that f has to be in the perp space of L_λ.

(4) Now he complicates things by changing the BC's to be α and β at the two ends. He finds the solution of $L_\lambda v = 0$ as shown in 4.17. We can of course expand v(x) on the $\varphi_n(x)$ of our $\alpha=\beta=0$ homo problem. The coefficients of v(x) are then v_n. It is easy to see where A and B come from -- in B, the second parts must be retained. Now C is not obvious to me. Where is this coming from? It seems disconnected from what just went before. Well, if we combine A and B we get this:

$$\lambda \int dx\, v\varphi_n = = - \int dx\, v\varphi_n'' + v\varphi_n' |_0^\ell$$

$$\lambda \int dx\, v\varphi_n = = - \int dx\, v\{ - \lambda_n^2 \varphi_n \} + v\varphi_n' |_0^\ell$$

$$\lambda v_n = + \lambda_n^2 v_n + v\varphi_n' |_0^\ell$$

$$v_n [\lambda_n^2 - \lambda] = - v\varphi_n' |_0^\ell = -v(\ell)\, \varphi_n'(\ell) + v(0)\, \varphi_n'(0)$$

$$= - \beta\, \varphi_n'(\ell) + \alpha\, \varphi_n'(0)$$

$$= - \beta\, \varphi_n'(\ell) + \alpha\, \varphi_n'(0)$$

But we know that

$$\varphi_n'(x) = \{\sqrt{2/\ell}\, (n\pi/\ell) \cos(n\pi x/\ell)\}$$
$$\varphi_n'(\ell) = \{\sqrt{2/\ell}\, (n\pi/\ell) \cos(n\pi\ell)\}$$
$$\varphi_n'(0) = \{\sqrt{2/\ell}\, (n\pi/\ell) \}$$

so we get from above

$$v_n [\lambda_n^2 - \lambda] = - \beta \{ \{\sqrt{2/\ell}\, (n\pi/\ell) \cos(n\pi\ell)\} + \alpha \{\sqrt{2/\ell}\, (n\pi/\ell) \}$$

$$= \sqrt{2/\ell}\, (n\pi/\ell) (\alpha - \beta \cos(n\pi\ell) \} \qquad \text{// and this is result C}$$

So from this 4.18 is the solution for v(x) expanded onto the $\varphi_n(x)$ and of course this is the same as the expression for v(x) given in 4.17.

[p266] Suppose in 4.18 we set $\lambda = 0$ and $\alpha = \beta = 1$. From 4.17 we know that LHS = 1 (first take ratio of sines as (x/ℓ) but then this is times (1-1) so second term = 0.) The RHS of 4.18 becomes

$(2/\ell)\Sigma_n(n\pi/\ell)\ (1-\cos\pi n)\ (\ell/\pi n)^2 \sin(n\pi x/\ell)$

$(2/\ell)\Sigma_n\ (1-\cos\pi n)\ (\ell/\pi n) \sin(n\pi x/\ell)$

$\Sigma_n\ (1-\cos\pi n)\ (\ 2/\pi n) \sin(n\pi x/\ell)$

For n = even, we have $\cos\pi n = 1$ so no contribution. For n = odd, get $\cos\pi n = -1$, so

$= \Sigma_{n=1,3,5}\ (4/\pi n) \sin(n\pi x/\ell) = 1$ independent of ℓ !

which he claims is a "well known result" -- the sine Fourier series for the function "1".

Continuing now within "remark 4". The series 4.18 is obviously 0 at x = 0,ℓ but that is not what our BC's say in 4.16, so 4.18 disagrees with 4.17 at the end points, but agrees on the open interior of the interval! This seems very strange to me. The series solution is not continuous at the end points, and we do NOT have uniform convergence on the closed interval.

Finally, 4.18 has 1/n terms as is only slowly log n convergent, so probably not very useful.

(5) Suppose for 4.16 we just insert $v(x) = \Sigma v_n\varphi_n(x)$. We get result A which says $v_n = 0$! But this is wrong, and the error in our ways was the term-by-term differentiation of the expansion series. So this is just a warning that you have to be careful. Integration is always a better method that doing term by term differentiation.

The whole idea here of using the φ_n which have the wrong BC's in our expansion for v seems a bad one to me, and we see the kinds of things that happen!

Exercises [p 267]

Exercise 4.1. (a) From page 265 C, the consistency condition must be $\alpha - \beta\cos(k\pi) = 0$ since LHS = 0, and in this case v_k is undetermined. So this says $\alpha - (-1)^k\beta = 0$ or that $\alpha = (-1)^k\beta$. That's all I did.

Exercise 4.2. We solve here the same $L_\lambda g_\lambda = \delta_\xi$ problem, but now the BC's for g are different, they are derivatives instead of values. Ie, we just have different BC's here than we had before. Probably all our previous results are more or less altered by replacing sin→cos and in this way we get a cos expansion for δ. In other words, the solutions here must be φ_n with cos instead of sine. This seems a straightforward problem, and it appears it was assigned to me once upon a time.

Exercise 4.3. Same problem but now with some "mixed" BC's. Now have both sin and cos in our results, of which there are many quoted on page 268. Again, very doable I think.

4.2 The General Regular Boundary Value Problem (268)

Pre-Note added 3.13.10. I have a checkmark by equation 4.28a, but today it is "not obvious" where this is coming from, and these notes have no comments on it! The answer requires going back to Chapter 1 page 69 equation 1.65 known as Green's Formula where the J thing there is the "bilinear concomitant" gizmo. A derivation of 1.65 is what this page is doing, it is just the parts integration

activity, no rocket science. So we derive 1.65 in situ. Now what about the "differential form" 1.66 ? This just represents the algebra of the *integrand* inside the part integration. Consider the simpler case: vu'+v'u = (uv)' so we say vDu = - Dv u + d/dx(uv). In this simple case, we have L = D and L* = -D so vLu = uL*v + ∂_x(uv) and then we would say J = uv. So in a sense Lagrange's Identity (1.66) is the basic item, and you then integrate it to get 1.65. So the answer is that this is all just "algebra" including multiple parts integrations.

We now replace the $L = -D^2$ of our example with $L = -D[pD] + q$ which we note has the formally self adjoint form that was shown page 70 A where $a_0 = -p$ and $q = a_2$. And we also replace $L_\lambda = L - \lambda$ with the slightly more complicated $L_\lambda = L - \lambda s(x)$ where s(x) is a "weight function" which must be positive on the interval. He also wants p(x) to be positive as well, reason not given for these requirements. The requirements are on the open interval (a,b) only. So bottom of page 268 shows our three equations, each one written out in two useful ways. Functions p and q must be real and continuous.

[269] If (a,b) is finite and if the above conditions are valid on the closed interval (as well as open), then our ODE problem is **regular**, else it is **singular**.

The BC's are standard unmixed ones as shown, one at each end. We now allow functions f and u to be complex (were real in Chap 1), so of course we use the full inner product. Simple algebra gives 4.29 and the RHS is 0 because each W is 0. This is forced by the form of the BC's! For example, assume that u and v are solutions to one of our equations at point a. Then we have:

$$W(u,\bar{v};a) = \begin{vmatrix} u(a) & \bar{v}(a) \\ u'(a) & \bar{v}'(a) \end{vmatrix} \qquad \text{but } \alpha_1 u(a) + \alpha_2 u'(a) = 0 \text{ and same for } \bar{v} \quad (\alpha_i \text{ are real})$$

This means that the first column is a multiple of the second column, so W = 0. But we know from Abel p 60 that the W then vanishes everywhere. Of course our other BC says it vanishes at b as well.

Note 1 added 12.4.09. Lest this not be clear to stupid people (you rang), the BC's are these

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0$$
$$\alpha_1 \bar{v}(a) + \alpha_2 \bar{v}'(a) = 0$$

which we write as

$$\alpha_1 \begin{pmatrix} u(a) \\ \bar{v}(a) \end{pmatrix} + \alpha_2 \begin{pmatrix} u'(a) \\ \bar{v}'(a) \end{pmatrix} = 0$$

And THIS says that the two ROWS of our matrix are DEPENDENT, therefore the det vanishes. Another way to say this is that we have

$$\begin{pmatrix} u(a) & u'(a) \\ \bar{v}(a) & \bar{v}'(a) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since we are claiming we have nonzero numbers α_1 and α_2, it must be true that the inverse of this matrix does not exist, and that means det M = 0 hence det M^T = 0 hence Wronksian = 0.

Note2added 12.4.09. On page 269 Stak says "For any u and v". But when it gets to the Wronskian issue, he says p 270 top " if u and v satisfy the homo BC's. Only then does the Wronskian vanish by the argument given above. The larger picture is this: you think of L and L* as operator and its adjoint. The operator is "formally" self-adjoint of the parts stuff is right. But it is "fully" select adjoint only on a certain space of functions $D_A = D_{A*}$ which consists of functions which are differentiable AND which meet the two BC's. "Vanishing at the BC's" is part of the *definition* of the domain of A = L. So our whole world of functions is this. So in all this proof stuff, we really mean "for any u and v which are in our space of functions" which includes the requirement that u and v separately respect both homo BC's.

_____digression _____

Repair! *Why* is the first column a multiple of the second column? This is not obvious to me now.

Answer: Consider this problem: $a^T x = 0$ and $a^T y = 0$ where $a = (a_1, a_2)$, $x = (x_1, x_2)$ same for y. These two conditions are like the BC's above. Think $a_i = \alpha_i$ and **x** = (u,u') and **y** = ($\bar{v}$,$\bar{v}$'). What can we say about **x** and **y** just given that $a^T x = a^T y = 0$. In a 2D space, if vectors **x** and **y** are both perpendicular to vector **a**, then we must have **x** and **y** along the same line, so that **x** = k**y,** so that vectors **x** and **y** are linearly dependent. Therefore det(**x**,**y**) = 0 . Another way to say this. The two equations $a^T x = a^T y = 0$ say this:

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \qquad \text{or} \qquad M\mathbf{a} = \mathbf{0}$$

If detM ≠ 0, then **a** = 0. But **a** ≠ 0, therefore we must have detM = 0 and so **x** and **y** must be linearly dependent.

Comment: When functions like u and v can be complex, we can talk about W(u,$\bar{u}$; x). This Wronskian is in general NOT zero, so u(x) and $\bar{u}$(x) are in general independent functions. We can apply our entire Repair discussion above to the case v = u. So let's summarize all this stuff:

Theorem: Suppose functions u and v both satisfy BC $\alpha_1 u(a) + \alpha_2 u'(a) = 0$ with α_i = real. Then the following facts are true: (note that $\bar{u}$ and $\bar{v}$ must satisfy this BC as well)

1. W(u,v; a) = 0 and therefore W(u,v; x) = 0

2. W(u,$\bar{v}$; a) = 0 and therefore W(u,$\bar{v}$; x) = 0

If we set v = u, item 1 above is trivially true since the two columns are identical, and this has nothing at all to do with the BC. However, item 2 is not trivially true and is only true if the BC is applied:

3. W(u,$\bar{u}$; a) = 0 and therefore W(u, $\bar{u}$; x) = 0

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[270] Therefore, we obtain our symmetric form for <Lu,v> as in 4.340 (Hermitian). I think L might also be self-adjoint, but Stak does not seem concerned, so symmetric it is! But self-adjoint is what you need for L to claim that the Green's is symmetric, so I will just assume L is self-adjoint.

Comments on the EV Problem:
(1) the EV's are real
(2) the EF's for different λ_i are orthogonal with weight s(x)
(3) number of EV's and EF's is at most denumerable infinity since separable HS.

[271] Next, find the Green's general form for $L_\lambda g_\lambda = \delta_\xi$. Of course assume same BC's. Assume homo solution w on the left and z on the right. By requiring A, we meet BC's in B. Then we can write form C because we know that g must be symmetric, and this is the only way we can make our w,z product be symmetric.

Now here come the jump condition. We know that

$$dg/dx|^{x \text{ on the right}} = A\, w(\xi)\, z'(x)|^{\xi} = Awz'$$

$$dg/dx|^{x \text{ on the left}} = A\, w'(x)|^{\xi}\, z(\xi) = Aw'z$$

$$\text{so } \Delta dg/dx = A(wz' - w'z) = A \begin{vmatrix} w & z \\ w' & z' \end{vmatrix} = A\, W(w,z;\xi) = 1/a_0(\xi) \text{ from p 62 } = -1/p(\xi)$$

For L formally SA, we get the simplified Abel form page 72 which says $W = C/a_0 = -C/p$. But now he is going to change the sign of C to make us have $W = C/p(\xi)$ and we will just use this C. Then the above becomes

$$A\, W(w,z;\xi) = AC/p(\xi) = -1/p(\xi) \Rightarrow A = -1/C.$$

and we then have 4.35 with C as in the following line.

[272] So, the poles of the Green's function are now coming from $C = C(\lambda)$ as in 4.35. We always implicitly use the idea that the homo solutions like w are analytic in free parameter λ. Then 4.36 states the symmetry of g. There should be a bar over one of the g's for general λ, but if λ = real, then you can omit the bar.

Now, where does p 272 B come from? He confused me by replacing λ with $\lambda' - \theta$ and not keeping the prime. Our EV problem says $L\varphi = \lambda s\varphi$ and this is what the Green's for, so $\varphi = \lambda G s\varphi$ where s just goes along for the ride, and this last equation is B. I then agree that if you define k as in C, and ψ as in C and μ as in C, then this integral equation becomes 4.37 which is our old Chapter 3 friend. Notice that k is symmetric. How do we know that μ=0 is not an EV? It would mean $\lambda = \infty$ and I guess we are just assuming λ is some finite value, fine. So, from our Chapter 3 theory, we know that the set of eigenfunctions ψ form a complete set. We know that $\mu = 0$ is the only allowed accumulation point, which translates to $\lambda = \infty$ as an accumulation point. He claims without proof that can only have a finite number of negative λ_n eigenvalues, so really the accumulation point is $+\infty$ as shown in E.

Recall in passing how our "famous orthogonal polynomials" tend to have some weight function s(x), so this is fitting into the Stakgold theory at this point.

[273] Now consider the inhomo $L_\lambda u = f$ which we solve as usual as $u = Gf$ as in A where we use the Green's g we found (in general terms) in 4.35. But it is more interesting to write $u = \Sigma_n u_n \varphi_n$ where the φ_n are our homo EF's (including s). It is easy to grind down through this page of equations and we happily end up with solution for u_n as in F, and so u(x) as in 4.41. Just as we did in the example, we consider $L_\lambda g = \delta$ and out pops (4.42) where one of the φ has a bar over it. You can see that this makes g in general by Hermitian as I show at the top of page 272 (which Stak calls symmetric).

Now would we derive 4.43? It is just the statement (p272) that the ψ_n form a complete set. But we can follow the path he gives,

$$u(x) = \Sigma_n u_n \varphi_n(x) = \Sigma_n \{ \int dx'\, s(x')\, u(x') \bar{\varphi}_n(x') \}\, \varphi_n(x) = \int dx'\, u(x') \{ s(x') \Sigma_n \bar{\varphi}_n(x')\, \varphi_n(x) \}$$

so it must be that $\{ s(x') \Sigma_n \bar{\varphi}_n(x')\, \varphi_n(x) \} = \delta(x-x')$ which is 4.43, you can bar either one.

[274] The λ integral of g then follows at once. We are just repeating *everything* that happened in our example!

Looking at p 272 A, it is pretty clear that W = 0 at zeros λ_i of $C(\lambda)$. But from 4.35 these λ_i are the poles of g and then from 4.42 we see that these λ_i are the EV's of our EV problem! If W = 0 at λ_n, then the two functions w and z are linearly dependent as in 4.45.

Now the next step is to jam 4.35 into our λ integral and model the zero of $C(\lambda)$ near λ_n in the obvious manner. The residue of g is shown in A, and we then get either of the forms in B, and boom, we now see that the w and z evaluated at $\lambda = \lambda_n$ are in fact our φ_n. So if we found say w, then we know φ_n up to the constant k_n which makes it be normalized.

Example: [275] We already had our huge introductory example, so here we do the example which appeared in Exercise 4.2 where the BC's are on slopes instead of values. We just grind through page 275 and everything is just applying our theorem to this example. We even have a value for the normalizer factor k_n at page bottom. Here we are examining our known w and z functions at the point $\lambda = \lambda_n$. Then 4.46 tells us our exact normalized EF's $\varphi_n(x)$. Notice the factor of 2 involved for φ_0 which comes from where I wrote OK on page 275. Our general $\delta(x-\xi)$ expansion is 4.43 applied to this case (s = 1).

This δ expansion appears in Exercise 4.2 on page 267 as (4.19), and also as 1.22a way back in Chapter 1 found by a different method. So here we see all the results popping out.

My summary of the Regular Boundary Value Problem.

1. The **general form** of the ODE system must be as shown bottom page 269 with all the "**conditions**" there listed. L = L* always, and $L_\lambda = L_\lambda^*$ when λ real.

2. The **BC's** must be the **unmixed** ones, one for x=a and one for x=b, as shown 4.26, again with the "conditions" as listed there. These determine the domain D_L.

3. The difference between <Lu,v> and <u,Lv> can be expressed in terms of Wronskians at the endpoints of the interval, such as $W(u,\bar{v}; a)$. If u and v are in the space of functions D_L which satisfy the BC's just stated, it is easy to show that both Wronskians vanish, and therefore that <Lu,v> = <u,Lv>, which says that **L is "symmetric" on D_L.**

4. The EV's of $L\varphi = \lambda s\varphi$ are **real**, the EF's are **orthogonal with weight s(x),** and the number of EV's is **denumerable**. The spectrum of EV's therefore cannot fill the entire real axis. EV problem is also $L_\lambda \varphi = 0$.

5. The "symmetric" **Green's** for $L_\lambda g = \delta_\xi$ can be constructed as (1/C) $w(x_<)\, z(x_>)$ where, as usual, w satisfies just the left side BC, and z satisfies just the right side B, and C = p W[w,z; x] . Everything here is a function of λ.

6. One can recast the EV problem $L\varphi = \lambda s\varphi$ into an **integral equation**, and then use the theory of integral equations to claim that the set of EF's $\psi_n(x) = \sqrt{s(x)}\, \varphi_n(x)$ form a **complete orthonormal set**, and that the only possibly accumulation point for λ is at $\lambda = +\infty$. That is, $L\varphi_n(x) = \lambda_n s(x)\varphi_n(x)$. We don't of course know exactly what the EV's λ_n are at this point, but they are denumerable and accumulate at $+\infty$ if they are infinite in number. [the spectrum is discrete!]

7. To say that the $\psi_n(x)$ form a complete orthonormal set is also to say that;

$$\delta(x-\xi) = \Sigma_n\, \psi_n(x)\, \bar{\psi}_n(\xi) = s(x) \quad \Sigma_n\, \varphi_n(x)\, \bar{\varphi}_n(\xi) \qquad // \text{ 4.43}$$

8. If $\lambda \neq$ EV, one can solve the inhomo $L_\lambda u = f$ to get result shown in 4.41:

$$u(x) = \Sigma_n\, \varphi_n(x) \langle f,\varphi_n\rangle/(\lambda_n - \lambda) \qquad // \text{ notice that inner product does not contain s}$$

As usual, if $\lambda = \lambda_n$, we need $\langle f,\varphi_n\rangle = 0$ as our consistency condition.
If we apply this result to $L_\lambda g = \delta_\xi$, we get 4.42,

$$g(x|\xi;\lambda) = \Sigma_n\, \varphi_n(x)\, \bar{\varphi}_n(\xi)/(\lambda_n - \lambda) \qquad // \text{ here are your EV's and normalized EF's !}$$

9. If we apply great circle $\oint d\lambda$ to the above formula (and use our δ result above) we get

$$-(1/2\pi i) \oint d\lambda\, g(x|\xi;\lambda) = \delta(x-\xi)/s(x) = \Sigma_n\, \varphi_n(x)\, \bar{\varphi}_n(\xi)$$

10. Now that we see from 8 that $g(x|\xi;\lambda)$ has a pole at each eigenvalue, when we write
$g(x|\xi;\lambda) = w(x_<;\lambda) z(x_>;\lambda)/C(\lambda)$ it is not hard to conclude that $C(\lambda)$ must have a simple zero at each eigenvalue, and that near such an eigenvalue $\lambda \approx \lambda_n$ we will have $C(\lambda) \approx C'(\lambda_n)(\lambda-\lambda_n)$. Since we are then saying $C(\lambda_n) = 0$, we must from p 272A have $W[w(x_<;\lambda_n) z(x_>;\lambda_n); x] = 0$ which says the functions w and z are linearly dependent when $\lambda = \lambda_n$, and we write $w(x,\lambda_n) = k_n\, z(x,\lambda_n)$ as in 4.45. It is then only a small step to relate the functions w and z to our functions φ_n:

$$\varphi_n(x) = \pm\, [k_n C'(\lambda_n)]^{-1/2}\, w(x,\lambda_n) = \pm\, [k_n/C'(\lambda_n)]^{1/2}\, z(x,\lambda_n)$$

Degeneracy Question: From our integral equation theory we knew that the multiplicity of an eigenvalue μ had to be finite, except for the case $\mu = 0$ which could have an infinite multiplicity. In our ODE work to

this point, he seems to be assuming that the λ_n all have multiplicity 1. We saw this to be true in our prototype long example, and it seems to be true in our various examples. It seems odd that he has never commented on this fact. I am sure degeneracy will arise when we go to > 1D problems. So I will just state this question and then let it ride for the timing being.

Exercises: (276-283)

Exercise 4.4 Here we just show that L_λ is symmetric only when λ = real.

Exercise 4.5 In this problem we consider the EV problem $L\varphi = \lambda s\varphi$ with *initial-value* BC's (not our standard form for this section). The claim here is that you can recast the ODE into a Volterra integral equation. I am not sure what the main point of doing this is.

Exercise 4.6 An exercise where we show what I claimed above, that $C(\lambda)$ has zeros at "isolated" points on the real axis.

Exercise 4.7 Consider the general EV problem $Lu = \lambda su$. We are to show that there can be no negative EV's under certain conditions. Then we are supposed to find a lower bound for operator L, where we use $||| L |||^2 = \max_u \langle Lu,u\rangle/\langle u,u\rangle$. We are given a detailed string of hints as to how to show this. We conclude after hint 5 that $||| L ||| \geq$ A and this tells us that $\langle Lu,u\rangle/\langle su,u\rangle \geq B$ which says $\lambda_n \geq B$. The point is that you can only have a finite number of negative eigenvalues, and so $\lambda = +\infty$ can be the only accumulation point.

Exercise 4.8 Here we compute the "normalization factor' $\int dx\ s(x)\ |w_n(x)|^2$ and somehow the result is supposed to jibe with 4.46. But in 4.46 k_n was just a proportionality factor, so not sure what the point is here.

Exercise 4.9 Here we look at a specific L where $p = x^2$ and $q = 0$ on the interval (1,e) with simple vanish BC's at each end. (a) First claim is that all EV's are positive. (b) general solution form (ignoring BC's) is given as shown for general $\lambda < 1/4$, but in this case you cannot satisfy the BC's. For $\lambda > 1/4$ we can find EF's and EV's as shown, it turns out $\lambda_n = 1/4 + (n\pi)^2$. The EF form is sin(... log(x)). This exercise just lets "the student" "work with" a specific ODE and learn some things about it.

Exercise 4.10 The claim here is that you can do a change of variables from $x \rightarrow t = \int^x dx'\ 1/p(x')$ and this causes the new ODE operator in variable t to have this form: $D_t^2 + Q(t) - \lambda S(t)$ where Q and S are the simple functions shown. Any linear term D_t has been killed off, and this is called "**the normal form**" of the ODE. I can dimly recall doing something like this in various past efforts, perhaps the Laplace Method and perhaps the WBK approximation.

Exercise 4.11 This huge exercise fills 3 entire pages of the text. We have $L = D^2$ so the ODE is pretty simple. In the past (Chap 1 p 15 Exercise 1.2) we dealt with **heat** problems $-kAD^2u = f(x)$, but this was an inhomo problem with heat supplied as f(x). We did this again p 85 Example 1 with insulated ends meaning u' = 0 at each end. But here we are doing the problem with $-D^2\varphi - \lambda\varphi = 0$ so we are talking some

kind of heat EV problem that is not very clear to me. At the rod's left end we set $\varphi = 0$, zero temperature. But we parameterize the right end BC with β so it can be "any" of "the usual suspects": zero temperature, insulated, or radiating heat. Of course $\varphi = 0$ is always a solution of the ODE and BC's, the trivial solution. The claim here is that there are certain values of λ which allow a non-trivial steady-state temperature distribution in the rod! I do not understand how this is physically possible unless heat is being supplied. You would think if would just equalize to a uniform temperature. I guess you can think of this as an inhomo problem in the sense $-D^2\varphi = \lambda\varphi$ where $\lambda\varphi$ is the heat source in effect, so we find some self-consistent solutions.

OK, let's now forget the physical interpretation and just look at this as a simple EV problem. By inspection we can write down our w and z functions and compute their W as on page 279 A. We know this is a symmetric L problem so eigenvalues are all on the real axis, so we do the three cases. We know that W = 0 at EV's, and we have our W in p 279 A.

With λ=0 W vanishes when $\beta = \hat{\beta}$, the value such that $\tan(\hat{\beta}) = -\boldsymbol{\ell}$ where we somehow thing of $\boldsymbol{\ell}$ as having no dimension (unclear how that works). When $\lambda = 0$ we know that $\varphi = Ax + B$ and B=0 at the left end, so EF is $\varphi = Ax$.

When $\lambda < 0$ there is a single eigenvalue $\lambda_0 < 0$ corresponding to a certain r_0 which is the transcendental equation solution shown in Fig 4.1. As $\lambda \to 0$, $r_0 \to 0$ and $\beta \to \hat{\beta}$. In this case, $\sqrt{\lambda}$ = imaginary, and our EF is of the form $\sinh(r_o x/\boldsymbol{\ell})$ and the normalization factor N_0 is a complicated function of the BC constant β, as shown in p 280 A.

When $\lambda > 0$, in contrast, there are an infinite number of EV's, $\sqrt{\lambda}$ = real, and they are found again from a transcendental equation solution situation in Fig 4.2, We have $\sqrt{\lambda} = r/\boldsymbol{\ell}$ and the r_n are the EV points of interest. I figured out what is going on in that figure by plotting $\tan\beta$ versus β in pencil at page top 281 and numbering the cases. The normalizers are N_n as in 281 A. The EF's are $\sin(r_n x/\boldsymbol{\ell})$.

So here is a little review:

$\lambda = 0$	$\beta = \hat{\beta}$	Ax
$\lambda < 0$	EV λ_o exists only if $\hat{\beta} < \beta < \pi$.	$\sinh(r_o x/\boldsymbol{\ell})$
$\lambda > 0$	∞ EV's for any β	$\sin(r_n x/\boldsymbol{\ell})$

Now, we want to think about $\delta(x-\xi)$ expansions.

If $\beta = \hat{\beta}$ we have $\lambda_0 = 0$ as an EV along with all the λ_n positive EV's and sin functions. We have s(x) = 1 in this problem, by the way. So our δ sum is going to include that sin products plus a term of the form $x\xi$ (since we have Ax as EF) and this is p 282 B where he has computed the N_0 factor for the λ=0 EV and installed it.

If $\hat{\beta} < \beta < \pi$, we still get all those sin terms, but we now have a single sinh EF and its EV and that is why we see the sinh product term in C.

If $\hat{\beta} < \beta$ then we don't get the isolated EV and EF, so we have only the sin products as shown in A.
So these are "pretty fancy results" given the simplicity of our ODE with β-parameterized right-end BC's.

His final act is to take the large-n limits.

Comment: This is the first ODE problem we have considered in this book where the EV's had to be determined from a **transcendental** equation. In Saxon p 121 we had the particle in a square well QM problem which is this: $D^2\psi + [\ 2m/\hbar^2(V_0-\varepsilon)]\psi\ = 0$. In that problem we have a square well in 1D centered at the origin, and we find reasonable solutions in our three regions, and we match ψ and ψ' at the boundaries, and we get transcendental equations for the eigenvalues. I think we can regard the operator L as essentially the same, but the BC's are very different that those of our "heat problem" above. But it is the act of enforcing the BC's in either case that gives you the transcendental issue.

Exercise 4.12 The L here has p = x and q = 0 and so $L\varphi = \lambda x\varphi$. Without the BC's, we have just the **Bessel order 0 equation** which we ran into in Chapter 1 page 75, so solutions are $J_o(kx)$ and $N_o(kx)$ where $k^2 = \lambda$. But now we require solutions to vanish at both ends a and b. We end up with $k_n^2 = \lambda_n$ as determined by a certain equation p 282 D, and all we know is that the k_i are positive. So this is similar to our transcendental situation above, but now we have special functions involved instead of elementary functions. The EF's are stated, lincombs of the two J and N.

Exercise 4.13 A similar ODE is considered, the EV problem. We are supposed to cast this into our standard form and study the eigenfunctions. OK, fine, nothing special here.

4.3 Introduction to the Singular Boundary Value Problem: Examples (283)

Remember that there are lots of reasons for being "singular", anything that is dropped from the set of conditions that makes an ODE system be "regular". The first "singular" issue to be brought up here is the case that $a = -\infty$ and/or $b = +\infty$. If we just look at $L = D^2$ and $L_\lambda = D^2 - \lambda$, we get our usual solutions and none can be normalized in the full L_2 on $(-\infty,\infty)$. If $Im(\lambda) > 0$, we can have a decaying expo on $(0,\infty)$ and a similar thing on $(-\infty,0)$. For real λ, no solutions in either case (all in L_2 that is).

[284] We look here at inhomo $L_\lambda u = f$ on $[0,\infty)$ and $u(0) = 0$. We don't really have a BC at the high end, just requiring L_2 (so we can have a HS) is enough. He makes these claims:

(a) If λ is off the positive real axis, there is always a unique solution. On $[0,\infty)$ there are restrictions on f.
(b) The Green's method "still works"
(c) there is "sort of" an eigenfunction expansion. Our system here in fact has *no* EV's or EF's due to the low end $u(0) = 0$ which kills off our only expo friend.

[285] We can solve $L_\lambda g = \delta_\xi$ in our usual manner for the problem above and we get 4.53 where we apply the usual jump condition at $x = \xi$. We have sine on the left and expo decay on the right. Our only BC is that $g(0)=0$ at the left end and sine does that. And the expo decay keeps things finite.

Now comes a new idea: the solution $g(x|\xi; \lambda)$ is seen here not to have poles, but to have a continuous **branch cut** on the positive real axis. We shall see later that $b = \infty$ can be considered limit $\ell \to \infty$ and all those poles we had before "**coalesce**" into this simple branch cut. By being careful with the definition of $\sqrt{\lambda}$, he computes the discontinuity across the cut which he writes as [g](x). As 4.54 shows, it is imaginary.

Now, why should 4.55 "still be true"? Well, if we think of the limit concerning poles becoming a branch cut, it seems pretty compelling. After all, the result is independent of ℓ in the finite-ℓ situation.

[286] Now look at the **contour** on page 286. In the discrete poles situation, the contour shown gives zero since it contains no singularities and can be shrunk to zero. But we can think of this as a full great circle plus the part around the real axis. So our full great circle contour integral $\oint d\lambda\ g(x|\xi;\lambda)$ which is around the great circle he calls C can be set equal to the negative of the integral on contours $C_+ + C_-$. In our earlier case, this $C_+ + C_-$ picked up all the pole residues, but here it picks up the cut discontinuity and we can write 4.57 with all plus signs now. But we just computed [g] in 4.54, so throw it in to get 4.58 rewritten in simpler variable as 4.58a. Our previous sum of an eigenfunction product is somehow now an integral of an "eigenfunction product". I sense that the sin solution is OK as a "generalized function" so we really do have eigenfunctions in the sense of distributions, but he is not wandering off on those details in the middle of this storm of equations.

[287] But one thing is pretty clear. Once you have 4.58a, it is pretty easy to talk about a **Fourier Sine Transform**, and he does that on the first half of this page 287. Now stare at 4.60. f(x) can be even or odd or neither and we get some F(ν) here, where ν is the "conjugate variable". I think if you were to apply the full FT to an odd function, you end up with this sine transform thing. Notice that $F(\nu) = -F(-\nu)$ for any f(x), so this thing is "not as general" as the full FT.

But I get the general feeling that the following fact is true: Any ODE of the type we are working with in this chapter gives rise to a δ expansion, and that in turn implies some kind of "integral transform". There must be some group theory underlying this fact, but Stak is not talking about group theory.

[287,288] Now we are going to redo everything in terms of an $\ell \to \infty$ limit. Notice that large ℓ makes the eigenvalue spacing denser. I have followed all the steps now on page 288. We start with the discrete sine δ form in 4.62 and in the limit it becomes an integral and boom, there is out sine transform. A similar limit in the lower half of page 288 gives us the **Parseval's equality** 4.63. For me, this just says that the norm is the same regardless of the representation in which you compute it, see pencil page bottom. I think the s on F_s means this is the sine transform projection.

Now that we have verified our 4.59 idea by taking limits on ℓ, we are sort of backwards justifying our contour idea. Each derivation is just slightly wobbly, but two wobbly things are probably correct.

[289] The first act here is to take the finite-ℓ version of g(x|ξ; λ) from way back on page 262 and take the $\ell\to\infty$ limit of it, and sure enough, we get our hand-constructed Green's in 4.53. Right in this limit, when you start you have the discrete poles, and when you are done, you really do have that branch cut.

Theorem 1: This says that our usual solution method u = Gf works for the singular system in 4.64 where $b = +\infty$ and we only have our one BC. We assume that λ is not on that cut, ie, $\lambda \neq$ "EV".

The proof is similar to past "careful differentiation" proofs Stak has done. We break the interval up into two parts to get the two terms in A for u'(x), then each of those terms generates two terms in B. Then we use $L_\lambda g = 0$ in either side, and then the jump condition, and we end up with our 4.64 ODE, so we have shown that u = Gf really is a solution. There is no homo extra piece that you might possibly expect because u(0) = 0 kills such a homo expo.

[290] **Theorem 2:** The solution u(x) of our little ODE system just mentioned can be written as a certain double integral of f(x) which is just our Fourier Sine expansion. But now compare 4.66 A to 4.4 and they are the same except sum → integral. The double integral is like sum + integral in 4.4. Then, just as we did on page 261, we apply this to the case $L_\lambda g = \delta$ and we can then write g as an integral version of 4.8 which was that "bilinear expansion" for g.

My summary of the Introductory Singular Boundary Value Problem.

1. The ODE is just $L_\lambda = -D^2 - \lambda$, but the interval is taken as $[0,\infty)$ and the only BC is u(0)=0. With this BC, the only possibly EF (expo decay) is ruled out, so there are *no L_2 eigenfunctions at all*! However, we might in some sense think of $\varphi_\nu(x) = \sqrt{2/\pi}\sin(\nu x)$ with $\nu^2 = \lambda$ as a "non-normalizable" EF. There is surely some connection to "distribution theory" here, but Stakgold has chosen not to shine any light on it at this time. Note that this $\varphi_\nu(x)$ meets the BC requirement that u(0) = 0.

2. A good Green's Function $g = (1/\nu)\sin(\nu x_<)\exp(i\nu x_>)$ can be constructed. First, note that

$$\nu = \sqrt{\lambda} = |\lambda|^{1/2} e^{i\theta/2} \quad \text{where } \theta = \arg(\lambda) \text{ runs 0 to } 2\pi.$$

Then

$$\exp(i\nu x) = \exp(ix\,|\lambda|^{1/2}\{\cos\theta/2 + i\sin\theta/2\}) = \exp(ix\,|\lambda|^{1/2}\cos\theta/2)\exp(-x\,|\lambda|^{1/2}\sin\theta/2).$$

The only place this exp(iνx) will be non-integrable is when θ = 0 which is the positive real λ axis. So, this Green's is square integrable for all other λ.

3. The above Green's has a branch cut on the positive real λ axis with this discontinuity:

$$[g] = (2i/\nu)\sin(\nu x)\sin(\nu\xi) \qquad \text{where here } \nu \text{ just means } |\lambda|^{1/2} \text{ since } \theta = 0.$$

4. The great circle integral theorem is still valid:

$$(1/2\pi i)\oint d\lambda\, g(x|\xi;\lambda) = -\delta(x-\xi)$$

5. We distort the contour to pick up the discontinuity along real λ and we get this "completeness",

$$\delta(x-\xi) = \int d\nu\, \varphi_\nu(x)\varphi_\nu(\xi) \qquad (4.58a) \qquad \text{where} \quad \varphi_\nu(x) = \sqrt{2/\pi}\sin(\nu x)$$

which we can compare to our previous result (and s = 1 in our introductory singular problem)

$$\delta(x-\xi) = \Sigma_n\, \psi_n(x)\,\bar{\psi}_n(\xi) = s(x)\,\Sigma_n\, \varphi_n(x)\,\bar{\varphi}_n(\xi) \qquad (4.43)$$

6. The Green's function can be written in this form [derived in 10 below]

$$g(x|\xi;\lambda) = \int d\nu\, \varphi_\nu(x)\varphi_\nu(\xi)/(\nu^2 - \lambda) \qquad (4.67)$$

which we compare to our previous result

$$g(x|\xi;\lambda) = \Sigma_n \varphi_n(x)\,\overline{\varphi}_n(\xi)/(\lambda_n - \lambda) \qquad (4.42)$$

7. The above completeness relation serves as the basis of the Fourier Sine Transform:

$$F_s[\nu] = \int_0^\infty dx\, f(x)\,\varphi_\nu(x) \qquad (4.60) \qquad \text{projection}$$

$$f(x) = \int_0^\infty d\nu\, F_s[\nu]\,\varphi_\nu(x) \qquad (4.61) \qquad \text{expansion}$$

$$\int_0^\infty dx\, |f(x)|^2 = \int_0^\infty d\nu\, |F_s[\nu]|^2 \qquad (4.63) \qquad \text{Parseval's equality}$$

8. All of the above results can be obtains using $\ell \to \infty$.

9. The inhomo equation $L_\lambda u = f$ has the unique solution $u = Gf$ with the above Green's. This is true for λ off the positive real axis which then means g is square integrable so the integral Gf will converge. This is **Theorem 1**, and $u = Gf$ is our "normal" solution to the inhomo problem.

10. The inhomo $L_\lambda u(x) = (-D^2 - \lambda)\,u(x) = f(x)$ can be diagonalized by the Fourier Sine Transform to give:

$$\nu^2 U_s[\nu] - \lambda\, U_s[\nu] = F_s[\nu]$$

which can be solved

$$U_s[\nu] = F_s[\nu] / (\nu^2 - \lambda) \qquad \text{where } F_s[\nu] = \int_0^\infty dx\, f(x)\,\varphi_\nu(x)$$

then

$$u(x) = \int_0^\infty d\nu\, U_s[\nu]\,\varphi_\nu(x)$$

This is similar to our normal use of the Laplace Transform. The ODE is converted into a simple polynomial thing as derivatives become powers of the conjugate variable. (This is all **Theorem 2**)

11. If we apply the above to $L_\lambda g = \delta_\xi$, we get $\delta_s[\nu] = \varphi_\nu(\xi)$ and then $g = \int_0^\infty d\nu\, \varphi_\nu(x)\,\varphi_\nu(\xi) / (\nu^2 - \lambda)$ which is the result we quoted above in item 6.

12. One gets the impression that every ODE of this type with $b = \infty$ will give some kind of "transform" similar to the Fourier Sine Transform associated with the above example singular problem.

Exercises: (290-295, 5 pages)

Exercise 4.14 A highly digressionary exercise, surely a setup for later exercise(s) below. We consider the general integral equation u = Kf. We know that if K is HS, then K is bounded as in 3.3 p 193. Here we want a "softer" boundedness situation. Suppose each *single* integral of k(x,ξ) is bounded by $\boldsymbol{\ell}$. Then in this exercise he shows $\| K \| \leq \boldsymbol{\ell}$ so K is also bounded.

Exercise 4.15 OK, right here we apply the above to K = G of our introductory problem. We find that G is bounded in the sense shown above, there is some constant $\boldsymbol{\ell}$ such that the single integrals are finite. We know this just because we know there is expo decay. So we have $\| G \| \leq \boldsymbol{\ell}$ for some number $\boldsymbol{\ell}$. Stak says that G is not Hilbert-Schmidt, but is still bounded. It is not obvious to me why it is not H-S. Above I showed that we have:

$$\exp(i\nu x) = \exp(ix\,|\lambda|^{1/2}\cos\theta/2)\,\exp(-\,x\,|\lambda|^{1/2}\sin\theta/2) \;=\; e^{i\alpha x}\,e^{-\beta x} \;= e^{-\gamma x}$$

Squaring this gives $|\exp(i\nu x)|^2 = e^{-2\beta x}$. If we do a single tail integral of this we get the constant 1/2β. But then the second tail integral is an integral of a constant which diverges, so that is why,.

Exercise 4.16 We take here our exact same example ODE system and look at $L_\lambda u = f$ inhomo. We assume that f(x) = 0 beyond $x = \boldsymbol{\ell}$. The nullspace of L_λ (somehow in our distributional sense) is 1D and has that EF $\varphi_\nu(x) = \sqrt{2/\pi}\sin(\nu x)$ noted above, so we expect that f is restricted by the consistency condition which says $\langle f,\varphi_\nu\rangle = 0$ so that f is in the perp space of L_λ. If ν = 0, nullspace function is Ax so we then get $\langle f,x\rangle = 0$ as the consistency condition.

Exercise 4.17 Here we consider the same system, but $L_\lambda u = f$ where $f = e^{-x}$. We are supposed to prove the solution u(x) shown is correct. For λ > 0 this is probably non-normalizable due to the trig functions. For other λ, you will need to choose A to make it normalizable so we will want

$$A\,[\,e^{i\nu x} - e^{-i\nu x}] + (1+\lambda)^{-1}\,[e^{i\nu x} + e^{-i\nu x}] \;\rightarrow \text{expo decay} \qquad\qquad \nu = \sqrt{\lambda} \;= |\lambda|^{1/2}e^{i\theta/2}$$

$$\Rightarrow -A + (1+\lambda)^{-1} = 0 \qquad A = (1+\lambda)^{-1} \qquad\qquad \text{need } e^{-i\nu x} \text{ to be gone.}$$

Notice also here that the A term is the homo nullspace "EF".

Exercise 4.18 A strange one. r(x) is sort of a test function, square integrable on any finite (a,b). And then M is a subset of L_2 containing f(x) for which $\int dx\, r(x)f(x) = 0$. The strange claim is that if $\int |r|^2 = \infty$, then M is dense in L_2. The idea is that saying $\langle f,r\rangle = 0$ does not restrict f very much if the integral is ∞, since we can get arbitrarily close to any g in L_2 with such an f.

Exercise 4.19 This is another "math thing" that will be used later. Let's not do this unless necessary to understand whatever comes "later".

Exercise 4.20 An operator thing. We showed in 4.15 that G was bounded, but $G = L_\lambda^{-1}$. Let's now quote some things from Chapter 2: We talked there about $B \equiv (A - \lambda I)$ and made this classification

(1) the **resolvent set of A** (B is **regular**)
(2) the **point spectrum of A** (λ_i are the eigenvalues, B has some nullspace, B^{-1} does not exist.)
(3) the **continuous spectrum of A** (B^{-1} unbounded, R_B dense in H)
(4) the **residual spectrum of A** ($H - R_B = (R_B)^{\perp}$ has dimension called **the deficiency of λ.**)

B Regular requires:
(a) $Bx=0 \Rightarrow x=0$, as with matrices; and this implies that B^{-1} exists;
(b) the range must be *the entire* Hilbert space, so $Bx = f$ has a solution for any f in H;
(c) B^{-1} must be bounded (= continuous).

Classifying Closed Operators.

These are the only possibilities, one regular and three singular:
(1) A is regular (meets the three criteria listed above)
(2) A^{-1} does not exist, nontrivial $Ax=0$; [violates criterion (a)]
(3) A^{-1} exists *and* A^{-1} is unbounded [violates criterion (c) and (b)]
In this case, we know from our drawing that $R_A \neq \overline{R}_A = H$
(4) A^{-1} exists but $\overline{R_A} \neq H$. [violates criterion (b)] $R_A \neq H$

So in our present situation, we have A = L and B = L_λ. We just showed in 4.15 that when λ is not on the positive real axis, G = L_λ^{-1} is bounded. We also know that $L_\lambda u = 0$ has only the trivial solution, so we seem to have conditions (a) and (c) for operator L_λ to be "regular". When λ in $[0,\infty)$, G = $L_\lambda^{-1} = B^{-1}$ is unbounded, and this says such λ values are in the "continuous spectrum" of L. The facts needed to clinch this analysis are those "math things" that were shown in earlier exercises.

So Stak is trying now to tie our ODE operators to our "general theory" of operators in infinite dimensional Hilbert Spaces, as presented in Chapter 2 of this book.

Exercise 4.21 Here we just redo everything of our example case but we use u'(0) = 0 instead of u(0) = 0. The result is that sin → cos and we get the **Fourier Cosine Transform** stuff and all the good results are listed off here!

Exercise 4.22 (Full Fourier Transform, two ways)

Here we "repeat our example" yet again, but now we work on the full $(-\infty,\infty)$ interval. We have the same L and L_λ as before. As usual, there are no real EF's. The Green's is now shown in p 294 A where we have expo decay in either direction, and $g(x|\xi;\ \lambda) = (i/2\nu)\ \exp(+i\nu\ |x-\xi|)$, where we have the "well behaved" +iv in the exponential, see earlier work above. Notice in this case we impose NO BC's at all, they are effectively imposed by the $+\infty$ and $-\infty$ endpoints. Recall from above that our δ expansion was

$$\delta(x-\xi) = \int d\nu\ \varphi_\nu(x)\varphi_\nu(\xi) \qquad (4.58a) \qquad \text{where} \quad \varphi_\nu(x) = \sqrt{2/\pi}\ \sin(\nu x)$$

In this problem, I expect it will come out being this

$$\delta(x-\xi) = \int d\nu\ \varphi_\nu(x)\overline{\varphi}_\nu(\xi) \qquad \text{where} \quad \varphi_\nu(x) = (1/2)\sqrt{2/\pi}\ e^{+i\nu x} = \sqrt{1/2\pi}\ e^{+i\nu x}$$

and in fact this is just what 4.75 says with full range integral, and this is the basis of the normal Fourier Transform I am very familiar with. We can write it out like so

$$2\pi\delta(x-\xi) = \int_{-\infty}^{\infty} d\nu\, e^{+i\nu(x-\xi)} = \int_{-\infty}^{\infty} d\nu\, \{ \cos[\nu(x-\xi)] + i \sin[\nu(x-\xi)] \}$$

$$= 2 \int_{0}^{\infty} d\nu \cos[\nu(x-\xi)] = 2 \int_{0}^{\infty} d\nu\, \{ \cos(\nu x)\cos(\nu\xi) + \sin(\nu x)\sin(\nu\xi) \}$$

and this is what (4.73) says. This gives the "non-expo" version of the Fourier Transform where you have to compute both a sine and a cosine projection as in 4.74 (with full range integrals!)

Now, I just guessed the answers here. You can derive this using the g-contour rule for δ, and you then need to compute the discontinuity [g] from $g(x|\xi;\lambda) = (i/2\nu)\exp(+i\nu|x-\xi|)$ and integrate it 0 to ∞ and that must yield the result 4.73. So let's do it:

$\nu = \sqrt{\lambda} = |\lambda|^{1/2} e^{i\theta/2}$ where $\theta = \arg(\lambda)$ runs 0 to 2π.
$\nu^{+} = +|\lambda|^{1/2}$
$\nu^{-} = -|\lambda|^{1/2}$
$g^{+} = (i/2\nu^{+})\exp(+i\nu^{+}|x-\xi|) = (i/2\nu)\exp(+i\nu|x-\xi|)$
$g^{-} = (i/2\nu^{-})\exp(+i\nu^{-}|x-\xi|) = -(i/2\nu)\exp(-i\nu|x-\xi|)$
$g^{+} - g^{-} = (i/2\nu)\{ \exp(+i\nu|x-\xi|) + \exp(-i\nu|x-\xi|) \}$

But this function of $|x-\xi|$ is even in this variable, ie, we have $F(|x-\xi|) = F(-|x-\xi|)$ and so we can just replace $|x-\xi|$ with either $x-\xi$ or $\xi-x$, use the first and we get

$$g^{+} - g^{-} = (i/2\nu)\{ \exp(+i\nu(x-\xi)) + \exp(-i\nu(x-\xi)\} = (i/2\nu)\, 2\cos[\nu(x-\xi)] = [g]$$

Then our integral of page 286 says

$$\delta(x-\xi) = -(1/2\pi i)\oint d\lambda\, g(x|\xi;\lambda) = +(1/2\pi i)\int d\lambda\, [g] = (1/2\pi i)(i/2)\, 2\int d\lambda \cos[\sqrt{\lambda}\,(x-\xi)]$$

$$= (1/2\pi)\int_{0}^{\infty} d\lambda \cos[\sqrt{\lambda}\,(x-\xi)] \qquad \text{but } \lambda = \nu^2 \text{ so } d\lambda = 2\nu d\nu \text{ and get}$$

$$= (1/\pi)\int_{0}^{\infty} d\nu \cos[\nu(x-\xi)] \qquad = 4.73.$$

I am sure we could more rigorously obtain these results starting with the interval $(-\ell,\ell)$ where we would have discrete poles on the positive λ axis which then coalesce into the branch cut.

Exercise 4.23 We are asked to redo our example problem with u(0) = A, a complex number. I think the "cosine transform" problem with u'(0) = 0 is really the same as this problem.

Exercise 4.24 Now do it again with the BC being u'(0) + h u(0) = 0. He claims this BC has a heat conduction in rod interpretation. I think in this case there is a λ pole at λ = -h and then the usual branch cut on the positive λ axis. This is then why the δ expression shown has the "extra" term if you write it when h > 0. If h<0, there is no pole and you don't get this extra term.

4.4 The General Singular Boundary Value Problem (295)

The setup:

We set up the problem on (a,b) allowing these might go infinite, we take our general formally self-adjoint L as usual so L = L*, and we have all our "regular conditions" understood, with the idea that we might violate one or more of them to be "singular". We introduce a new Hilbert Space H_s which is just $\langle u,v\rangle_s$ with s-weighted integrals, no big deal. This of course implies a certain norm $\|u\|_s$ and if u has such a finite norm, then the function u is "of **finite s-norm** over (a,b)", totally reasonable.

[296] Now we write our "usual equations" like A making use of the form of the ODE. Equation B is then the integrated form of A, BUT, for now we imagine a finite interval (a_0, b_0) within (a,b) to handle possible limit situations, so that is why B has these subscripted animals. We then imagine in C that we can separately take endpoint limits. In all these equations A,B,C the functions u are just "twice differentiable. But for the limit in C to exist, we have to add a condition on u and v: both u and Au must have finite s-norms (and same for v). When we restrict our u and v to this smaller space of functions, we call this space D: finite s-norm, but Au is also finite s-norm. He claims D is a subspace of H_s.

[297] Now instead of generic u and v, we assume u = v = φ which is some (assumed to exist) EF of Lφ = λsφ. Our general results A,B,C now become A',B',C' as shown here. So φ must be of finite s-norm.

Weyl's Theorem: [1910] Let "x=b be a singular point" (this can have several meanings). Two claims are made: { my comments are now obsolete, see meta notes on Weyl)

(1) Suppose for *some* λ every solution of our ODE (no BC's) has finite s-norm. Then for *any other* λ (call it μ) the same is true. [It is not clear to me what "every solution" means here. For n=2 ODE it could mean that "both solutions" have finite-s norm.] [Somehow this is the limit-circle case.]

(2) For any λ off the real axis, our ODE has at least one s-norm solution. [I think in this case, there is in fact ONLY one solution, and this is the "limit point" case. You cannot impose any b=∞ extra conditions.]

This from http://books.google.com/books?id=a8jk-dmOOHgC: This book has much more detail that I want on this subject but here is some history:

1. Introduction

Regular boundary conditions of the type discussed in Part I do not make sense at a singular endpoint because, in general, the solutions and their quasi-derivatives are not defined there. What should take their place? The answer depends on the nature of the singularity: limit-point (LP) or limit-circle (LC). This terminology was introduced by Hermann Weyl in 1910 [**607**] in connection with a geometric construction of concentric circles in the plane each one contained in the preceding one. In the limit these circles converge to a circle - the limit-circle (LC) case - or to a point - the limit-point (LP) case. For the latter case no boundary condition is required or allowed, but the LC case requires boundary conditions to determine self-adjoint extensions. This paper of Weyl is one of the most widely quoted papers in analysis and can be credited with starting the theory of *singular* SLP. Its extension to the fourth and higher order cases had to wait until about 1950 when Glazman published results from his dissertation under M.G. Krein; see the classic book by Naimark [**487**] for a detailed exposition of this work. This landmark paper of Glazman corrected some fundamental errors in earlier works and provided a Hilbert space framework for the study of second and higher order boundary value problems. Although this approach does not make use of the Weyl circles the LP, LC terminology has survived and is still universally used today.

Proof of Weyl's Theorem Part (1)

Hopefully in this proof, we well gain clarity on what the theorem is actually saying. I see that I went through this in some detail at one time.

We start off assuming that u_1 and u_2 are two independent solutions of our ODE problem for some given value of λ, and we assume $u_{1,2}$ have finite s-norm. We then want to consider a function v which we assume is a solution to the same ODE with a different value of λ which he calls μ. Now on my first pass, I just want to see some of the landscape and what we are proving. He starts by defining a function $\varphi_\xi(x)$ in terms of u_1 and u_2, and then he rolls out his Volterra solution (which can handle ∞ endpoints!) from way back in Chapter 1 as 1.50 on page 63, but here we apply it not to the inhomo, but to our EV problem. As he says, ψ is some u_1 u_2 lincomb which solves $L_\lambda u = 0$ which is the homo part of page 297 D, and our integral is the "particular" part. I have not proven p 298 A, just trying to see what it is, what is in it. It is a solution to p 297D. So right off the bat, we have a candidate solution of our ODE for $\mu \neq \lambda$, and that candidate is v(x). All the rest of page 298 is for one purpose: to show that this v(x) is of finite s-norm. Thus, he has proven claim (1) of Weyl's theorem as stated above. And now we know that "every" means "both". As I noted in pencil, the term limit-circle never appears here, this is a historical artifact from the way Weyl did things in 1910. Why were both u_1 and u_2 necessary in this proof? I think if either was missing, the function $\varphi_\xi(x) \equiv 0$ and then things fall apart.

Proof of Weyl's Theorem Part (2)

The opener here is similar to past openings. We come up with φ and ψ as independent solutions of our ODE but they each satisfy a certain general-looking BC at x = a (only), but the constants α_1 and α_2 are arranged so φ and ψ *are independent* by the Wronskian test. Then $u = \varphi + m\psi$ is a family of solutions of the ODE. We want to apply a BC to u at b (with $\beta_{1,2}$), but we do it for the moment at b_0 rather than b, since we don't know whether u even exists for the limiting value b. Now by my notated argument re pp 269-270, the BC 4.82 forces 4.82a because it makes the W = 0. This is because the elements of the first column of W are dependent by the BC.

Now it is a simple matter to rewrite 4.82a as shown in 4.83 and 4.83a (I have done it all in pencil). Now this means the square bracket in 4.83a is 0. We write it as:

$$W(\psi,\bar{\psi})\ \{\ m\bar{m}\ +\ m\ W(\psi,\bar{\varphi})/W(\psi,\bar{\psi})\ +\ \bar{m}\ W(\varphi,\ \bar{\psi})/W(\psi,\bar{\psi})\ +\ W(\varphi,\bar{\varphi})\ /W(\psi,\bar{\psi})\ \ \}$$

We then claim that

$$\{\} = (m\text{-}A)(\bar{m}\text{-}\ \bar{A})\ -\ r^2\ \ =\ \ m\bar{m}\ -\ m\ \bar{A}\ -\ \bar{m}A\ +\ A\bar{A}\ -\ r^2$$

and comparing this requires

$$\bar{A} = -\ W(\psi,\bar{\varphi})/W(\psi,\bar{\psi}) \qquad 1$$
$$A\ = -W(\varphi,\ \bar{\psi})/W(\psi,\bar{\psi}) \qquad 2 \qquad \text{// this is page 299 D.}$$
$$A\bar{A}\ -\ r^2\ = W(\varphi,\bar{\varphi})\ /W(\psi,\bar{\psi}) \qquad 3$$

Are the first two lines consistent? Start with line 2 and CC to get

$$\bar{A} = -\ \bar{W}(\varphi,\ \bar{\psi})\ /\ \bar{W}(\psi,\bar{\psi}) = -\ W(\bar{\varphi},\psi)/\ W(\bar{\psi},\psi) = -\ [-W(\psi,\bar{\varphi})]/[-W(\psi,\bar{\psi})] = -\ W(\psi,\bar{\varphi})/W(\psi,\bar{\psi})\ \ = A$$

so yes, it works. Meanwhile, we have

$$r^2 = A\bar{A}\ -\ W(\varphi,\bar{\varphi})\ /W(\psi,\bar{\psi})\ \ =\ \ W(\varphi,\ \bar{\psi})/W(\psi,\bar{\psi})\ *\ W(\psi,\bar{\varphi})/W(\psi,\bar{\psi})\ -\ W(\varphi,\bar{\varphi})\ /W(\psi,\bar{\psi})$$

$$= W(\varphi,\ \bar{\psi})/W(\psi,\bar{\psi})\ *\ W(\psi,\bar{\varphi})/W(\psi,\bar{\psi})\ -\ W(\varphi,\bar{\varphi})\ W(\psi,\bar{\psi})\ /[W(\psi,\bar{\psi})\ W(\psi,\bar{\psi})]$$

$$= \{\ W(\varphi,\ \bar{\psi})W(\psi,\bar{\varphi})\ -\ W(\varphi,\bar{\varphi})\ W(\psi,\bar{\psi})\}\ /W^2(\psi,\bar{\psi}) \qquad \text{// as shown p 300 A}$$

If we CC r^2 we get it back again, so we know it is real. But it is not obvious to me that $r^2 > 0$, but I suppose we could show this perhaps using CSI.

Well he is going to show this for r right now. I agree with B, C seems reasonable, and D we know from p 299 top, so we get 4.84 left which says yes, $r^2 > 0$ and the right gives r.

Now on the bottom of page 300 Stak goes on to show in H the equation of the circle written in a very strange manner -- m is inside u on the LHS and Im(m) appears on the right. To get this, he first shows 4.85. But we know that when m is "on the circle", $W(u,\bar{u}; b) = 0$. Thus, "on the circle" the RHS of 4.85 is zero, and this gives H. I don't know how to show that I gives the "inside" of our circle, but OK, it has to be one or the other!

Now come some punch lines! As b_o increases, our radius r in 4.84 gets smaller because we are integrating a positive thing $|\psi|^2$s. If this $|\psi|^2$s integral diverges as $b_0 \to b$, we approach r = 0 and our circle has shrunk to a point. If the integral is finite, we approach a limiting circle instead. In either case, our limiting thing (circle or point) lies inside the original b_o equation I, so we end up with page 301 B which says that our solution u(x) has finite s-norm. Thus, we have shown that we can find at least one s-norm solution for $\mathrm{Im}\lambda \neq 0$, and that solution is our u(x), and this was the claim of part 2 of Weyl's Theorem.

Now in the limit-circle case, we have a finite $|\psi|^2$s integral. This means that $\psi(x)$ and u(x) are both viable solutions. But the general solution ("any solution") must be a lincomb of these two, and therefore in the circle case, all solutions are s-norm. So "limit circle" says we have two s-norm solutions.

In the limit-point case, ψ is not a solution, so there is only one solution, u(x) !

Time to back up and see what has really happened here!

Review of Part (1) of proof.

First, let's look at what Stak has proved in his two parts: In the first part, he showed this to be true:

(1) Suppose for *some* λ (call it λ=λ) our ODE (with no BC's) has two independent solutions which are of finite s-norm. Then for *any other* λ (call it λ=μ), there are also two independent solutions of finite s-norm. [At least for λ=μ, any solution that exists will be finite s-norm. See below.]

I think this is what he has shown. Let's review his work on this subject. He takes u_1 and u_2 to be the two independent finite s-norm solutions of our ODE when $\lambda = \lambda$. He then claims that *any* solution of the same ODE with $\lambda = \mu$ must satisfy equation 297 D which we can think of as saying $L_\lambda v = f$ with $f = (\mu-\lambda)sv$. We know that the homo solution of this equation is $A u_1 + B u_2 \equiv \psi$. The particular solution given in 1.51 is the main term in 298 A. But now since $f = (\mu-\lambda)sv$, our equation 298A is an integral equation for v. If we were to apply L_λ to both sides in order to "check" the validity of this integral equation, we would have $L_\lambda \psi = 0$, so the ψ term would contribute nothing. We would then try to show that the L_λ(main term) = $(\mu-\lambda)sv$, and that was the work that led to 1.51 so we don't want to redo that.

So my point is that, in page 298 A, the $\psi(x)$ is an *arbitrary mix* of u_1 and u_2. But these are independent functions. Suppose you set $\psi = 0$ and you solve the integral equation and you get some solution function V(x). Then the general solution of page 297 D is this:

$$v(x) = A\, u_1(x) + B\, u_2(x) + V(x)$$

So here then are two viable solutions of the problem having $\lambda = \mu$:

$$v_1(x) = A\, u_1(x) + V(x)$$
$$v_2(x) = B\, u_2(x) + V(x)$$

For these to be *de*pendent, we would have to have $Au_1 = Bu_2$, but this cannot be since $u_{1,2}$ are *in*dependent. Therefore, once we learn that V(x) has finite s-norm, we then have found two independent solutions for our problem with $\lambda = \mu$ which are both finite s-norm, since we assumed at the start that both $u_{1,2}$ were finite s-norm. So interestingly, changing λ away from λ just causes this V(x) term to appear, and we expect it to be a function of μ and λ, and probably is proportional to $(\mu-\lambda)$, or at least we expect it to vanish when $\lambda = \mu$. We don't care much about the details of the solution V(x).

How do we know that our V(x) integral equation (with $\psi = 0$, say) has a solution? Well, on page 63 we showed that it has a solution for any f(x) you come up with. Well, the argument is a little circular here. Suppose you magically knew the solution ahead of time and put it in on the RHS. Then the equation would be true. But I have to assume a solution exists in order to "put it on the RHS". So if V(x) exists, it solves this integral equation. Our theorem really only says that "every solution" at λ=μ is finite s-norm, so if no solution exists, we are still OK. The theorem only has to be true for solutions that exist. But I think we could somehow show that V(x) exists.

Notice that no boundary conditions were used in this proof. There might be BC's at the point a and or b, but we ignored them. At least at point a, our ODE system has a BC.

Restate (1): If the ODE has two finite s-norm solutions for one particular value of λ, then for any other value λ any solution must be finite s-norm.

There is nothing here about points and circles!

Review of Part (2) of proof.

The issue here I think is only the question of whether the ODE system has 1 or 2 solutions.

We start here with functions φ and ψ. These are both solutions of our ODE (without any BC's) but we then impose a specific endpoint BC at x = a on φ, and a different BC on ψ. So these functions φ and ψ are really solutions to two different ODE "systems", but all we care about here is the ODE itself.

We are really taking b to be some finite b_0 at this point, and *that* is how we know that we have two ODE solutions. The two BC's force φ and ψ to be independent solutions of the ODE. It is interesting that we don't have to actually *find* these functions, we just know they exist.

Now at the other endpoint b_0 we impose our second BC (4.82) with $\beta_{1,2}$ and we expect this will "nail down" our solution. If we assume our solution is $u = \varphi + m\psi$, we expect our second BC will determine the constant m. In any event, we know that $W(u,\bar{u}; b_o) = 0$ just from the imposed BC.

Now when we do out the algebra for $p(b_0)W(u,\bar{u}; b_o) = 0$, we find that there is not a specific m that works, but rather a locus of points m which lie on a circle in the complex-m plane with radius r and centered at some point A which is a function of φ, ψ and b_0. I think the reason we don't get a specific m is that, at the endpoint a, we have not applied the *same* BC to φ and ψ. We somehow think that if φ and ψ have the same BC's at endpoints a and b_0, then there must be a unique solution which would mean one value of m that works. But in this work we don't include BC's in our ODE "system", and we are free to impose BC's as we want, separately for φ and ψ, in order to control their nature as we like.

The radius of the circle is shown in (4.84).

Now all the above was just setup for the limit $b_0 \rightarrow b$ we want to take. Notice that he does not say b is going to be ∞ or $-\infty$, just some value b. I guess from the form of r in 4.84 that we assume we approach b from the lower side. So as we approach the limit, the radius gets smaller.

Now is the place where we get our dichotomy: if the s $|\psi|^2$ integral diverges in 4.84 as $b_0 \rightarrow b$, then we know that ψ is NOT a viable solution in the limit. In this case, our radius is going to $r \rightarrow 0$ and we are in the "point" situation. We had a circle as our locus of m which made our φ and ψ do what we wanted. But now m is just a single point m_c, and only that specific $u = \varphi + m_c\psi$ is a solution that is s-norm finite. We know this special solution is finite norm because that is what page 301 B says.

Notice that the center of the circle is a function of b_0 according to p 299 D, so as we move toward the limit, the radius r and the center A both change, so we don't have a set of concentric circles here. He claims without proof that any circle is at least contained in a previous circle as you increase b_0.

On the other hand (other half of our dichotomy), maybe the s $|\psi|^2$ approaches a finite value as $b_0 \rightarrow b$. In this case, we end up still with a locus of points m on a circle and all points "work". In this case, we know that $u = \varphi + m\psi$ is a viable finite s-norm solution, and so is ψ ! We know that φ and ψ are independent, so we know that u and ψ are independent solutions of the ODE. We have sort of constructed φ and ψ which make all this true.

So what then is our conclusion?

(2a) If $\int dx\ s\ |\psi|^2$ diverges as $b_0 \to b$, there is exactly one solution to our "ODE without specified BC's". We specified some BC's just to help us in our proof. The solution is $u = \varphi + m_c\psi$ where m_c is the limit-point of our shrinking circle as $b_0 \to b$. On the other hand, if $\int dx\ s\ |\psi|^2$ is finite as $b_0 \to b$, then there are two finite-s-norm independent solutions of our ODE. This is the limit-circle case. In either case, there does exist at least one solution (when $\mathrm{Im}\lambda \neq 0$).

Final Conclusions from Weyl's Theorem

For a given L,s,a,b, we are in one or the other of these two cases when b is a singular endpoint:

Case 1: Suppose for our ODE we are in a **limit-point** situation as $b_0 \to b$ for some λ off the real axis. We know that for any such λ there is only ONE finite-s-norm solution to the ODE, regardless of BC's.

Case 2: Suppose for our ODE we are in a **limit-circle** situation as $b_0 \to b$ for some λ off the real axis. In this case we know there are TWO finite-s-norm solutions to the ODE, regardless of BC's. But in this case, we can apply the "first part" of our theorem to conclude that there are then two finite-s-norm solutions as we do $b_0 \to b$ for any λ in the complex λ plane (including on the real axis).

The two cases are mutually exclusive. Which case you are in depends on L, a, b. It is sort of an "alternative theorem" idea.

Examples (301) // they are all excellent!

Example 1. Let's consider our standard $L = -D^2$ problem on $(0,\infty)$, no BC's. If $\mathrm{Im}\lambda \neq 0$, *one* of the expos will be a viable normalizable (finite s-norm, s=1 here) solution. From above, for his branch choice,

$$\exp(+i\sqrt{\lambda}\,x) = \exp(ix\,|\lambda|^{1/2}\cos\theta/2)\,\exp(-x\,|\lambda|^{1/2}\sin\theta/2) \;=\; e^{i\alpha x}\,e^{-\beta x}$$

If λ on the negative real axis, $\theta = \pi$, $\sin(\theta/2) = 1$, solution is OK, again, just this one solution.
If λ on the positive real axis, there is no finite-s norm solution.
Therefore, for this L, s, a, b we must be in the limit-point case!

Example 2. Bessel n=0. (a) Here s = x and p = x and interval is (0,b). Since p(0) = 0, a=0 is a singular endpoint. For λ=0, however, it is easy to find two finite s-norm solutions. We can then apply Part (1) of the Weyl theorem to conclude that for any λ there will be two finite s-norm solutions, and these are the $J_o(\sqrt{\lambda}\,x)$ and $N_o(\sqrt{\lambda}\,x)$ functions. So this is limit-circle case for a = 0.

(b) If you change the interval to (a,∞), then b is a singular point. For $\lambda = 0$, both our previous solutions are non finite s norm. To be in limit-circle, both would have to in fact be finite s norm, so we must be in limit-point. Remember that one solution must exist for $\mathrm{Im}\lambda \neq 0$, so none need exist for $\lambda = 0$ or on the real axis. In this case it is one of the Hankel's that is the single finite s norm solution.

Question added 3.12.10. WHY is the $H^{(1)}(\sqrt{\lambda}\, x)$ function the only finite s-norm function in (b) above? As I look at the large-z behavior of the various Bessel functions, it seems that they all go as $1/\sqrt{z}$ times either a cosine in z, or an expo in iz, so in all cases the other factor is oscillatory. It seems to me offhand that therefore all the Bessel functions of J,N,H variety would "work" at the high end! Finite z-norm means $\int dx\, s(x)\, |u|^2$ has to be finite at the high end *for λ off the real axis!* That is the key point. Also, we know s(z) = z, so $1/\sqrt{z}$ is not going to give us convergence, we need expo convergence somehow.

If we back up to Stak page 262, we see the function $\sqrt{\lambda}$ defined not in the usual way, but with the branch cut taken off to the right, so the cut is on the positive real axis! I think this is a very important part of the theory. The discrete positive poles coalesce into a branch cut in this region. So this means that we think of λ as having arg(λ) in the range (0,2π), and that means arg($\sqrt{\lambda}$) lies in (0,π) which means that Im($\sqrt{\lambda}$) > 0 for λ not on the positive real axis. That means $\sqrt{\lambda}\, x$ has a positive imaginary part for ANY λ in the complete λ plane as just described. That means that exp(+iz) = exp(+i$\sqrt{\lambda}\, x$)) will have expo decay for any λ off the real axis. This is the large-z behavior of $H^{(1)}(\sqrt{\lambda}\, x)$ and NOT $H^{(2)}(\sqrt{\lambda}\, x)$ or any other Bessel function type (see p 364 A&S) ! So only $H^{(1)}(\sqrt{\lambda}\, x)$ will have expo decay for any λ off the real axis, and therefore only $H^{(1)}(\sqrt{\lambda}\, x)$ will be finite s-norm at the high end.

Example 3. Hermite. Here s=1 p=1 q=x^2 and (-∞,∞). The trick is always to pick some λ for which you can find simple solutions. Here, λ = -1 yields two solutions, but at least one of them is non-finite s norm at both end points. Each endpoint is separately a singular point. This has to then be limit-point at each end.

Example 4: 2D Laplace Radial. Here s = 1/x and p=x and (0,∞). So here we have two singular points to think about. We take λ = 0 and identify two solutions. They both diverge at both endpoints, so here we are in limit-point at both endpoints, independently. Here it is the s = 1/x that causes divergence at the lower endpoint for the $u_1 = 1$ solution (for example), and you get ln(∞) divergence at the upper end as well. For Imλ ≠ 0, the single solution is $x^{-\mathrm{isqrt}(\lambda)}$.

Example 5. Legendre. Here p = 1-x^2 s = 1 and (-1,1), Both endpoints are thus singular. Pick λ = 0, there are two simple solutions. At each endpoint separately, you have finite-s norm for each solution, so both endpoints are the limit-circle case. Here λ = -$\ell(\ell+1)$ see Schaum p 146, so λ=0 means $\ell = 0$ and one solution is then going to be $P_0(x) = 1$ as noted here. The other solution is $Q_0(x) = (1/2)\ln[(1+x)/(1-x)]$ just as he claims. So, we know that for any ℓ (any λ), there will always be two normalizable solutions on this interval! I guess that is something I did not know -- the Q_0 is always normalizable. If is not of course a polynomial. We don't seem to use these much in practice.

The Green's Function when endpoint b is singular (303)

First, at endpoint "a" he assigns to g the BC that he assigned to ψ in the Weyl part 2 proof, 299B. This is certainly consistent with putting ψ on the left side of p 303A for g. Remember the u,v_ξ method where you put some homo solution u on the left, and some other one on the right, and make things right at the matching point x = ξ. So here, we put ψ on the left with its BC at x=a. What should go on the right for g shown in p 303 A ? We know it has to be a solution of homo. Normally it would match the BC at x=b, but here that BC is in effect that the right-side function has to be finite s-norm when b=∞. That is why our special u is the right candidate for the right side. So now I like p 303A,

Digression for general fact: Suppose $g = f(x_<)\, h(x_>)$. Then

$$dg/dx^+ = f(\xi)h'(\xi)$$
$$dg/dx^- = f'(\xi)h(\xi)$$
$$dg/dx^+ - dg/dx^- = f(\xi)h'(\xi) - f'(\xi)h(\xi) = W[\, f,h;\, \xi\,]$$

So, the jump is always just the Wronskian and we must set it to $1/a = -1/p$.

Fact #2: $W[\, a, b+c;\, x] = W[\, a, b;\, x] + W[\, a, c;\, x]$ // linear in either term

I am completely happy with the evaluation of constant A from the jump condition as done bottom of page 303

[304] Recall that m is shown page 299 4.83. The solutions ψ and φ are always analytic in λ. Therefore, the denominator of m is analytic in λ, call it $d(\lambda)$. An analytic function can of course vanish at some λ_1. At such a λ, we could expand $d(\lambda)$ in a power series since it is analytic. Then $d(\lambda) = 0 + (\lambda-\lambda_1)d'(\lambda_1) + \ldots$ but this says that $d(\lambda)$ vanishes at "simple zeros", not something like $(\lambda-\lambda_1)^{1/2}$, say. This means that $m(\lambda)$ can only have poles (regular problem) and is therefore meromorphic.

Comment about BC at the regular end. Back on page 299 where we started developing "the theory" which leads to limit-circle and limit-point, we assumed a certain BC A and B for φ and ψ. If we look only at the ψ BC p 299A, it is a *completely general* unmixed BC of the form $B_a(\psi) = A\psi(a) + B\psi'(a) = 0$. It is true that the two BC's A and B are "correlated" in such a way as to force $W(\varphi,\psi) \neq 0$ so φ and ψ are independent. But in most of our work, we will never even mention φ and you will see ψ occurring at a regular endpoint. So the point is that the regular endpoint condition $B_a(\psi) = 0$ really is fully general. Remember that in our initial formulation even of the "regular boundary value problem" on page 268, we required an unmixed BC at each regular endpoint as shown in 4.26 on page 269. So here when we have only one regular endpoint, we stay with this plan.

Limit-Point Analysis

We end up here with $u = \varphi + m_b(\lambda)\, \psi$ where m_b is "the limit point", and u then has finite s norm, as $b_0 \to b$. Now, back on page 303 A he wrote $\hat{g}$ to indicate this was for b_0, and g will apply to b. But we know all about what happens to u in 303 A in our limit, so no need for fancy notation.

It would not surprise me to learn that a bunch of poles of $m(\lambda,b_0)$ might coalesce into a branch cut when we get to the limit $m(\lambda,b)$. Now recall from our earlier work above:

$$(1/2\pi i) \oint d\lambda\, g(x|\xi;\lambda) = -\,\delta(x-\xi)/s(x) = -\int d\nu\, \varphi_\nu(x)\, \overline{\varphi}_\nu(\xi) \qquad \text{// } b=\infty \text{ singular problem}$$

$$(1/2\pi i) \oint d\lambda\, g(x|\xi;\lambda) = -\,\delta(x-\xi)/s(x) = -\sum_n \varphi_n(x)\, \overline{\varphi}_n(\xi) \qquad \text{// regular problem}$$

Now, notice from p 303 (4.93) that $g(\lambda) \sim m_b(\lambda)$ in terms of singularities in λ. Stak did not quite make this point clearly. We know that the great circle integral will equal the indented contour shown on page

286 (ignoring sign) and that this indented thing must pick up any poles of g as well as the branch cut discontinuity if present. If g has BOTH poles and a branch cut, I expect a form as shown in (4.95), based on what we see above for the two separate situations. Such a problem would have a "mixed" spectrum.

$$(1/2\pi i) \oint d\lambda\, g(x|\xi;\lambda) = -\delta(x-\xi)/s(x) = -\sum_n \varphi_n(x)\,\bar{\varphi}_n(\xi) - \int d\nu\, \varphi_\nu(x)\,\bar{\varphi}_\nu(\xi) \qquad // (4.95)$$

Suppose we absorb s(x) equally into the φ and call them ψ. Then we have

$$\delta(x-\xi) = \sum_n \psi_n(x)\,\bar{\psi}_n(\xi) + \int d\nu\, \psi_\nu(x)\,\bar{\psi}_\nu(\xi)$$

To me, this clearly says that "a complete set" of eigenfunctions must include both contributions. If there is no continuous part, then the ψ_n form a complete set (the φ_n are complete with weight s). He hedges a little on this fact bottom page 304, but I am very happy with it. [His hedge is this: you have to get your ODE into proper integral equation form, then you have to show that the kernel is symmetric and HS. Only then do you have a real proof that the EF's form a complete set.]

Examples of Limit-Point Cases (p 305)

Example 1. (p 305) **Bessel's Equation of general order ν (the Fourier-Bessel Transform)**

This is a long example. If we compare Stak's equation to the real Bessel equation on p 136 Schaum, we find that our equation is

$$x^2\varphi'' + x\varphi' + [\lambda x^2 - \nu^2]\,\varphi = 0 \qquad \Rightarrow \qquad n = \nu \quad \text{and } \lambda = -1$$

So we have to set λ = 1 in order to get the real Bessel equation. We have s = x and p = x and interval is (0,1) at least for this example. Since p(0) = 0, endpoint 0 is singular. ν is ≥ 0 as Schaum confirms.

If set λ = 0, then the above equation has simple solutions $\varphi = x^{\pm\nu}$, $\varphi' = \pm\nu x^{\pm\nu-1}$, $\varphi'' = \pm\nu(\pm\nu-1)x^{\pm\nu-2}$ so we have

$$x^2\varphi'' + x\varphi' + [-\lambda x^2 - \nu^2]\,\varphi =$$

$$x^2\,[\pm\nu(\pm\nu-1)x^{\pm\nu-2}] + x[\pm\nu x^{\pm\nu-1}] + [-\lambda x^2 - \nu^2]\,x^{\pm\nu} =$$

$$[\pm\nu(\pm\nu-1)x^{\pm\nu}] + [\pm\nu x^{\pm\nu}] + [-\lambda x^2 - \nu^2]\,x^{\pm\nu} =$$

$$\{\pm\nu(\pm\nu-1) \pm\nu - \nu^2\}\,x^{\pm\nu} = \{\nu^2 -/+ \nu \pm\nu - \nu^2\}\,x^{\pm\nu} = 0 \qquad \text{ok}$$

Now if ν < 1, both solutions are finite s-norm, no problems at x=0 endpoint, and we are limit-circle. But we care right now about ν ≥ 1 which then causes the $x^{-\nu}$ solution to not be finite s-norm, so we are then in the limit-point case which we are now studying. Fine.

Now he constructs a Green's, and I always have to ponder this a bit. He imposes BC φ(1)=0 at the upper regular endpoint, and I agree that the lincomb shown in the right side of D meets this BC. This is a

homo solution that meets the right side BC. At the left end we know that $x^{-\nu}$ is bad, not finite s-norm at that end, so we pick x^{ν} . Fine. He computes C = 1/(2ν) offline, I accept. So happy with p 305 D.

Next, he converts the ODE problem to an **integral equation** "in the usual manner".

$L\varphi = \lambda s\varphi \quad \Rightarrow \quad \varphi = L^{-1}(\lambda s\varphi) = G(\lambda s\varphi)$ which is p 305 E

and he then casts this into "standard form" as in F. He claims that this kernel k is HS. We know it is symmetric since L is always self-adjoint and symmetric (green's is symmetric, etc). So, we know that our eigenfunctions form a complete set. Assume the discrete set of EV's is λ_i.

So next question: what are our eigenfunctions? Go back to the above:

$$x^2\varphi'' + x\varphi' + [\lambda x^2 - \nu^2]\,\varphi = 0$$

We need a variable change to get this into Bessel form including λ, so try

$\Phi(y) = \varphi(x) \qquad y = \sqrt{\lambda}\, x$

$D_x\varphi = D_y\,\Phi\,\sqrt{\lambda}$

$D_{xx}\varphi = D_{yy}\,\Phi\,\lambda$

$$x^2\varphi'' + x\varphi' + [\lambda x^2 - \nu^2]\,\varphi = 0$$

$$\{y/\sqrt{\lambda}\}^2\,\{D_{yy}\,\Phi\,\lambda\} + \{y/\sqrt{\lambda}\}\{D_y\,\Phi\,\sqrt{\lambda}\} + [y^2 - \nu^2]\,\Phi = 0$$

$$y^2 D_{yy}\Phi + y D_y\,\Phi + [y^2 - \nu^2]\,\Phi = 0$$

The solutions of this thing are $\varphi(x) = \Phi(y) = J_\nu(y) = J_\nu(\sqrt{\lambda}\, x)$. But our BC φ(1)=0 says $J_\nu(\sqrt{\lambda}) = 0$, so this is where the EV's come from! We have then $J_\nu(\sqrt{\lambda_i}) = 0$. We know this sine-like function has an infinite number of zeros, and each one gives an EV as shown.

$\varphi_i(x) = J_\nu(\sqrt{\lambda_i}\, x)$ // not normalized

What is with the $N_\nu(\sqrt{\lambda}\, x)$ functions, why are we ignoring them? I think we could show that these are not s-norm at the x=0 end. We are limit-point here, and there is only one solution and it is $J_\nu(\sqrt{\lambda}\, x)$. If we go on to insist on our BC at x=0, then we get the $\varphi_i(x) = J_\nu(\sqrt{\lambda_i}\, x)$.

Comment: we are using to perhaps saying that $J_n(x)$, n=0,1,2... is a "complete set", at least this is what we do for Legendre functions. But in what we are doing in this example, the "order" ν is just a constant.

Now, in p 305 D we had the Green's for λ = 0, but now we want Green's for arbitrary λ. On the left side we have to use $J_\nu(\sqrt{\lambda}\, x)$ because only it is s-norm finite at the singular endpoint x = 0. On the right we must use the lincomb shown as $Z_\nu(\sqrt{\lambda}\, x)$ which matches φ(1)=0 very obviously. Then as usual we

need to find the constant A, and as usual this means we need the Wronskian shown in p 306 F. So we finally end up with p 306 I, I agree.

[307] So we want to know where the poles and cuts of g might be. He shows that the cut in the denominator cancels that in the numerator, so we end up with poles at the eigenvalues λ_i whose square roots we know are the zeros of $J_\nu(x)$. On this page our main effort is to compute the residues of the poles in $g(\lambda)$. I did all the math and we get the residue entirely in terms of J_ν as shown in H. We can then find our expression for $\delta(x-\xi)$ by doing the g contour integral and this gives I. From this, we can then read off the normalized EF's of this problem! If you talk about normalized with respect to the inner product without s = x, then you include $\sqrt{x}$ in each EF as in J. But if we use the s-weighted inner product, we omit it and then we get

$$\varphi_i(x) = J_\nu(\sqrt{\lambda_i}\, x) \; * \; \{ \sqrt{2} \,/\, J_\nu'(\sqrt{\lambda_i}\,) \} \qquad \text{// normalized}$$

[308] So, we have a complete set of eigenfunctions, we have our $\delta(x-\xi)$ as in I, and for this problem (as with every problem) we get a "custom" transform pair. Here it is called the **Fourier-Bessel Transform**, because the projection is onto Bessel functions. In the Fourier-Sine Transform, we projected onto sine functions, and so on. I have never even heard of much less used this transform! This thing is like a Fourier Series, not a Fourier Integral Transform, but there is a corresponding integral transform. It is the Hankel transform and has integral up to ∞, which typically causes the continuum aspect.

Comments on Example 1: It is the Bessel equation when $\lambda = 1$, but has elementary solutions when $\lambda = 0$ which let us quickly determine that we have the limit-point case when $\nu \geq 1$. Stak finds the Green's function $g_\lambda = g$ by imposing $\varphi(1)=0$ and just goofing around. He then uses the usual Greens contour integral for $\delta(x-\xi)$ and from this he finds exactly how the complete set of basis functions $\varphi_i(x)$ are normalized! He never had to do any integrals to find this out, he used the usual and simple Bessel Wronskian. The spectrum is completely discrete, the λ_i are obtained from zeros of J_ν, the N_ν are not solutions for this problem's interval.

Again, this limit-point example has a discrete spectrum, there is no continuous piece.

Example 2. (p 308) **The Mellin Transform**

We start off here reconsidering Example 4 on page 302 which was that "radial equation of 2D Laplace" with interval $(0,\infty)$ and both ends were limit-point singular. Although the solutions were not mentioned in Example 4, they are very simple: $x^{\pm i\sqrt{\lambda}} = \exp[\, \pm i\sqrt{\lambda}\, \ln(x)\,]$. He confirms again using these general solutions that both endpoints are limit-point (but we knew it already). He writes $\sqrt{\lambda} = \alpha + i\beta$ and $\beta > 0$ for his way of doing the branch cut (as usual). So, for our Green's function we take at each end the only solution that is s-norm finite there (so no need to impose any external BC's on this problem). So by inspection we have the Green's written as E and then as F where the constant is evaluated from the jump rule. And as usual, we want to do the Green's contour shown in G. This Green's has a branch cut on the real λ axis and he computes its discontinuity [g] in the usual manner as shown p 309 D, and then our rule is E. He does the disc integral and gets 4.106 which is fairly simple looking, though it has two terms which can be folded into a single term with a full integration range as in 4.106a.

As usual, each $\delta(x-\xi)$ "form" gives rise to a transform. In this case, since spectrum is continuous, both directions are integrals, and the two directions are given in 4.107 and 4.108. They do look strange to me but I know I have used them somewhere once upon a time in some form. Mellin-Barnes?

He points out that the simple change of variable $y = \ln(x)$ changes this problem to the result Fourier Integral Transform problem. Again, I always wonder in what circumstances you might choose to use a particular transform pair. Surely the geometry or something else must suggest the best choice, but we get no comments (at this point, maybe next volume).

Conclusion: This Mellin example has a purely continuous spectrum, no discrete part.

(end of examples, resume text)

[310] Here he considers the inhomo $L_\lambda u = f$ and the usual answer is $u = Gf$ (including s factor). He reinforces the notion that at a limit-point endpoint, you won't need to (or be able to) apply a BC, since there is already only one solution "there". Secondly, he reminds us that a solution u of the inhomo can be expanded on the complete set of EF's, but that set may have both discrete and continuous parts and you in general will have to include them both. [He has not really proven this idea yet.]

Limit-Circle Analysis

As usual, a is regular, b is singular in (a,b). We "impose" on g the same condition 4.92 at x = a, as in all these limit-circle situations. This is the "ψ end" of g, the ψ BC, we have done this all before. The big question is what to do at the "b end" where we don't have a unique solution u, since we are limit-circle. His approach is going to be to try and maintain 4.111 in the limit. On page 299 near A I show that at the a endpoint we have $p(a)W(\psi,\bar{\psi}; a) = 0$ which follows directly from the form of the BC B. So it certainly seems reasonable to seek $p(b)W(u,\bar{u}; b) = 0$ at the right endpoint. He talks about approaching this goal by starting with a right-end BC written at b_0, but he says this has technical problems (but I suspect it could be done), so he is going to sort of require this goal 4.111 by fiat, without having a "normal" BC there. It does not really matter whether you "impose" a normal BC or you "impose" 4.111. The question is: does this imposition "work".

[311] This limit (or equation) 4.111 forces u to be "on the limit circle". Remember that for any particular λ, say λ_0, the circle is a locus of solutions, not a unique solution as in the limit point case. I agree that ψ and "any u on the circle" are independent. The "self-adjoint Abel" (p 72) always tells us that $pW = -C$ where u,v are independent, so we just scale our function ψ as needed to get $C = 1$ (for our given λ_0). This is 4.112.

If we put an A constant in front of (4.113) we would have jump = $AW = -1/p$, but $W = -1/p$ so $A = 1$, so that is why the constant in 4.113 is 1, and that is why he choose to normalize ψ such that 4.112 is true. Notice in 4.113 that we use ψ at "its end", and u at "its end", but u is not unique since on the circle. Each point on the circle delivers a different Green's function.

[continue p 311 below the pencil line] We now consider inhomo $L_{\lambda 0}v = sf$ with our usual BC at the "a" end. We have picked some λ_o . We try as a solution $v = Gsf$ in the usual manner as shown p 311 A, and rewrite this as in B. Again, at this point, u is one of the infinite limit-circle solutions that correspond to points m on the limit-circle, while ψ is the fixed solution determined by the a-end BC. I think that C is

correct in that the two derivatives with respect to the Volterra endpoints will cancel. If you then just think of $v(x) = u(x)A(x) + \psi(x)B(x)$, etc, it is easy to derive result D (did this on scratch paper).

Now what happens if we start with this result D and take the limit $x \to b$, our singular endpoint value. Notice that we don't have a b_0 anywhere, we have already done $b_0 \to b$ and that is why we are on the limit-circle. If we do this limit, notice that the first term in D becomes 0 due to our "imposed condition" 4.111. The second term in D is also 0 because it is integral b to b. Thus, we get 4.116 which relates to the "cross Wronskian" of the solution v of our inhomo, and the u on the limit circle as found in the Green's.

We have shown 4.116 results from our Green's function and our system definition 4.114. He suggests that we add 4.116 as a BC to our ODE system 4.114. I still think u is not unique because we could use any point still on the limit circle. So I ask: in what sense is the solution p 311 A "unique" ? I can see for sure that it is "one solution" (as he says in the text). Once we pick a particular u, 4.116 puts a condition on v. So I am just a little hazy on uniqueness here.

[312] He shows that the solution we just obtained for v(x) is finite s-norm. Part of this demonstration is showing that our g is in fact HS. We already know that ψ and u are finite s-norm (in the case of u, that is what the limit-circle says).

Now at the page 312 pencil line. We consider now the homo equation which is exactly 4.117 with f = 0, and we have included our new "BC" same as in 4.117, that BC being 4.116. Yes, we know that our function u is a solution of this homo equation corresponding to some point on the limit circle.

We can cast this problem 4.119 into the integral equation shown in 4.120 where g is the g we have been using all along, and we can recast that into our standard Chapter 3 form and k is symmetric and HS so we get our happy conclusion: there are denumerable EV's and their EF's form a complete set!

Review of Limit-Circle Analysis.

1. We start with the **Green's problem** p 310A where b is a singular endpoint and a is not. We assume that this problem is a limit-circle problem at the b endpoint. We pick some particular λ_0 and for that λ_0 we pick some particular "point on the limit circle". This means we pick some particular m to use in the equation $u = \varphi + m\psi$ where φ solves our homo problem with BC 299 A, and ψ with BC 200 B.

At this point, instead of imposing a BC at the b end like 299 (4.82) as $b_0 \to b$, we directly impose condition 4.111 which mimics at the b end for u the condition we are imposing at the a end for ψ.

So, we now have our exact Green's function 4.113 which solves our Green's problem. Thus, in line 310A where the Green's Problem is stated we should really add these two "BC" conditions:

$B_a(g) = 0$ at x=a same as $p(a)\, W[g,\bar{g}; a] = 0 = p(a)\, W[\psi, \bar{\psi}; a]$

----- at x=b (limit $b_0 \to b$): $p(b)\, W[g,\bar{g}; b] = 0 = p(b)\, W[u, \bar{u}\,; b]$

I put ---- because we avoided using some kind of $B_{b0}(g) = 0$ at the b end.

So, we have a well-defined Green's Problem and a well-defined Green's Function g for our λ_0. It is true that if we move to another point on the limit circle, our Green's changes because u changes (because m changes). But this is not important, because ANY Green's function is going to work for us, so we just accept that we have picked some particular m. Notice that we never talk about the function φ of $u = \varphi + m\psi$. It is only this linear combination u that we care about.

2. We now consider the **Inhomo Problem** 4.114 $L_{\lambda 0}v = sf$, where solution is called "v". The BC at the left end is the same as for the Green's problem. It turns out that the correct BC at the b endpoint is what we have above in the Green's problem, but we replace the first u in the Wronskian with v. So for the inhomo problem we have this BC situation:

$B_a(v) = 0$ at x=a same as $p(a)\, W[v,\bar{v}; a] = 0$

----- at x=b (limit $b_0 \to b$): $p(b)\, W[v,\bar{u}; b] = 0$

where of course u is our "selected" limit-circle solution of the *homo* problem. The solution v is obtained in the "usual manner" as v = Gsf (311 A) where the g is the g we found in 1 above. We find that this solution is finite s norm. In all our earlier work and examples, we deal only with the *homo* equation and we found that there were either 1 or 2 (point or circle) homo solutions of finite s norm. Here for the first time (I think) we are commenting that the solution to the *inhomo* problem is also finite s norm. We do have to assume (p 312 B arrow) that f is finite s-norm. We already know that ψ and u will be finite s norm.

3. Finally, we consider the **Homo Problem** 4.119 with exactly the same BC's as in 2 above. Here he calls the solution w, so we have $L_{\lambda 0}w = 0$. All we do here is cast this into integral equation form, show the kernel is symmetric and HS, and conclude that our set of EF's are "denumerable" and "form a complete set". So here we have the promised conclusion that in the limit-circle case, we don't get a branch cut, we get only a denumerable set of poles for g. So there is no continuous spectrum, the EF's of the discrete EV's form a complete set.

Recall our two limit-point examples: The Bessel one gave a purely discrete spectrum, while the Mellin one gave a purely continuous spectrum, and general you could get mixed. In limit-circle you only get discrete, we just proved it!

Example: Bessel with ν= 0 and with parameter λ (limit-circle example, p 313)

Precomment: in this example, we will use ψ = log(x) and φ = -1+Alog(x) as the "independent functions" for the limit-point/circle analysis as on p 299. Then u = φ + mψ = (-1+Alogx)+mlogx = -1 + (A+m)logx = -1 + A'log(x). [Just out of curiosity, what is the "A" shown p 299 D which is the center of our circle. Well, $W(\varphi,\bar{\psi}) = W(\varphi,\psi) = -1/x$. And $W(\psi,\psi) = 0$ so we get "A" = $-\infty/0$ so "A" = ∞, not very convenient for me.] If we want to use the solution "u" as we have elsewhere, then u = -1 + (A+m)log(x). But that would require m=0 and then u = φ. That suggests that somehow m=0 is on the limit circle. This could be arranged with careful selection of β_1 and β_2 in 4.83. So I guess I will just imagine that m=0 is somehow on the limit circle for this problem, and then we can use φ in place of u, such as in 4.123 g = φ ψ.

On page 305 we saw that the Bessel equation (with parameter λ) is limit-point when $\nu \geq 1$ but is limit circle when $\nu < 1$. In that example only the limit-point case was developed. Here we set ν = 0 and we are in the limit-circle case. Interval is still (0,1) and x=0 is the singular point at issue. The solution is called "w" and we assert the same w(1)=0 BC at the regular endpoint x = 1. He works first with λ = 0 (which is not then the actual Bessel equation which recall is λ = 1), and finds the two indep solutions. This example was also treated on p 301 where 1 and log(x) were found to be the λ=0 solutions. Here he takes variations of these solutions. The solution ψ = log(x) satisfies w(1)=0 and so this will be used at the non-singular endpoint (which is now on the right!). He picks a specific φ = -1 + Alog(x), I guess of u = φ+mψ fame

(maybe not, maybe these are just two solutions φ and ψ). If A is real, he claims φ satisfies our x=0 singular endpoint condition which here is "a" not "b" so 4.111 says $p(a)W(\varphi,\bar{\varphi}; a) = 0$ in the limit $a \to 0$. So I think φ is playing the role of u in our text earlier. The green's function then has ψ at the regular end (now on the right) and has φ at the left end (a=0) as shown in 4.123. As was done in 4.113, things are normalized to give a unit constant in the Green's function. This Greens is the $\lambda = 0$ Green' !

Now write our ODE with $f = \lambda x w$ as the "driving term" and we get $w = \lambda G(xw)$ as the "solution, which is of course an integral equation at this gives 4.124, with the λ=0 Green's used. At the singular a end, we now have 4.125 which is just 4.116 where we have replaced $\bar{u}$ with $\bar{\varphi}$ and since φ is real, we replace $\bar{u}$ with φ. But now we know exactly what function φ is, so we can write out 4.125 in detail, where w is still unknown.

[314] We now convert our little integral equation to standard Chapter 3 form as in 4.127. The kernel is symmetric and HS so the EF's of this problem form a complete (discrete) set (weight s(x) = x). The case A = 0 is the usual one considered, and in this case the BC at x=0 becomes xw'(x) = 0, which has a familiar look to me. The normalized solutions are as we found on p 307.

The General A case. The solution has to be a lincomb of J and N, so

$$w(x) = C J_o(y) + D N_o(y) \qquad y = \sqrt{\lambda}\, x$$

$$w'(x) = \sqrt{\lambda}\ \{ C J_o'(y) + D N_o'(y) \}$$

Our BC is 4.125a p 313 which is this

$$Aw = (A \ln x - 1)xw'$$

$$A \{C J_o(y) + D N_o(y)\} = (A \ln x - 1)x[\sqrt{\lambda}\ \{ C J_o'(y) + D N_o'(y) \}\ ]$$

$$A \{C\ 1 + D\{(2/\pi)(\ln x + \beta)\} = (A \ln x - 1)x[\sqrt{\lambda}\ \{ C\ 0 + D\ (2/[\pi\sqrt{\lambda}\, x])\ \}\ ] \qquad \beta = [...]$$

$$A \{C + D\{(2/\pi)(\ln x + \beta)\} = (A \ln x - 1)\ D\ (2/\pi)$$

$$A\{C\pi/2 + D\ (\ln x + \beta)\} = (A \ln x - 1)\ D$$

$$CA\pi/2 + AD\ln x + AD\beta = AD\ln x - D$$

$$CA\pi/2 + AD\beta = - D$$

$$CA\pi/2 = -D(1+A\beta)$$

$$D = - C(A\pi/2)/\ (1+A\beta) \qquad \text{// which is p 314 G}$$

$$(D/C) = -(A\pi/2)/\ (1+A\beta)$$

$$(C/D) = -(1+A\beta)/\ (A\pi/2)$$

Now we also need w(1) = 0 so that

$$w(1) = C\, J_o(\sqrt{\lambda}) + D\, N_o(\sqrt{\lambda}) = 0$$

$$(C/D)\, J_o(\sqrt{\lambda}) + N_o(\sqrt{\lambda}) = 0$$

$$-(A\pi/2)\,(C/D)\, J_o(\sqrt{\lambda}) - (A\pi/2) N_o(\sqrt{\lambda}) = 0$$

$$(1+A\beta) J_o(\sqrt{\lambda}) - (A\pi/2) N_o(\sqrt{\lambda}) = 0$$

$$J_o(\sqrt{\lambda}) + A\,[\,\gamma + \log(\sqrt{\lambda}/2)\,]\, J_o(\sqrt{\lambda}) - (A\pi/2) N_o(\sqrt{\lambda}) = 0 \qquad \text{// page 315 A, corrected!}$$

So you have to solve this holy mess for your eigenvalues λ_i. Then

$$w_i(x) = C\, J_o(y) + D N_o(y)$$
$$\sim J_o(y) + (D/C)\, N_o(y)$$

$$= J_o(y) - (A\pi/2)/(1+A\beta)\, N_o(y) \qquad \text{where } \lambda = \lambda_i$$

His comment here is that each real value of A gives a separate problem, and that when $A \neq 0$, this is the whole solution, but he claims no practical use.

Exercises (315-322) // and these takes us to the end of this volume

Exercise 4.25 Bessel for $0 \le \nu \le 1$. In our big example above, we did the limit-circle with Bessel ν=0. Here we are supposed to redo it for limit-circle $0 \le \nu \le 1$. The claim is that the Fourier-Bessel transform is still valid. I think in this case that limit-point solution he called Z_ν is still a solution, so probably everything goes through the same on pages 306-308. Also, hard to imagine that you could not just "continue" the transform in ν to any ν you wanted, as long as the projection converges.

Exercise 4.26. Hankel Transform (of order $\nu = 0$). Here we continue with the Bessel $\nu = 0$ example, but now on $(0,\infty)$ instead of on (0,1). He shows that at 0 we are limit circle as in our example, but at ∞ we are limit point. At the 0 end with A=0 we still have the BC 4.128 so we still want a J_0 at that end. We know that only the Hankel is a solution for the right half of the Green's (recall previous work p 302 top), so the Green's seems right as shown in p315 B. Since these Bessel functions are of known normalization, we cannot just set the Green's constant to 1, we have to compute it, and he claims the result shown there. In this problem there is a branch cut from $J_o(\sqrt{\lambda}\, x)$ that is not cancelled by the constant in g (as happened earlier), and I suspect the Hankel has no cut. He must then compute [g] and then do the g contour integral to get 4.130 which is then all continuous. The corresponding integral (both ways) transform is called the **Hankel Transform (of order $\nu = 0$ here).**

Exercise 4.27. Hankel Transform for order ν . We now repeat the previous exercise with $0 \le \nu < 1$. This puts the 0 endpoint into the same limit-circle case as above, so we choose J_ν at the left end of the

Green's, and the Hankel at the right end as above, but now for general ν. We are again supposed to compute [g] and we end up with the **Hankel Transform for order ν** and the previous exercise is just the special case ν = 0

[On 7.18.11 I did the details of the Hankel transform part of this exercise 4.27 (page 316), see Transforms folder, Hankel Transform.]

Comment: In the above we have limit-circle at x=0 and limit-point at x=∞. We know that limit-circle with a regular other endpoint produces a purely discrete spectrum, and that the limit-point at the other end can produce a mixed one. In this example, it seems that the ∞ end "wins out" and in fact we get only a continuous spectrum.

Exercise 4.28. Mellin Sine Transform On page 308 Example 2 we did Mellin on (0,∞). Here we repeat for (0,1) instead. The endpoint 1 is regular and endpoint 0 is limit point, as was shown back on page 302 Example 4. Our instructions are to find the Green's (it is shown). We insist now that g(1) = 0 so the Green's is a little different from what it was in the (0,∞) case. We have the usual branch cut, compute [g], and we end up with a strange transform as shown which he makes up the name **Mellin Sine Transform**. As in the previous case, a change of variable changes this into something more familiar: the Fourier Since Transform.

Exercise 4.29. Mellin Cosine Transform Repeat the above but with g'(1) = 0 to develop the MCT.

Exercise 4.30. Kantorovich-Lebedev Transform. First, go back to p 305 (4.96) which shows Bessel's ODE for order ν if λ = 1, so the Bessel's is this: $-(x\varphi')' + (\nu^2/x)\,\varphi - x\varphi = 0$. With "variable λ" we get $-(x\varphi')' + (\nu^2/x)\,\varphi - \lambda x\varphi = 0$. Let's now rename ν^2 to be μ, so that problem was this: $-(x\varphi')' + (\mu/x)\,\varphi - \lambda x\varphi = 0$.

Now we are told to suddenly switch roles $\mu \leftrightarrow -\lambda$ to get $-(x\varphi')' + \mu x\varphi - (\lambda/x)\,\varphi = 0$. If λ = 0, we then have $-(x\varphi')' + \mu x\varphi = 0$. If we set μ=-1 we would say this was Bessel's equation for order 0. But instead we want to consider μ = +1 which means the xφ term "has the wrong sign". From page 331, we know that the solutions to this equation would be the I_0 and K_0 type Bessel functions. If we then generalize so μ > 0 but not necessarily μ=1, we get solutions $I_0(\sqrt{\mu}\,x)$and $K_0(\sqrt{\mu}\,x)$.

Now, at the ∞ end we know that only $K_0(\sqrt{\mu}\,x)$ is finite s norm, so that is limit-point. And at the 0 end things are also limit-point. He claims the Green's for general λ is as shown in D which seems reasonable -- I guess I must be the only solution allowed at the 0 endpoint. This g has a branch cut, and we end up with yet another expansion for δ(x-ξ) and thus yet another transform to which he gives some Russian names: the **Kantorovich-Lebedev Transform**. It is the $K_{i\nu}$ functions which appear in this thing, and μ is just a fixed parameter which you could set to μ=1.

Comment: for each ODE and each endpoint pair, you will get "some kind of transform" and by now we have seen quite a few. I suspect he is going to quote these and use them in volume 2!

[On 7.18.11 I did collected and summarized the details of the KL transform, see Transforms folder, the KL transform.doc.]

[318] **Exercise 4.31. The Modified Green's Function** .

Question regarding p 318 A: Suppose I find a solution g to this equation that meets BC's. What happens if I add to this solution $5\varphi_1(x)$? Well, $(L-\lambda s)5\varphi_1 = 5s(\lambda_1-\lambda)\varphi_1$. So unless $\lambda = \lambda_1$, this *alters* the equation so that the function $g_1(x|\xi;\lambda) = g(x|\xi;\lambda) + 5\varphi_1(x)$ is NOT a solution of A.

Derive 4.142: $g_M = g_M(\lambda_1)$ $L\varphi_n = \lambda_n s\varphi_n$

$$Lg_M - \lambda_1 s g_M = (L-\lambda_1 s)\,\Sigma_{n=2}^{\infty}\varphi_n(x)\,\bar{\varphi}_n(\xi)\,/\,(\lambda_n-\lambda_1)$$

$$= \Sigma_{n=2}^{\infty}[(L-\lambda_1 s)\varphi_n(x)]\,\bar{\varphi}_n(\xi)\,/\,(\lambda_n-\lambda_1)$$

$$= \Sigma_{n=2}^{\infty}[(\lambda_n s-\lambda_1 s)\varphi_n(x)]\,\bar{\varphi}_n(\xi)\,/\,(\lambda_n-\lambda_1)$$

$$= s\,\Sigma_{n=2}^{\infty}[(\lambda_n-\lambda_1)\varphi_n(x)]\,\bar{\varphi}_n(\xi)\,/\,(\lambda_n-\lambda_1)$$

$$= s(x)\;\Sigma_{n=2}^{\infty}\varphi_n(x)\,\bar{\varphi}_n(\xi)$$

But we know that the φ_n are a complete set for the homo problem, so we know that

$$\delta(x-\xi) = s(x)\,\Sigma_n\,\varphi_n(x)\,\bar{\varphi}_n(\xi) = s(x)\,\varphi_1(x)\,\bar{\varphi}_1(\xi) + s(x)\;\Sigma_{n=2}^{\infty}\varphi_n(x)\,\bar{\varphi}_n(\xi)$$

Therefore, continuing the above, we get

$$Lg_M - \lambda_1 s g_M = s(x)\;\Sigma_{n=2}^{\infty}\varphi_n(x)\,\bar{\varphi}_n(\xi) = \delta(x-\xi) - s(x)\,\varphi_1(x)\,\bar{\varphi}_1(\xi)$$

which is the first part of 4.142. As for the BC part, consider

$B_a(g_M) = B_a(g - \varphi_1(x)\,\bar{\varphi}_1(\xi)/(\lambda_1-\lambda)\,) = 0$ since $B_a(g) = 0$ and $B_a(\varphi_n) = 0$ as in 4.140

Thus, we have proved all parts of 4.142.

Question: suppose I find a solution g_M of 4.142. What happens if I add $7\,\varphi_2(x)$ to that solution?

$$(L-\lambda_1 s)\,7\varphi_2(x) = 7\,(\lambda_2 s-\lambda_1 s)\,\varphi_2(x) = 7s\,(\lambda_2-\lambda_1)\,\varphi_2(x) \neq 0$$

So, adding such a term DOES alter the equation, even though such an extra term respects the BC's. However, if we add a multiple of φ_1 then we get no alteration. So I agree with the statement made right after 4.142 that you can add $C\,\varphi_1(x)$ to any solution g_M.

Comments on this Exercise. I accept the last few steps, now step back and try to see what is being said here. Sometimes, he says, we are interested in studying a problem where $\lambda = \lambda_1$, some eigenvalue. Normally we don't do this. But if we want or need to do this, then we can use the g_M modified Green's function. This could be any generic problem where we have some L_λ as shown in 4.140. We have a homo, inhomo, and Green's equation to think about, as always. That is to say, homo = 4.140, Green's = p 318 A, inhomo not stated but we know what it is. We are always going to want to know if the φ_n of the

homo form a complete set. For everything regular, we know this will be the case from earlier work, but for singular cases, we have to do an *ad hoc* check in each case. If in such a case we need to use some $\lambda = \lambda_1$ which is an EV, we can still do that "check" as follows: construct the modified Green's g_M as instructed in this exercise, then test to see if the corresponding kernel as shown in p 319 A is Hilbert-Schmidt! If it is, then our homo φ_n are a complete set. I suspect we shall see this concept used in one or more upcoming exercises.

Exercise 4.32 The Legendre Functions (319). The first part of this discussion is on page 319. We write the Legendre equation this time with $\lambda = \ell(\ell+1)$ in normal notation, but general λ now. We try a power series solution for u(x) and get the usual recursion relation. We then find that, for any λ, there appears to be both an even and an odd candidate solution since the recursion steps by 2. However, in either case, if the series does not terminate, the power series always diverges at one of the interval endpoints and is therefore not an eigenfunction, since EF's have to be finite on the interval (is this true, or just have finite norm?). Well, if we are interested in "bounded solutions" on the full interval, we must have truncation. Thus, as the recursion form shows, we must have $\lambda = \ell(\ell+1)$ where ℓ is an integer (he calls it k). If ℓ is an even integer, then in 4.147 you are only truncating the even candidate series at some $n = \ell$. The odd candidate solution still diverges since it does not truncate. So for even ℓ you only get one solution which is even, and for odd ℓ you get one solution which is the odd candidate, and these are the $P_\ell(x)$.

[320, exercise 4.32 continues] We know that $\lambda = 0$ is one of our EV's for this problem ($\ell = 0$), so we could say $\lambda_1 = 0$ and $\varphi_1 = 1/\sqrt{2}$ in the language of our previous Exercise. If we want to work with $\lambda = 0$, then we must us the modified Green's formalism and this gives us 4.149, fine. He then claims he can solve this thing for g_M and he gets 4.150 which is manifestly symmetric. (But we could add some φ_1 to it and it would still be symmetric since φ_1 = constant.) Now as instructed, we want to see if g_M is Hilbert Schmidt (note that s(x) = 1 in this problem). He claims it is, so we may conclude that the $P_\ell(x)$ are in fact a complete set. Note that $\mu = 0$ is an accumulation point for the "standard form" integral equation stated in 3.20 B, consistent with our knowledge that no other value of μ can be such a point. He then states the normalization integral of the P's.

Finally, we come to the question of why we only want to consider "bounded solutions". This is a situation like 4.128 on page 314 where it just happens that "bounded solutions" corresponds to the official singular boundary condition involving the Wronskian which he states here at both of our limit-circle endpoints x = 1 and -1 .

Exercise 4.33 The Hermite Equation (320). Earlier on page 302 we studied this ODE a little and showed it was limit-point at each end of $(-\infty,\infty)$. For $\lambda = -1$ we know some independent solutions (involving I think the error integral), and he then comes up with two solutions which are "OK" at one or the other end (ie, integrable there) so the Greens for $\lambda = -1$ is as shown in p 320C.

[321, exercise 4.33 continues] We are supposed to show that our integral equation kernel is HS so that the EF's of our Hermite ODE form a complete set, though we don't yet know what they are. We convert the equation slightly differently from before to get a new ODE and we try series solutions for it. We again conclude that solutions diverge at our endpoints unless the series truncates. I think the series will be even more expo than that the friendly factor $\exp(-x^2/2)$ factor so that u(x) will diverge, which we know is not the limit point solution at our singular points $\pm\infty$. So we are then supposed to derive the Rodriguez formula for the Hermite Polynomials and then our solutions of the Hermite equation are

$u_k(x) = \exp(-x^2/2)H_k(x)$. He then wraps up with something that looks like a generalized generating function with parameter α. So, fine, we are just grinding out various facts about the Hermites here.

Exercise 4.34 Bessel's of order 0, with parameter λ, and on (a,∞) (321). We are asked to find "the spectral decomposition" for this "operator" (meaning ODE system). We would need to do the following: (1) construct a Green's. (2) examine its λ analytic structure to see if poles or cuts or both; (3) maybe along the way think about whether the endpoint is LC or LP (I would guess LP) ; (4) write a δ(x-ξ) formula which then shows the "spectral structure" ; (5) present this as a Transform formula. We are given no hints now because we are supposed to be Big Boys now.

Exercise 4.35 Particle in a finite square well potential. Saxon p 121 solves this, and I know that we have the mixed spectrum. I don't off hand know what the Green's is or how to write out δ(x-ξ) , but that is what this problem asks the reader to do. A good way to end this book! The Green's would be written radially as sin/cos inside and expo decay outside. The idea of matching BC's as done in QM is a little foreign to the Stak methods. This problem differs from others Stak has considered in that it has as part of L a discontinuous function, so it is not quite in his standard form. We could consider this discontinuous potential as the driving function f(x) where we recall that piecewise continuous is always OK. So maybe get the Green's with f(x) = 0. I can imagine several different ways to approach this problem. Since λ = E, I expect to get the usual branch cut on the positive λ axis, and I expect to get a finite number of poles on the negative λ axis. We did see a problem like this on page 295 Exercise 4.24. In fact it looks very similar to our current well problem in terms of the Green's shown there. In that case we only had one pole when h > 0, but in the current problem we will have several poles. It would be interesting to see how the multiple poles appear here, so maybe some day I will do this problem.

End of Stakgold Volume I, today is April 29, 2009. This is a pretty big milestone for me, I have always wanted to get through this book in my modern era with full notes. Meta notes for Chapter not yet written.

What happens to orthogonality in the singular case?

Question Added 3.13.10. In the *singular* problem, what becomes of the idea that

(1) eigenvalues are real
(2) eigenfunctions are orthogonal?

It seems that these subjects are never mentioned, though they get major mention in the regular problem. In the Bessel example where we are singular at x = 0, we do get a discrete spectrum and we get the usual orthogonality of the J's. But in general how does this work?

(1) eigenvalues are real ?

If I look at page 270 at the proof that eigenvalues are real, we need $L\varphi = \lambda s\varphi$ and we need the fact that $\langle Lu,u\rangle$ is real. But that requires us to be able to "swing around" L from one side to the other of the inner product. That in turn is possible because the "parts" vanish as shown bottom of page 269. And in turn, the parts vanish because our domain of functions is on in which all u satisfy the inhomo BC's of 4.26. We assume L is "formally" self adjoint, so it is the vanishing of these parts that is the big deal. Note in

passing that we start with 4.28a or b on page 269, which is that thing Lagrange's Identity that we derived from parts integration back on page 69 (and today I added a note in my Chapter 1 notes on this topic).

We now fast-forward to page 295 and the general singular problem. We consider the swing-around possibility for operator A = L/s(x). We accept the Hilbert Space of finite s-norm functions. Then on page 296 we investigate the "parts" situation when we "swing around" operator A. We start off with our same Lagrange Identity noted above here called A, and its integral form B which is Green's Formula. Again, these come from page 69 which is nothing but "integration by parts algebra".

Stak does all this first on the interval (a_0,b_0) and will later take the limit to the singular endpoints of interval (a,b). So we have those same parts terms in 4.79. Now the big question: do we have some kind of inhomo BC that is going to make these things be zero? In his text, at this point Stak says we need to restrict to functions u such that u and Au are finite-s norm. In the regular case we restricted to u which satisfied the inhomo BC's! He calls the restricted function space D in both cases. We now finish out the bottom of page 296 C and his only claim is that the two limits exist, he is not claiming they are each 0 as in the regular case.

Page 297 continuing. Look at result A'. In our regular BV problem, we would integrate both sides and get 0 on the right from our inhomo BC, and we would conclude that λ was real (proof that eigenvalues are real for self-adjoint L). B' is just a restatement and in the regular case we would have $\text{Im}(\lambda) = 0$. In C' we have taken our limit to (a,b). But now we stop motion and we go off and do the Weyl Theorem which is a huge digression, and the logic flow to which we move from C' gets lost in all the dust. In our regular case, we know that if $\text{Im}(\lambda) \neq 0$, there are NO solutions of the EV problem $L\varphi=\lambda s\varphi$. We know this because we know that all EV's must be real. But Weyl is telling us something very different for the singular case. If $\text{Im}(\lambda) \neq 0$, there is *always* at least one eigenfunction, and there may in fact be two! These are the famous LP and LC cases.

So I guess we have obtained the answer to our first question: In the singular case, eigenvalues are *not* required to be real! The more important aspect is that the one or two solutions are finite s norm.

(2) eigenfunctions are orthogonal?

Now what about our regular problem **orthogonality**? The proof on page 270 for the regular case requires the swing-around ability of L in the inner product. If EV's are allowed to be complex, then we might have $\lambda = \overline{\mu}$ by chance (where λ and μ are different eigenvalues). So this derivation is definitely not obviously going to go through for the singular case.

So how does this proceed? On page 299 we have a being a regular point so we DO have inhomo BC's at the a endpoint as shown p 299 A and B for the two functions φ and ψ. We then sort of conjecture an inhomo BC at the b end as in 4.82, before the limit. I am off reading my meta notes on Chap 4 on this subject right now. And then I read my meta-meta notes. My conclusion is this: in either the LC or LP case, we end up with in effect $p(b)W(u,\overline{u}; b) = 0$ as suggested in 4.86. This is for u being the one limit point solution, or any limit circle solution. It turns out though not clearly stated in the book that we also have $p(a)W(u,\overline{u}; a) = 0$, see meta notes.

But, does this have any connection with p 296 4.79 where we are trying to swing A around from one side to the other? On page 296 the Wronskians involve separate finite s norm functions u and v, whereas our results just stated involve only one function u. To really show that the limits in the swing over parts are 0, we need to have an inhomo BC at both endpoints. Equation 4.82 certainly looks like our desired inhomo BC at the singular b end. But is this still true in the limit? I have already given a counterexample on page 297 B' which was this: Weyl says there is always one EF for general complex λ off the real axis.

In this case, the two objects on the right of 4.80 cannot possibly both be zero since Im(λ) ≠ 0 and since we assume our EF is finite s norm.

Suppose we restrict our interest to real λ. In this case, Weyl Part 2 gives no conclusion at all, so we no longer know that there is at least one finite s norm solution. But suppose we study a particular ODE and we learn that for some λ there are in fact two finite s norm solutions. Then I think we could conclude that for λ real, we always have two finite s norm solutions. And it might be that we are limit point and there is only one finite s norm solution even for real λ.

So maybe for real λ, we CAN say that both those parts things are zero, that we get "swing around", and that different eigenfunctions with different real λ are orthogonal somehow. Not dead yet!

Look at the Bessel example on page 301 with ν = 0 which is singular at the 0 endpoint. He tests with λ = 0 and finds two finite s norm solutions, and he then concludes that for any λ there are two, and he writes them down as linear combination of J and N. But nothing about orthogonality.

So after my long review here on this question, I am no closer to answering the question than when I started. This is typical Stakgold situation.

Let's keep scanning Stak starting on page 303 where he gets into the Green's function with λ present in the operator. We use the special magic function "u" at the singular end in the Green's function as shown page 303A. We get involved here with the analyticity of m(λ) in 4.93. On page 304 he is discussing only the limit point case where we know exactly what function to use for the singular half of our Green's function. He then suggests that m(λ) will have a cut and or poles. He then writes the completeness δ thing and talks about the mixed spectrum possibility shown in 4.95. Now he claims "if there is a pure point spectrum, then the EF's with weight s form a complete set" (this claim not proved by Stak). His next claim is that if an endpoint is limit circle, then you always have a pure point spectrum. Still nothing is said about orthogonality with a continuous spectrum.

I now realize that "I have been here before" on this orthogonality question or a related one, but I doubt I can find the notes, they could be anywhere.

(3) eigenfunctions are orthogonal: Bessel example.

Question: Doesn't the δ completeness relation *imply* some kind or continuous orthogonality?

Let's take as an example the Hankel transform on page 315 bottom. I see above that I did not "do" this problem, and I presume that $\mu = \sqrt{\lambda}$. In any event, here is the claim:

$$F_H(\mu) = \int_0^\infty dx\, x J_0(\mu x) f(x) \qquad f(x) = \int_0^\infty d\mu\, \mu J_0(\mu x)\, F_H(\mu)$$

So let's first insert the projection into the expansion:

$$f(x) = \int_0^\infty d\mu\, \mu J_0(\mu x)\, F_H(\mu)$$

$$= \int_0^\infty d\mu\, \mu J_0(\mu x) \left\{ \int_0^\infty dx'\, x' J_0(\mu x') f(x') \right\}$$

$$= \int_0^\infty dx' \, f(x') \quad \{ x' \int_0^\infty d\mu \, \mu J_0(\mu x') J_0(\mu x) \}$$

It must be then that

$$\{ x' \int_0^\infty d\mu \, \mu J_0(\mu x') J_0(\mu x) \} = \delta(x-x')$$

or

$$\delta(x-x')/x = \int_0^\infty d\mu \, \mu J_0(\mu x') J_0(\mu x) \qquad \text{// which is 4.130.}$$

Second, let's insert the expansion into the projection:

$$F_H(\mu) = \int_0^\infty dx \, x J_0(\mu x) f(x)$$

$$= \int_0^\infty dx \, x J_0(\mu x) \int_0^\infty d\mu' \, \mu' J_0(\mu' x) F_H(\mu')$$

$$= \int_0^\infty d\mu' \, F_H(\mu') \{ \mu' \int_0^\infty dx \, x J_0(\mu x) J_0(\mu' x) \}$$

It must be then that

$$\{ \mu' \int_0^\infty dx \, x J_0(\mu x) J_0(\mu' x) \} = \delta(\mu-\mu')$$

or

$$\delta(\mu-\mu')/\mu = \int_0^\infty dx \, x J_0(\mu x) J_0(\mu' x)$$

and this then is our "orthogonality" of the eigenfunctions which Stak doesn't want to talk about. In this example, the projection and expansion are completely symmetrical. I am happy to report that the above "orthogonality" appears stated in Jackson p 77 as (3.112). Jackson pulls this thing out of nowhere, calling it an "integral relation", though the reader could obtain it as I did from the transform itself. However, Jackson does not derive the continuous transform, he just quotes it, whereas Stak derives it.

Suppose we now set $\mu = \sqrt{\lambda}$ then we can say

$$\delta(\mu-\mu') = \delta(\sqrt{\lambda} - \sqrt{\lambda'}) = 1/[(1/2)1/\sqrt{\lambda}] * \delta(\lambda-\lambda') = 2\sqrt{\lambda} \, \delta(\lambda-\lambda')$$

$$\delta(\mu-\mu')/\mu = 2\sqrt{\lambda} \, \delta(\lambda-\lambda')/\sqrt{\lambda} = 2\,\delta(\lambda-\lambda')$$

and then we have

$$\delta(\lambda-\lambda') = (1/2) \int_0^\infty dx \, x J_0(\sqrt{\lambda} x) J_0(\sqrt{\lambda'} x)$$

Can I verify this distributional result somewhere? Bateman doesn't show delta function things. Hard to find something like this. We can use Jackson's discrete result

recovery $\quad f(x) = \Sigma_{k=1}^{\infty} A_{k,\nu} \; J_\nu(x_{\nu,k}x/a)$ (3.96)

projection: $\quad A_{k,\nu} = 2/[a^2 J_{\nu+1}(x_{\nu n})^2] \; \int_0^a dx\, x\, f(x)\, J_\nu(x_{\nu,k}x/a)$ (3.97)

and put the expansion into the projection

$$A_{k,\nu} = 2/[a^2 J_{\nu+1}(x_{\nu n})^2] \; \int_0^a dx\, x\, f(x)\, J_\nu(x_{\nu,k}x/a)$$

$$= 2/[a^2 J_{\nu+1}(x_{\nu n})^2] \; \int_0^a dx\, x\, J_\nu(x_{\nu,k}x/a) \; \Sigma_{k'=1}^{\infty} A_{k',\nu} \; J_\nu(x_{\nu,k'}x/a)$$

$$= \Sigma_{k'=1}^{\infty} A_{k',\nu} \; \{ 2/[a^2 J_{\nu+1}(x_{\nu n})^2] \; \int_0^a dx\, x\, J_\nu(x_{\nu,k}x/a) \; J_\nu(x_{\nu,k'}x/a)\}$$

which tells us that

$$\delta_{k,k'} = 2/[a^2 J'_\nu(x_{\nu n})^2] \; \int_0^a dx\, x\, J_\nu(\sqrt{\lambda_{\nu k}}\, x)\, J_\nu(\sqrt{\lambda_{\nu k'}}\, x) \qquad x_{\nu,k}/a = \sqrt{\lambda_{\nu k}}$$

$$\delta(\lambda-\lambda') = (1/2) \int_0^{\infty} dx\, x\, J_\nu(\sqrt{\lambda}\, x) \; J_\nu(\sqrt{\lambda'}\, x)$$

where I have now generalized the work above from $\nu = 0$ to $\nu = \nu$. You wonder if the second cannot be found from the first by taking limit $a \to \infty$? I have tried a few things and nothing is very obvious, though I am sure this could be done.

What about convergence of these integrals? Are they just "formal" statements that don't converge? The first integral above, shifted to (0,1), appears in GR7 as 6.521 (see "Jackson on cylindricals" notes.), but this is not the case where we are concerned about convergence. We can get something close to the second line above here from GR7 p 684,

$$2.^{11} \quad \int_0^\infty x^{-\lambda} J_\nu(ax)\, J_\nu(bx)\, dx = \frac{a^\nu b^\nu\, \Gamma\left(\nu + \frac{1-\lambda}{2}\right)}{2^\lambda (a+b)^{2\nu-\lambda+1}\, \Gamma(\nu+1)\, \Gamma\left(\frac{1+\lambda}{2}\right)} \times F\left(\nu + \frac{1-\lambda}{2}, \nu + \frac{1}{2}; 2\nu+1; \frac{4ab}{(a+b)^2}\right)$$

$$[a > 0, \quad b > 0, \quad 2\,\mathrm{Re}\,\nu + 1 > \mathrm{Re}\,\lambda > -1] \quad \text{ET II 47(4)}$$

which is a little like Bateman p 92 bottom. It we set $\lambda = -1$ illegally, we get $\Gamma(0)$ on the bottom of the RHS so in general this says that the integral is 0 when $a \neq b$. But if $a = b$, then the F argument is 1 (Maple plot shows < 1 for $a \neq b$), and then bets are off. [I don't know about the ET reference above. Well, it is the

parts 4 and 5 of the Bateman series which I don't have!] I took a quick web look and found a PDF that might be related to ET, it is 15 MB from a places called ifile.it : http://ifile.it/0gl41xz/i_t_i.pdf
My download seems to be all of Volume 4 and covers Fourier, Laplace and Mellin in 391 pages! The site says **Date Added:** March 29 2009 so within the last year or so! The site does not have the other volume of the transforms, or in fact any of the other Bateman items. I found the second volume, but only in Russian! I found another book by Debnath and Bhatta on this subject. This at least has the Hankel transform, and many others, 700 good pages, a little bit on many different transforms.

My conclusion is this: If you restrict to real λ, then in the singular case you are going to find some kind of orthogonality in the continuous sense as in the example above. In our example, the spectrum is the entire positive real axis, but you might have a discrete or mixed spectrum and then I guess orthogonality is a bit more complicated. You can always find what orthogonality must look like from the δ completeness relation, which in turn you get from the Green's Function approach.