Stakgold Chapter 6 Meta Notes PhL begun 9.13.09

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Chapter Title: Potential Theory

6.1 Introduction (88)

Comment: In the first three sections of this chapter, we look for solutions to Laplace $\nabla^2 u=0$ where we specify the solution u (or perhaps $\partial_n u$) on a closed bounding surface. This specification is "the boundary value" of the solution. We do this in R_n for n = 2 and n = 3 mostly. This whole activity is very similar to finding a solution of an ODE where you insist on two boundary values, one at each endpoint of an interval (a,b), which endpoints comprise the bounding closed surface in the case n = 1. In these sections, there are no "sources", and there are no Green's functions used. Only starting in section 6.4 do we start talking about sources and Green's functions.

Notion of a well-posed problem: the solution to the problem should exist (not too few) and should be unique (not too many). The third item is new to me: the solution should be a continuous function of the various functions involved in the PDE and the BC's. If this is missing, you probably have a stability problem with your solution.

The Laplace and Poisson equation are defined, Poisson with a minus sign. The various integral theorems are stated: divergence and the two Green's identities. All old hat to the new Phil. These Green's theorems relating volume integrals to bounding surface integrals are very heavily used.

6.2 Interior Dirichlet Problem for the Unit Circle (90)

The phrase interior Dirichlet implies that data is specified on a closed boundary and you want to find the solution of your PDE inside that boundary. Here we are in R_2 with a unit circle as boundary, and with Laplace as our PDE. The boundary values are specified by f(φ). When u satisfies $\nabla^2 u = 0$ in some region, u is said to be **harmonic** in that region.

Separation of Variables [2D, r φ]

In the usual manner, we try $u(r,\varphi) = R(r)\,\Phi(\varphi)$ and identify a separation constant λ. The usual azimuthal conditions quantize this to be $\lambda_n = n^2$ with n = 0,1,2... General solution is then lincomb of $R_n(r)\,\Phi_n(\varphi)$ where $\Phi_n(\varphi) = a_n e^{i\varphi n} + b_n e^{-i\varphi n}$ and $R_n(r) = (1-\delta_{n,0})\,[\,A_n r^n + B_n r^{-n}\,] + \delta_{n,0}\,[\,C + D\ln(r)\,]$. In this section, D = 0 because we are only interested in solutions inside the unit circle finite at r=0 (Theorem 1) or outside with the solution bounded at r=∞ (Theorem 2), in which case $R_n(r) = A_n r^n + B_n r^{-n}$ where $C = A_0+B_0$. So our general solution is this, where we could set any set of the coefficients to 1,

$$u(r,\varphi) = \Sigma_{n=0}^{\infty}[\,a_n e^{i\varphi n} + b_n e^{-i\varphi n}][\,A_n r^n + B_n r^{-n}\,]$$

In order to do simple Fourier series work, it is nicer if this series has the form $\Sigma_{n=-\infty}^{+\infty}$. In the raw notes, I show that the following form is just as general as that shown above,

$$u(r,\varphi) = \Sigma_{n=-\infty}^{+\infty} e^{i\varphi n}\,[\,A'_n r^n + B'_n r^{-n}\,]$$

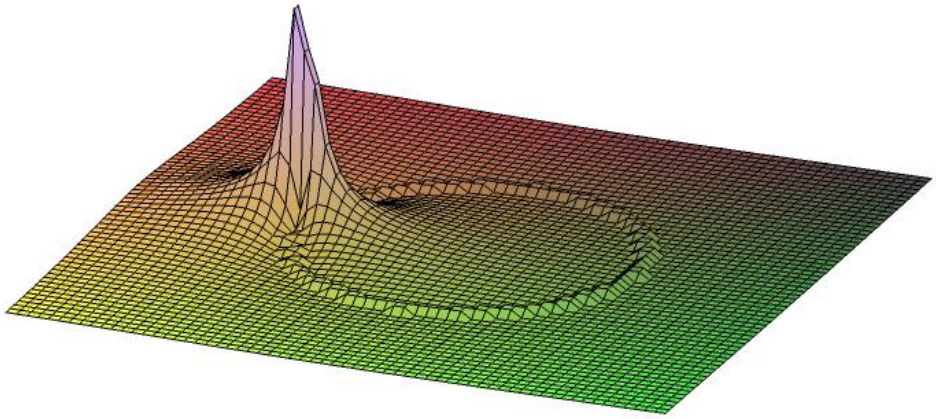
and when we apply this to the unit circle, our most general solution can be written in this Fourier friendly manner:

$$u(r,\varphi) = \Sigma_{n=-\infty}^{+\infty} f_n^{\pm} e^{i\varphi n} r^{\pm|n|}$$ + for interior, – for exterior

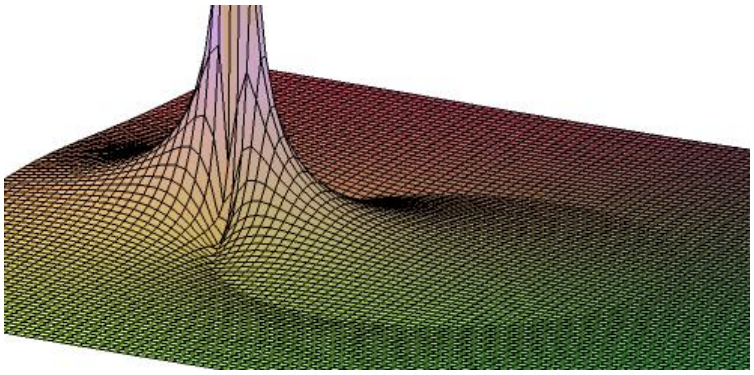
That these are the solutions is what Theorems 1 and 2 on page 93 say (a = 1).

For either solution, then, we have a Fourier Series transform pair as in 6.9 and 6.10 p 94. If we combine these equations together, we end up with the famous 6.11 which is written for the inside + case. The - case is the same except for an overall minus sign (he does not show this case here). [I show this in the raw notes under Exercise 6.2, he confirms it in Exercise 6.4.]

The interpretation of 6.11 is important. If you were to set $f(\varphi) = \delta(\varphi-\varphi_1)$ as a "unit impulse BC" (do not confuse with a unit source!), then the solution to the Laplace equation inside the unit circle is just the integrand of 6.11, which function is called **the Poisson Kernel**. For r>1, 6.11 is also the solution to the problem if you add an overall sign. You can then plot this solution in Maple and it looks like this:



where I have added a "marker" to show where the ring lies. If I remove that marker, the plot is



You see here a "dimple" around the ring that becomes more pronounced as we get nearer the δ. This can be interpreted as an effective ring line source, but we are not yet ready for that idea. Remember that u is being pinned to the value 0 all around the ring except at the one angle where it is ∞.

Notice by the way that at any point a "harmonic function" must have equal and opposite curvature in the two directions, Stak never mentions this simple visual fact. The pioneers would like Maple.

Verification of the Solution of (6.5).

Putting on his rigorous math hat, Stak shows that 6.11 (written as 6.9) does solve our Laplace problem as posed in 6.5. He then shows the solution is unique, and finally he shows on p 96 that the solution has a "continuous dependence on f(φ)", which is the BC.

Verification of the Solution of (6.4).

Here he does a similar thing for the posing 6.4. I am not very interested in these details right now.

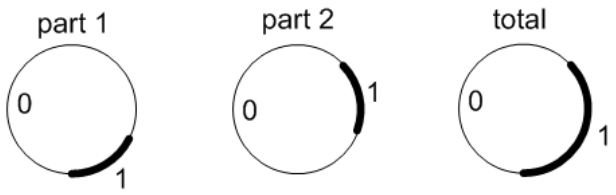
6.3 Some properties of harmonic functions

We must always keep in mind the distinction between a "boundary value" and a "source". We are going to worry about sources later, here we have no sources (no charges in ES), just boundary values for the potential on some boundary.

A. Mean Value Theorem and the "part problem method"

The MVT for a unit disk is stated in 6.18 and follows at once from 6.11 by setting r = 0. The potential at the center is the average of the potential around the ring.

Stak gives an alternative derivation using the "part problem method". Suppose you have a solution for f = 1 on some finite arc, and f=0 on the remaining arc of the circle. Call that the part 1 solution. And then imagine some other part 2 arc-abutting solution. If you add these two solutions, they satisfy Laplace *and* they meet the BC of the larger arc. The picture is this:



If you repeat this giving each arc a potential value f_i, you find using symmetry arguments that

$$u_{total}(r=0) = \Sigma_i (1/N)\,(f_i/1) = (1/N)\,\Sigma_i f_i \qquad \text{which} \rightarrow (1/2\pi) \int d\varphi\, f(\varphi)$$

which gives the MVT. You can think about "part problems" in any number of dimensions.

Note that the MVT only applies to a circular boundary (2D), not to an arbitrarily shaped boundary. It applies to any mathematical "circle and center point" you pick in an open solution region. The MVT only applies to regular polygons which have constant u on each edge.

B. The Maximum Principle for Harmonic Functions (101)

Using the MVT, you can show that both the maximum and the minimum value of a harmonic function on some region R must occur on the boundary of that region. Here is another proof of the min-max I just thought of: if a max occurs inside, then that max is a hilltop. The two orthogonal curvatures have the same sign, which violates the property that all Laplace solutions must have: opposite curvatures at all points. So you cannot even have a "local hilltop" in an open region.

C. Uniqueness and BC dependence continuity of the solution

Lemma: if u = 0 on a boundary of R, then u = 0 in all of R.

This implies uniqueness at once. He then goes on here to show that the solution has a continuous dependence on the BV data.

Exercises: (103) Stak develops many main line ideas in these problems, so we need to review them. Basically, these exercises form a huge chapter within a chapter that the reader has to flesh out.

Exercise 6.1. Compute four infinite sums. All are used later in the chapter!

Exercise 6.2. Evaluate a Poisson Kernel integral with f = 1, then interpret the result. (2D)

Exercise 6.3. Solution outside the unit circle, exterior Dirichlet. (2D)

Several things to note here. If $u(r,\varphi)$ is harmonic, so is $u(1/r,\varphi)$. The MVT also says that the value at ∞ is the average around the ring. This $r \rightarrow 1/r$ maps the interior solution to the exterior one and we find that overall sign change noted above for 6.11 for $r > 1$. [Jackson 3D inversion theory says $\Phi'(\mathbf{r}) = (a/r)\,\Phi(\mathbf{r'}) = (r'/a)\,\Phi(\mathbf{r'})$ where $a = 1$, so $\Phi'(\mathbf{r}) = (1/r)\,\Phi(1/r)$, where Φ' is the outside potential and Φ is the inside potential. *But in 2D*, inversion theory must say $\Phi'(\mathbf{r}) = \Phi(1/r)$!]

Exercise 6.4. Symmetric polygon problem. (2D)

Here the *whole* problem is a regular polygon with n edges, and u = 1 on k of those edges, and u = 0 on n-k edges. The solution at the center is u(0) = k/n and is found using the "*parts* problem method" and the symmetry of a polygon under specific rotations. [Let edge = b. In this case, the σ integral of u would be k*1*b, the boundary length is nb so MVT says u(0) = kb/kn = k/n, the correct answer from MVT.]

Exercise 6.5. Work things through for Helmholtz in place of Laplace. (2D) (105)

Helmholtz is $(\nabla^2+k^2)u=0$ and is the wave equation for constant frequency. Laplace is k = 0. (a) The radial ODE gives Bessel $J_n(kr)$ instead of r^n, and we still have the $e^{in\varphi}$ stuff. If f = 1 on the circle, solution must be $J_0(kr)/ J_0(ka)$ since no φ dependence ($J_0(0) = 1$). In Laplace we got u = 1 in this situation. (b) The MVT for Helmholtz is as shown 6.24, just a different norm factor. And (c) is the general Fourier expansion in this case which we can compare to 6.9 for Laplace. [You can also think of Helmholtz as the Laplace eigenvalue problem, $-\nabla^2 u = k^2 u$ so $\lambda = k^2$. More EV stuff later in this chapter.]

Exercise 6.6. The ∇^4 guy

This is a sort of n=4 Laplace equation, called **biharmonic**. We put value = f(φ) and radial derivative = g(φ) around a ring, and then we look at some special cases. Using the part method, we can come up with a MVT for this case that involves both the f and g functions. Remember that higher order PDE's require more BV data to specify a solution.

Exercise 6.7. Here $L = \nabla^2 - P$ where P is a "positive function". (2D)

I am able to derive a form of the MVT for this situation on the unit disk. We are supposed also to show that the max and min values are on the boundary and solution is unique, both these for an arbitrarily shaped boundary. I wrote a lot of raw notes for this problem.

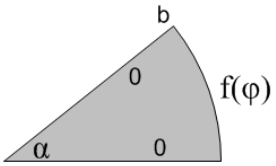
Exercise 6.8. Show uniqueness for $Lu = \nabla\bullet(k\,\nabla u)$.

This L operator appears in the static heat conduction situation, see Vol I p 328 A.10. The proof here is pretty simple, we mimic the Laplace uniqueness proof using a Green's function approach.

Exercise 6.9. Poisson Kernel and the distributional Dirichlet problem (2D)

$\nabla^2 u = 0$ with $\lim_{r\to 1} u = \delta(\varphi)$ is a "distributional Dirichlet problem" and its solution is the Poisson Kernel. He goes on to show that u' is then the solution of the same problem but with $\lim_{r\to 1} u = \delta'(\varphi)$.

Exercise 6.10. Dirichlet Laplace on a wedge (2D)



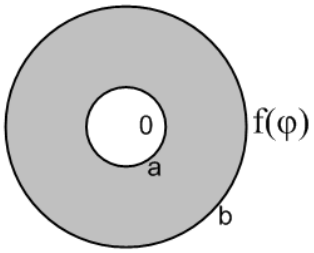
We separate variables as usual, constant is $\lambda = \sigma^2$, and find that $\Phi_n(\varphi) = A_n \sin(\sigma_n\varphi)$ in order to meet the two 0 edge BCs, where $\sigma = \sigma_n = (\pi/\alpha)n = \pi n/\alpha$. Radial form is $C_\sigma r^\sigma + D_\sigma r^{-\sigma}$ etc, but with our special σ values. So our general radial form is $R(r) = \Sigma_{n=1}^{\infty}(C_n r^{\pi n/\alpha} + D_n r^{-\pi n/\alpha}) + C_0 + D_0 \ln(r)$. But finite at r=0 limits this to $C_n r^{\pi n/\alpha}$ so general solution is $u = \Sigma_{n=1}^{\infty} b_n\, r^{\pi n/\alpha} \sin(\pi n\varphi/\alpha)$. A somewhat unusual Fourier analysis (see raw notes "wedge meets fourier series.doc") concludes that our final solution is this:

$$u(r,\varphi) = \Sigma_{n=1}^{\infty} b_n\ (r/b)^{\pi n/\alpha} \sin(\pi n\varphi/\alpha)$$
$$b_n = (2/\alpha) \int_0^{\alpha} d\varphi\, f(\varphi) \sin(\pi n\varphi/\alpha)$$

A special case is $\alpha = 2\pi$ with $f(\varphi) = \sin(\varphi/2)$, the solution in this case is $u(r,\varphi) = \sqrt{r}\ \sin(\varphi/2)$ which I think matches the general unit circle solution in this case, since the assumed f(φ) matches the slit. [We could instead consider $f(\varphi) = \cos\varphi$ in which case $b_n \sim \delta_{n,2}$ so $u \sim r\ \sin(\varphi)$ which is the wrong answer! This problem only allows f(φ) such that f(0) = 0. See later for the $f(\varphi) = \cos\varphi$ solution and plot of same!]

Comment: if f(φ) = 0, solution is u = 0 as expected. There is a Green's Function solution that is 0 on the entire wedge boundary, but that is a solution of Poisson with a δ source, not a solution of Laplace.

Exercise 6.11. Dirichlet Laplace on an annulus. (2D)



We start here with this general form,

$$u(r,\varphi) = \Sigma_{n=1}^{\infty} (A_n r^n + B_n r^{-n})(C_n \cos(\varphi n) + D_n \sin(\varphi n)) + (A_0 + B_0 \ln(r))$$

We can no longer throw out the cos(φn) terms, we have to keep both powers or r, and we have to keep the log and constant, so more complicated that just the r=b disk. Also, we have to keep the ln(r) term.

We first identify this with Schaum p 131 Fourier series with 2L = 2π and this lets us relate our capital letter coefficients to the Schaum coefficients.

$$a_0/2 = A_0 + B_0 \ln(b)$$
$$a_n = (A_n b^n + B_n b^{-n})C_n$$
$$b_n = (A_n b^n + B_n b^{-n})D_n$$

where

$$a_n = (1/\pi) \int_0^{2\pi} d\varphi\, f(\varphi) \cos(n\varphi) \qquad \text{valid also for n = 0,1,2...}$$

$$b_n = (1/\pi) \int_0^{2\pi} d\varphi\, f(\varphi) \sin(n\varphi) \qquad \text{valid also for n = 1,2...}$$

If we require u=0 on the inner r=a boundary, we get these requirements:

$$A_n a^n + B_n a^{-n} = 0 \qquad B_n = - A_n a^{2n}$$
$$A_0 + B_0 \ln(a) = 0 \qquad B_0 = -A_0/\ln(a)$$

which allows us to eliminate all the B_n coefficients, and we then have

$$a_0/2 = A_0[1 - \ln(b)/\ln(a)]$$
$$a_n = A_n C_n(b^n - a^{2n}b^{-n}) =$$
$$b_n = A_n D_n (b^n - a^{2n}b^{-n})$$

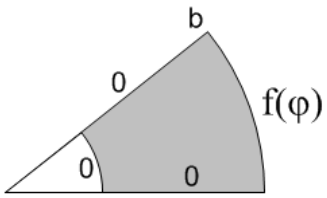
This then leads to a final solution, with the a_n and b_n integrals shown above:

$$u(r,\varphi) = \Sigma_{n=1}^{\infty} (r^n - a^{2n} r^{-n})/ (b^n - a^{2n}b^{-n}) [a_n \cos(n\varphi) + b_n \sin(n\varphi)]$$

$$+ (a_0/2) [1 - \ln(r)/\ln(a)] / [1 - \ln(b)/\ln(a)]$$

Exercise 6.12. Dirichlet Laplace on the curved quadrilateral. (2D)

Part a has this picture [needs osc in φ, "expo" in r]



wedge angle is α

This is obviously a horrible combination of the two previous problems. The starting position is this:

$R_n(r) = (1-\delta_{n,0})(C_n r^{n\pi/\alpha} + D_n r^{-n\pi/\alpha}) + \delta_{n,0}(C_0 + D_0 \ln(r))$
$\Phi_n(\varphi) = A_n \sin(n\pi\varphi/\alpha)$ // no cos due to BC u=0 at φ = 0, same in (b) to come

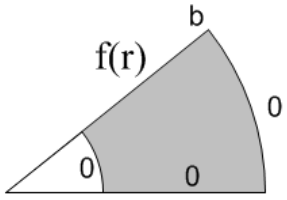
But $\Phi_n = 0$ for n=0 so that kills off the log and constant terms, so we get

$u(r,\varphi) = \Sigma_{n=1}^{\infty}(C_n r^{n\pi/\alpha} + D_n r^{-n\pi/\alpha}) \sin(n\pi\varphi/\alpha)$

We turn the crank (see raw notes) and out pops the solution,

$u(r,\varphi) = \Sigma_{n=1}^{\infty} b_n (r^{n\pi/\alpha} - a^{2n\pi/\alpha} r^{-n\pi/\alpha})/ (b^{n\pi/\alpha} - a^{2n\pi/\alpha} b^{-n\pi/\alpha}) \sin(n\pi\varphi/\alpha)$
where $b_n = (2/\alpha) \int_0^{\alpha} d\varphi\, f(\varphi) \sin(n\pi\varphi/\alpha)$

Part b has this picture [needs osc in r, "expo" in φ]



wedge angle is α, inner radius a, outer radius b

This problem is different in that the radial direction now sets the eigenvalues. We start with our general form

$u(r,\varphi) = \Sigma_{\sigma} \{ C_\sigma r^{\sigma} + D_\sigma r^{-\sigma}) \sin(\sigma\varphi)$

where we have no $A_0 + B_0 \ln(r)$ term because sin(σφ) kills it off for σ = 0. The r = a and r = b conditions that u = 0 tell us that

$C_\sigma b^{\sigma}a^{\sigma} + D_\sigma b^{\sigma}a^{-\sigma} = 0$
$C_\sigma b^{\sigma}a^{\sigma} + D_\sigma a^{\sigma}b^{-\sigma} = 0$ and similarly for A_σ and B_σ

Either set of conditions forces us to have σ eigenvalues as follows: (all are imaginary!)

$\sigma = n\, i\pi/ \ln(b/a)$ $n = 0, \pm 1, \pm 2, ...$

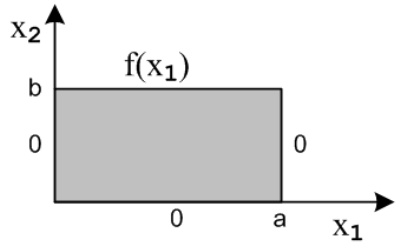
and when the dust settles we get this final solution

$$u(r,\varphi) = \Sigma_{n=1}^{\infty}\ b_n\ [\sinh(n\pi\varphi/k)/\sinh(n\pi\alpha/k)]\sin(n\pi\ \ln(r/a)/k) \qquad k = \ln(b/a)$$

$$b_n = (2/k)\ \int_0^k dx\ f(ae^x)\ \sin(n\pi x/k)$$

You see the sine in the radial direction which makes u = 0 on the two arcs. The sin(σφ) became a sinh function, and the rest comes from the usual Schaum p 131 comparison.

Exercise 6.13. Dirichlet Laplace on a rectangle. (2D) (108)



Here we do a Cartesian separation which as usual causes sine in one variable and expo in the other. We quickly arrive at this general *form* for a solution (cosh no good, cos no good)

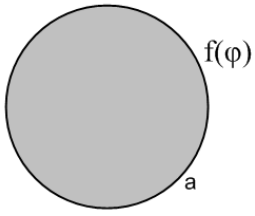
$$u(x_1, x_2) = \Sigma_{n=1}^{\infty} A_n\ \sinh(n\pi x_2/a)\ \sin(n\pi x_1/a) \qquad \sigma = n\pi/a$$

and then the usual Schaum comparison gives our general solution

$$u(x_1, x_2) = \Sigma_{n=1}^{\infty}\ b_n\ [\ \sinh(n\pi x_2/a)/\sinh(n\pi b/a)]\ \sin(n\pi x_1/a)$$

$$b_n = (2/a)\ \int_0^a dx_1\ f(x_1)\ \sin(n\pi x_1/a)$$

PL Exercise Added: Do the disk using the methods of the previous exercises.



That is to say, use sines and cosines keeping everything real. We then get the Schaum p 131 thing:

$$u(r,\varphi) = \Sigma_{n=1}^{\infty}\ (\ a_n\cos(n\varphi)\ + b_n\sin(n\varphi)\ )\ (r/a)^n\ + a_0/2$$

$$a_n = (1/\pi)\ \int_0^{2\pi} d\varphi\ f(\varphi)\ \cos(n\varphi) \qquad a_0/2 = (1/2\pi)\ \int_0^{2\pi} d\varphi\ f(\varphi)$$

$$b_n = (1/\pi)\ \int_0^{2\pi} d\varphi\ f(\varphi)\ \sin(n\varphi)$$

where the $a_0/2$ term is the Mean Value and $u(0,\varphi) = a_0/2$. [Now if $f(\varphi) = \cos\varphi$, get $a_n \sim \delta_{n,1}$ and get $b_n = 0$ so result is $u(r,\varphi) = (r/a)\cos\varphi$.]

Exercise 6.14. Complete orthogonal set -- don't care much about this one.

Exercise 6.15. Laplace solution inside a sphere whose surface has f(θ,φ) (3D) (108)

In 3D, separation of variables produces the general Laplace solution form

$$u(r,\theta,\varphi) = \Sigma_{n=0,1..}\ \Sigma_m\ a_{mn}\ r^n\ Y_{nm}(\theta,\varphi) \qquad (*)$$

where $n = \ell$, the traditional symbol which Stak avoids. The other radial term is r^{-n-1} which we drop for in-sphere action, see p 393 for pencil details. If we set r = 1, the inversion formula for a_{mn} is given in A, where now $u(1,\theta,\varphi) = f(\theta,\varphi)$, the boundary value. We then insert this inversion for a_{mn} into (*) above, but this result is not written out. If we evaluate this unstated result at θ=0, the Y_{nm} vanish for $m \neq 0$ and the result is C. The fancy Legendre sum in C is done using A.14 and we arrive at D. We then magically reinstall θ = θ and at the same time the integrand angle changes from θ' to γ. Accepting this detail for the moment, we then have E which is the analog of 6.11 in 2D. It gives the potential inside the unit sphere as an integral of the boundary value f(θ,φ) on the surface times a certain 3D Poisson kernel. He then writes out cosγ in 6.26.

The big step for me was going D to E which I now present as a general separate Exercise.

PL Exercise 6.15A. A theorem about full group space integrals and its application D→E .

1. The *full* 4π integral $I = \int d\Omega\ f(\Omega) = \int d\Omega'\ f(\Omega')$ where $\Omega' = R\Omega$ for any R. In other words, since I is a rotational scalar, you can compute the same result in any rotated reference frame. In group theory, this is called "the rearrangement theorem": as Ω takes on all group values, so does RΩ, and the measure dΩ is an invariant measure that makes this work for a continuous group: $\int dg\ f(g) = \int d(g_1 g) f(g)$.

2. **Theorem**: $g(\Omega_1) = \int d\Omega'\ F(\Omega', \Omega_1) \Rightarrow g(\Omega_2) = \int d\Omega'\ F(\Omega', \Omega_2)$ for any $\Omega_2 = R\Omega_1$. This says that if the equation $g(\Omega_1) = \int d\Omega'\ F(\Omega', \Omega_1)$ is true for some direction Ω_1 , it is true for all directions. The example of interest to us is $\Omega_1 = \Omega_0 = (0,\text{-})$ and $\Omega_2 = \Omega = (\theta,\varphi)$. If true for Ω_0, then true for Ω. You might think of this theorem as concerning $\mathbf{v}_1 = \mathbf{w}_1$ where both sides are vectors. Since this is a vector equation, it must be true regardless of the orientation of the vectors, so that $\mathbf{v}_2 = \mathbf{w}_2$ where $\mathbf{v}_2 = R\mathbf{v}_1$ etc.

3. We then apply our theorem to the special case $F(\Omega', \Omega_2) = f(\Omega')\ h(\hat{\mathbf{r}}' \bullet \hat{\mathbf{r}}_2)$. We can then conclude that if the first line below is true, the second line must also be true:

$$g(\Omega_0) = \int d\Omega'\ f(\Omega')\ h(\hat{\mathbf{r}}' \bullet \hat{\mathbf{r}}_0) \qquad \mathbf{r}' = (r,\theta',\varphi') \qquad \hat{\mathbf{r}}' \bullet \hat{\mathbf{r}}_0 = \cos\theta' \qquad \Omega_0 = (0,\text{-})$$

$$g(\Omega) = \int d\Omega' \, f(\Omega') \quad h(\hat{\mathbf{r}}' \bullet \hat{\mathbf{r}}) \qquad \mathbf{r} = (r,\theta,\varphi) \qquad \hat{\mathbf{r}}' \bullet \hat{\mathbf{r}} \equiv \cos\gamma \qquad \Omega = (\theta,\varphi)$$

This then is how we get from D to E ! It is trivial to compute cosγ in terms of θ,φ,θ',φ', see raw notes.

Exercise 6.16. Laplace solution *outside* a sphere whose surface has f(θ,φ) (3D) (110)

We start by replacing r^n by r^{-n-1} in the general form. Then our tricky (A.14) sum gets replaced by a slightly different sum I call (A.14)' and which I derive (more or less) in " stagold vol 2 Appendix A.doc". This sum has $r^n \rightarrow r^{-n-1}$ and an overall minus sign. When you DO the sum going from C to D (p 109), you get the exact same answer you got inside the sphere, except now you have an extra overall minus sign. The bottom line is that for r > 1, our solution is 6.26 but with a minus sign. This is exactly what happened in 2D with equation 6.11.

Exercise 6.17. Laplace solution for spherical annulus (3D)

Radii are b > a. Solution general form is now this:

$$u(r,\theta,\varphi) = \Sigma_{n=0} \Sigma_{\mathtt{all\ m}} \, (A_{nm} r^n + B_{nm} r^{-n-1}) \, Y_{nm}(\theta,\varphi)$$

The two boundary conditions are now

$$u(b,\theta,\varphi) = \Sigma_{n=0} \Sigma_{\mathtt{all\ m}} \, (A_{nm} b^n + B_{nm} b^{-n-1}) \, Y_{nm}(\theta,\varphi) = f_b(\theta,\varphi)$$
$$u(a,\theta,\varphi) = \Sigma_{n=0} \Sigma_{\mathtt{all\ m}} \, (A_{nm} a^n + B_{nm} a^{-n-1}) \, Y_{nm}(\theta,\varphi) = f_a(\theta,\varphi)$$

Rewrite these as

$$u(b,\theta,\varphi) = \Sigma_{n=0} \Sigma_{\mathtt{all\ m}} \, b_{nm} \, Y_{nm}(\theta,\varphi) = f_b(\theta,\varphi) \qquad b_{nm} = A_{nm} b^n + B_{nm} b^{-n-1}$$
$$u(a,\theta,\varphi) = \Sigma_{n=0} \Sigma_{\mathtt{all\ m}} \, a_{nm} \, Y_{nm}(\theta,\varphi) = f_a(\theta,\varphi) \qquad a_{nm} = A_{nm} a^n + B_{nm} a^{-n-1}$$

Apply page 395 E orthogonality with N_{nm} to get

$$b_{nm} = (1/N_{nm}) \int d\Omega \, Y_{nm}(\Omega)^* \, f_b(\theta,\varphi)$$

$$a_{nm} = (1/N_{nm}) \int d\Omega \, Y_{nm}(\Omega)^* \, f_a(\theta,\varphi)$$

Then reverse solve for A_{mn} and B_{mn} and you are done.

$$A_{nm} = + (a^{-n-1} b_{nm} - b^{-n-1} a_{nm}) / (a^{-n-1} b^n - a^n b^{-n-1})$$
$$B_{nm} = - (a^n b_{nm} - b^n a_{nm}) / (a^{-n-1} b^n - a^n b^{-n-1})$$

Then our solution is this:

$$u(r,\theta,\varphi) =$$

$$\Sigma_{n=0}\,\Sigma_{\text{all } m}\,[\,(a^{-n-1}b_{nm} - b^{-n-1}a_{nm})\,r^n - (a^n b_{nm} - b^n a_{nm})r^{-n-1}\,]\,/\,(a^{-n-1}b^n - a^n\,b^{-n-1})\;\;Y_{nm}(\theta,\varphi)$$

$$b_{nm} = (1/N_{nm})\int d\Omega\; Y_{nm}(\Omega)^*\; f_b(\theta,\varphi)$$

$$a_{nm} = (1/N_{nm})\int d\Omega\; Y_{nm}(\Omega)^*\; f_a(\theta,\varphi)$$

General Comment: Cauchy, Dirichlet and Neumann BC's

This is a subject that Stak sidesteps. Back in Chapter 5, we learned that if you took a piece of hypersurface C (boundary), and if you are dealing with a 2nd order PDE operator L, then if you specify u and $\partial_n u$ on that piece of boundary, it is very likely that your entire solution is determined. There were potential problems if your surface C was tangent to the characteristic curves of L, but Laplace is an elliptic equation and there are no such curves to worry about. Now in Chapter 6 we have learned that if we have a *closed* hypersurface or boundary, then specifying just u on the entire surface determines the solution (Dirichlet problem). If you tried to also specify $\partial_n u$, your problem would be overspecified and there would be no solution. So somehow the "general theory" of Chapter 5 does not apply to bounding surfaces which are closed. We do know that if such a surface is not closed, then u on that non-closed bounding surface does NOT give you the complete solution, since you can fill in the missing BV piece with any function you want and for each there will be a different solution. We shall learn I think that you can specify just $\partial_n u$ on a closed boundary and that will also determine the solution (Neumann problem).

Comment on Theory So Far versus Real World Problems (verbatim)

1. In **electrostatics**, we often have metal conductors floating around in space. Such conductors have surfaces which are at a constant potential, the reason being that were there any transverse E fields causing potential differences, currents would flow. So "conductor electrostatics" offers no problems of the Dirichlet type other than f=constant. However, the charge distributions which arise in conductor problems cause normal E fields which means cause normal potential gradients at surfaces. This offers itself to Neumann type BC problems.

We could consider electrostatics on insulating objects which can hold charge stably on their surfaces. In this situation, you could have transverse fields and thus you could have some f(θ,φ) function on a surface. You would have to construct it by micro-loading charge at each point on the surface to create the desired potential f on the surface. In this case there would of course be normal gradients determined by f, so you could not do "Cauchy" assignment of the data, and this seems to go with closed surfaces.

In both the above examples, we have charges (sources) active. Still, away from the surfaces, we do have the Laplace equation.

2. **Steady state heat flow** solutions result in stable Laplacian temperature distributions. So you could micro-control a temperature f(θ,φ) at each point on a surface and wait for steady state to arrive. The temperature distribution in a uniform conducting solid object contained by said surface would then be the solution to the kind of Dirichlet Laplace problems we have been considering.

3. In terms of **diffusion**, you could micro-stabilize a density of some molecules on a surface (that does not attract or repel molecules) and then I suppose in the vacuum space contained by the surface, you would obtain a density n determined by Laplace. Again I think we are talking steady state here.

4. As we learn later, incompressible ideal **fluid flow** in certain cases has $\mathbf{v} = \nabla\varphi$ where φ is Laplace.

6.4 Surface Layers (110) (all for n=3)

In the previous three sections, we studied the solutions of Laplace given imposed values of the solution on bounding surfaces -- the Dirichlet problem -- and this was done in the complete absence of sources. Now in this section we do the reverse: we study solutions of Laplace with sources (ie, Poisson) but we impose no boundary values. Eventually, we will combine BV's and sources together.

In this section, our only sources will be continuous ones (monopole and dipole) spread out on a closed boundary. Later we shall consider instead sources which are point sources, then we will talk Green's Functions. However, at the very start, in order to *develop* the continuous boundary sources, we momentarily consider the general Poisson and then isolated point monopole and dipole sources.

So this is a very major change in topic! I have made up my own subsections for this section because the section is large and dense and all the pieces just run together in the text. I present things in his ordering, but my section tags show how the items are related to each other (and are thus not in his order).

A1. Finding u from a source density function q (n=3) (111, 6.30 resulting u)

Well, if $q(\xi)$ is some charge distribution, we know from superposition that (E = fundy solution)

$$-\nabla^2 u = q(\mathbf{x}) \qquad \text{in "free space"} \qquad \text{(ie, with no boundary conditions)}$$

$$\Rightarrow \quad u(x) = \int d\xi\, E(x-\xi)\, q(\xi) \qquad E(x-\xi) = 1/4\pi|x-\xi|$$

E is the solution of $-\delta(\mathbf{r}) = \nabla^2 E$. In fact, here is a complete derivation of what E is:

$$-1 = \int dV\, \nabla\bullet(\nabla E) = \int d\mathbf{S} \bullet \nabla E = \int (\partial E/\partial r)\, dS = \int (\partial E/\partial r)\, r^{n-1} d\Omega_n = S_n(1)\, (\partial E/\partial r)\, r^{n-1}$$

For n = 3, $S_3(1) = 4\pi$ and E = k/r so $\partial E/\partial r = -k/r^2$ and then $S_n(1)\, (\partial E/\partial r)\, r^{n-1} = -4\pi k$ and $k = 1/4\pi$. This was one of the fundy solutions computed in Section 5.8.

B1. Suppose the source density function represents an isolated unit dipole. (112, 6.29)

We arrive at the dipole source equation $-\nabla^2 u = \partial_{\ell}\, \delta(x-\xi)$ by taking a limit with two point charges, and the name we give for the solution to this equation is $u = D_{\ell}(x|\xi)$. The dipole is two charges of size 1/h separated by distance h. Recall that p = qh and so here p = 1, a unit dipole. Here is that limit:

$$[g(\mathbf{x}+\mathbf{dx}) - g(\mathbf{x})] = \mathbf{dx} \bullet \nabla g \qquad \Rightarrow \qquad g(\mathbf{x}+\mathbf{dx})/dx - g(\mathbf{x})/dx = \hat{\mathbf{dx}} \bullet \nabla g$$

so $\lim_{h\to 0}\,[\,\delta(\mathbf{x}-\xi+h\hat{\boldsymbol{\ell}})/h - \delta(\mathbf{x}-\xi)/h\,] = \hat{\boldsymbol{\ell}}\bullet\nabla\,\delta(\mathbf{x}-\xi) \equiv \partial_{\ell}\,\delta(\mathbf{x}-\xi)$

Our equation is then $-\nabla^2 D_{\ell}(x|\xi) = \partial_{\ell}\,\delta(x-\xi)$ where $\boldsymbol{\ell}$ is associated with the $\hat{\boldsymbol{\ell}}$ unit vector, which is the direction of the dipole, so $\hat{\mathbf{p}} = \mathbf{p} = \hat{\boldsymbol{\ell}}$. You can see at once that in this example, q(x) is a distribution (as it is of course in the monopole Green's function definition).

So what is the solution of $-\nabla^2\, D_{\ell}(x|\xi) = \partial_{\ell}\,\delta(x-\xi)$? The first step involves doing the limit just done above, but for the solution, not the sources: [there are a few subtleties here]

$D_{\ell}(x|\xi)$= limit h→ 0 of the two single delta charge potentials

$= \lim_{h\to 0}\ (1/4\pi)\,[\,(1/h)/|\mathbf{x}-(\xi+h\hat{\boldsymbol{\ell}})| - (1/h)/|\mathbf{x}-\xi|\,]$

$= (1/4\pi)\lim_{h\to 0}\,\{\,[\,f(\xi+h\hat{\boldsymbol{\ell}}) - f(\xi)\,]/h\,\}$ $\qquad f(\xi) \equiv 1/|\mathbf{x}-\xi|$ = temp variable

$= (1/4\pi)\ \hat{\boldsymbol{\ell}}\bullet\nabla_{\xi} f(\xi) = (1/4\pi)\,\partial_{\ell} f(\xi)$

Thus, at this point we have our solution $D_{\ell}(x|\xi) = (1/4\pi)\,\partial_{\ell} f(\xi)$. We then go compute this object,

$\nabla_{\xi} f(\xi) = (\mathbf{x} \hat{-} \xi)/\,|\mathbf{x}-\xi|^2$

as shown in the raw notes. Then we insert this to get the following classic result

$D_{\ell}(x|\xi) = (1/4\pi)\,\partial_{\ell} f(\xi) = (1/4\pi)\hat{\boldsymbol{\ell}}\bullet\nabla_{\xi} f(\xi)$

$= (1/4\pi)\ \hat{\boldsymbol{\ell}}\bullet(\mathbf{x} \hat{-} \xi)/\,|\mathbf{x}-\xi|^2 = (1/4\pi)\cos(\mathbf{x} \hat{-} \xi,\ \hat{\boldsymbol{\ell}})/\,|\mathbf{x}-\xi|^2 = \cos\theta/4\pi r^2$

The dipole is located at position ξ, and θ is the angle between the dipole axis $\hat{\boldsymbol{\ell}}$ and the direction from ξ out to the observation point $\mathbf{x}$.

A2. Suppose our charge a(x) is spread on a surface σ (simple layer) (112, result 6.30 for u)

The result is extremely simple. We just superpose the potential of charges $a(\xi)\,dS_{\xi}$:

$u(\mathbf{x}) = \int_{\sigma} dS_{\xi}\ a(\xi) * 1/4\pi|\mathbf{x}-\xi|$ $\qquad$ n = 3 only $\qquad$ // $\int dS\ a\ 1/4\pi R$

and this is known as a **simple layer**. Here of course a has dimensions of charge per area. We have this simple result because there are no imposed boundary conditions. [Such imposed BC's would add homo solutions of Laplace tuned to meet the BC's.]

A3. Proof that u of A2 is finite for x actually on the surface (112)

You can see that you might get a divergence in our result above as $\mathbf{x} \to \xi$, but in fact that is not the case. Stak considers just the part of the surface near the point of contact and shows that in fact there is no divergence. I followed his proof, it is interesting and is based on $1/\rho * \rho d\rho$ over small tangent plane disk.

B2. A dipole layer b(x) on surface σ (113 result 6.31 for v)

I might have called the density p(ξ), but he uses b(ξ). Again, we just superpose our single dipole solutions and we then have an integral over σ. The angle of the dipole on the surface is assumed to be pointing out normal from the surface, that is what is meant by a "**double layer**". So then replace $\hat{\boldsymbol{\ell}} = \hat{\mathbf{n}}$ and we have our result as shown in 6.31. The angle is then between the normal at point ξ on σ and the ray from ξ to our observation point **x**. In other words, just the usual polar angle of the dipole.

$u(\mathbf{r}) = \int_\sigma dS'\, a(\mathbf{r}')\, 1/4\pi R$ $\quad \mathbf{R} = \mathbf{r} - \mathbf{r}'$ $\quad r'$ is on σ $\quad$ simple layer

$v(\mathbf{r}) = \int_\sigma dS'\, b(\mathbf{r}')\cos(\mathbf{R},\mathbf{n}')/4\pi R^2$ $\quad \mathbf{R} = \mathbf{r} - \mathbf{r}'$ $\quad r'$ is on σ $\quad$ double layer

B3. Proof that dipole u of B2 is finite for x actually on the surface. (113)

He just outlines the proof here, let's not worry about it. Same general idea as A3 discussion above.

C. Detail about x → s where s is on the surface σ which has a δ source. (113-115)

In A3 and B3 above, we saw that as you approach a surface having some reasonable charge or dipole distribution on it, nothing violent happens, everything is under control. BUT, what happens if right at the point you approach, you happen to hit a δ type charge or dipole density? Then something different does in fact happen, and that is the subject of this and the next sections.

Stak "models" this situation carefully. He is thinking of a curved surface that we are going to approach along the normal direction, but we know when we get close we can just think of the tangent plane. He arranges for the tangent plane to be the x-y plane and we approach from the $+x_3$ direction and the delta source is sitting right smack at the origin of our little tangent plane coordinate system.

(1) Approach to a charge δ. Now our usual 1/4πR thing he writes carefully again as $E(x_1,x_2, x_3)$ in this same coord system, and he considers $x_3 \to 0$ which gives $E(x_1,x_2, 0)$. If we integrate this against a test function over the surface plane, we find that E is locally integrable, sort of $1/4\pi\rho \ * \ \rho d\rho$ on a 2D disk. This means that we are allowed to take our limit $x_3 \to 0$ through a test function integral, as shown in 6.32, and this in turn means that we can write $E(x_1,x_2, x_3) \to E(x_1,x_2, 0)$ in the full "distribution sense".

(2) Approach to a dipole δ. In (1) we had $u = E(\mathbf{r}) = 1/4\pi r$ due to our δ charge at the origin and we studied $x_3 \to 0$. Here we have instead $u = D_{x3}(x|0) = \partial_3 E = \partial_3(1/4\pi r) = - x_3/4\pi r^3 \equiv F(x_1,x_2, x_3)$. You can see that this is more singular than our (1) situation and there could be trouble. If we integrate F against a test function now, we find that F has this property: $F(x_1,x_2, x_3) \to - \delta(x_1)\, \delta(x_2)/2$ as $x_3 \to 0$. If we miss the dipole delta, we get zero because $\cos\theta = 0$ on the flat surface.

We can now summarize these last two little sections: (E = 1/4πr = Green's)

(1) $\lim_{x3 \to 0\pm} \{ E(x_1, x_2, x_3) \} = E(x_1, x_2, 0)$ in full symbolic function sense.

(2) $\lim_{x3 \to 0\pm} \{ \partial_3 E(x_1, x_2, x_3)\} = \mp\delta(x_1)\, \delta(x_2)/2$ in full symbolic function sense

(3) Here Stak shows that all tangential derivatives are well behaved, so it is only the normal one we have to worry about, the ones studied above. [Stakgold is nothing if not thorough!]

D. Extend the Part C analysis from δ sources to layers on the flat x-y plane (115)

In C above we considered what happens when we approach an *isolated* δ charge or dipole. Here, instead, we approach a point on a charge or dipole distributed *layer*. This layer, we assume in this section only, lies on a completely flat surface, there is no curvature yet. This is the model we use here. We proved in sections A3 and B3 above that these limits gave finite results, here we are looking in more detail at what these finite results are.

(1) Simple Layer u and $\partial_3 u$. We start here with 6.30 which says $u(\mathbf{x}) = \int_\sigma dS_\xi \; a(\xi) \; (1/4\pi|\mathbf{x}-\xi|)$, an equation we just wrote 8" above. We make our point of interest still be the origin, and we ask about u as we approach on a direct line from $+x_3$. We write $u(0,0,x_3)$ = the integral with $x_1 = x_2 = 0$. We know we can take the $x_3 \to 0$ through the integral, and this gives us result 6.36. However, if we then compute $\partial_3 u(0,0,x_3)$ equals its integral and then take this same limit, we get - a(0,0)/2 ! The integrand is exactly the $- x_3/4\pi r^3$ thing we had above and it is what approaches $- \delta(x_1)\,\delta(x_2)/2$.

(2) Double Layer v and $\partial_3 v$(116 results 6.38 and 6.40). The solution v -- as I already noted earlier -- has the exact same form as the object $\partial_3 u$ above if we just replace the simple layer density a with double layer density b and have a + sign out front. Thus we just crib the limit in 6.37, change the sign and do a→b and we have 6.38. So this is nothing new. The result for $\partial_3 v$ starts as 6.39 and the limit as $x_3 \to 0$ is the huge mess shown in 6.40 which actually involves a contour integration on the surface because we threw in some Green's Theorem action. I don't think we ever need this $\partial_3 v$ result, just there for interest.

E. Now consider the actual curved surface σ. (117)

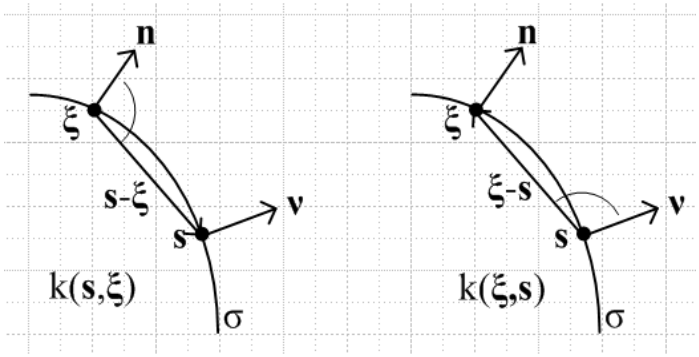
The idea here is that now consider our previous section D to model the tangent plane here.

(1) Simple Layer u and $\partial_\nu u$ (118 results 6.41 and 6.43) As before, we have $u(\mathbf{x}) = \int_\sigma dS_\xi \; a(\xi) \; (1/4\pi|\mathbf{x}-\xi|)$ but now σ is a curved surface. We imagine our contact point is at point **s** on the surface (instead of the origin). We break the σ integration into a core disk piece on that tangent plane right around s, and the non-core piece outside. For u, the core piece gives nothing and our result is boring (6.41). For $\partial_\nu u$, however, the core piece shows that $- \delta(x_1)\,\delta(x_2)/2$ type behavior and we "pick up" - a(s)/2 as an "extra term" as shown in 6.43. This says that, yes, you can take the limit right through the integration, provided you add this extra piece! I will state this fact again below.

(2) Double Layer v and $\partial_\nu v$ (119 results 6.44 and 6.49). Now for the first time we get a *difference* between $\partial_\nu u$ and v, and this difference only appears when σ is not flat. In 6.42, **ν** is the normal at point **s** and is fixed in the integral. But doing v on p 119A, now **n** is the local normal to σ and varies over the integration. In the core part, the surface really is flat and we get exactly the same core term, **ν** and **n** are

the same there. But the residual term still has **n** in it. Compare this 6.44 to 6.38. Again for flat σ the cos = 0 everywhere on σ. [Let's forget about $\partial_{\nu}v$ for these meta notes since don't think it gets used, 6.49.]

So, at this point our results for $\partial_{\nu}u$ and v are given by 6.43 and 6.44. These results are very similar, the first of course has charge density a, the second has dipole density b. The big difference is that 6.43 has **ν** in its cosine factor, while 6.44 has **n**. As just noted, **ν** is the fixed normal at point **s**, whereas **n** moves around as you do the integration. I had to draw this pretty detailed picture to get this understood:



For this reason, the integrals are not the same. If you combine the cosine factor with the $1/4\pi R^2$ factor, and if you define k in the way shown, you find that k(**ξ**,**s**) is what appears in 6.43, whereas k(**s**,**ξ**) is what appears in 6.44. It is the same k function in both cases, and the point is that k is not symmetric!

So here is where we end up:

$$u(\mathbf{x}) = \int_{\sigma} dS_{\xi}\, a(\xi)\, E(x|\xi) \qquad E(x|\xi) = 1/4\pi|\mathbf{x}-\xi|$$

$$v(\mathbf{x}) = \int_{\sigma} dS_{\xi}\, b(\xi)\, k(x,\xi) = \int_{\sigma} dS_{\xi}\, b(\xi)\, \partial_{\mathbf{n}\xi} E(x|\xi)$$

where $k(x,\xi) = \cos(x-\xi,n)/[4\pi|\mathbf{x}-\xi|^2] = \partial_{\mathbf{n}\xi}E(x|\xi)$ // see Noted added below

$\lim_{\mathbf{x}\to\mathbf{s}} \partial_{\nu}u = \mp a(s)/2 + \int_{\sigma} dS_{\xi}\, a(\xi)\, k(\xi,s)$ // 6.43 becomes 6.47 ν = normal

$\lim_{\mathbf{x}\to\mathbf{s}} v = \pm b(s)/2 + \int_{\sigma} dS_{\xi}\, b(\xi)\, k(s,\xi)$ // 6.44 becomes 6.46

In the above expressions for limits, here is what we are really saying:

$$\lim_{\mathbf{x}\to\mathbf{s}} \left[\int_{\sigma} dS_{\xi}\, a(\xi)\, (1/4\pi R(\mathbf{x},\xi)) \right] = \int_{\sigma} dS_{\xi}\, a(\xi) \lim_{\mathbf{x}\to\mathbf{s}} [(1/4\pi R(\mathbf{x},\xi))] \quad // \text{ u}$$

$$\lim_{\mathbf{x}\to\mathbf{s}} \left[\int_{\sigma} dS_{\xi}\, a(\xi)\, k(\xi,\mathbf{x}) \right] = \int_{\sigma} dS_{\xi}\, a(\xi) \lim_{\mathbf{x}\to\mathbf{s}} [\, k(\xi,\mathbf{x})\,] - a(\mathbf{s})/2 \quad // \partial_{\nu}u$$

$$\lim_{\mathbf{x}\to\mathbf{s}} \left[\int_{\sigma} dS_{\xi}\, b(\xi)\, k(\mathbf{x},\xi) \right] = \int_{\sigma} dS_{\xi}\, b(\xi) \lim_{\mathbf{x}\to\mathbf{s}} [\, k(\mathbf{x},\xi)\,] - b(\mathbf{s})/2 \quad // \text{ v}$$

Stakgold's effort here is to explain *why* you get that "**extra term**" when you interchange the order of limit and integration in the lower two cases. He has done this rigorously within distribution theory and that is what we went through.

Comment: *Why ?* have we done all this work on evaluating the four objects u, $\partial_{\mathbf{v}}$u, v, $\partial_{\mathbf{v}}$v as we approach the surface σ at point **s**, the first two cases being for a layer of simple source, the second of dipole source. The source functions recall were a and b on σ.

I think the answer is this, and I just give one example. Suppose we have a candidate Laplace solution v of the Neumann problem in the interior of our boundary. We want to take the limit of v as we approach the boundary on which we impose some value v(**s**). This limit ix exactly the third equation above and it is critical that we know about the presence of the term $-\,b(\mathbf{s})/2$.

Note added 10.22.09. Consider the left picture above, but imagine that **s** is replaced by general point **x** not on the boundary (just redraw the boundary so it does not pass through s). The angle marked in the picture would be the θ of cos(x-ξ,n). Suppose we draw a thin triangle where one edge is **x**-ξ,(length we call R) and the short edge is a little dn at the base of the **n** arrow. The third edge has length R+dR. The law of cosines tells us

$$(R+dR)^2 = R^2 + (dn)^2 - 2\,R\,dn\cos\theta$$

$$2RdR \approx -2R\,dn\cos\theta$$

$$\partial_{\mathbf{n}}R = -\cos\theta \qquad \text{this is the result we need!}$$

Then consider

$$\partial_{\mathbf{n}}E = (1/4\pi)\,\partial_{\mathbf{n}}(1/R) = -\,(1/4\pi R^2)\,\partial_{\mathbf{n}}R = +\,(1/4\pi R^2)\cos\theta = (1/4\pi R^2)\cos(x-\xi,n) = k(x,\xi)$$

where n is the normal at point ξ on the surface, so we say $\partial_{\mathbf{n}\xi}$. So the point of this whole Note is to remind ourselves of this last fact which I have included above.

Exercises (120) -- redo all the above for n = 2

Here we want to repeat for n=2 everything we did above for n=3. This task is broken out into three different exercises.

Exercise 6.18 In 2D space, find u and v due to layers on σ = C, a curve. (120)

A1. Finding u from a source density function q (n=2)

$-\nabla^2 u = q(\mathbf{x})$ in "free space" (ie, with no boundary conditions)

$$\Rightarrow \quad u(x) = \int_C d\xi\, E(x-\xi)\, q(\xi) \qquad E(x-\xi) = \ln(1/|x-\xi|)/2\pi \qquad \text{p 120}$$

Since our bounding surface is a curve, we write σ as closed contour C.

B1. Suppose the source density function represents an isolated unit dipole.

We still have $-\nabla^2 D_\ell(x|\xi) = \partial_\ell \delta(x-\xi)$ but now 2D ∇^2. The solution is

$D_\ell(x|\xi) = u_{sol}(x|\xi) = (1/2\pi)\partial_\ell f(\xi) = (1/2\pi)\hat{\ell} \bullet \nabla_\xi f(\xi)$ $\quad f(\xi) \equiv \ln(1/|\mathbf{x}-\xi|)$

$= (1/2\pi) \bullet (\mathbf{x} \hat{-} \xi)/|\mathbf{x}-\xi| = (1/2\pi)\cos(\mathbf{x} \hat{-} \xi, \hat{\ell})/|\mathbf{x}-\xi| = \cos\theta/2\pi r$

so same except denominator that was $4\pi r^2$ is now $2\pi r$.

A2. Suppose our charge a(x) is spread on a surface σ (simple layer) (112, result 6.30 for u)

$u(\mathbf{x}) = \int_C dS_\xi \; a(\xi) * \ln(1/|x-\xi|)/2\pi$ $\quad$ n = 2 only $\quad$ // $\int dS \; a \, (\ln(1/R)/2\pi)$

B2. A dipole layer b(x) on surface σ (113 result 6.31 for v)

$u(\mathbf{r}) = \int_C dS' \; a(\mathbf{r}') \ln(1/R)/2\pi$ $\quad \mathbf{R} = \mathbf{r} - \mathbf{r}'$ $\quad$ r' is on σ $\quad$ simple layer

$v(\mathbf{r}) = \int_C dS' \; b(\mathbf{r}')\cos(\mathbf{R},\mathbf{n}')/2\pi R$ $\quad \mathbf{R} = \mathbf{r} - \mathbf{r}'$ $\quad$ r' is on σ $\quad$ double layer

Exercise 6.19 In 2D space, compute the two distributional results for E (121)

C. Detail about $x \to s$ where s is on the surface σ which has a δ source. (113-115)

I will just quote the results I obtained in the raw notes. Now we have $E(x_1, x_2, x_3) = \ln(1/|\mathbf{x}|)/2\pi$ and as before our delta source is at the origin in our x_1-x_3 coordinate system and we approach from $+x_3$.

(1) $\lim_{x_3 \to 0\pm} \{ E(x_1, x_2, x_3) \} = E(x_1, x_2, 0)$ in full symbolic function sense.
(2) $\lim_{x_3 \to 0\pm} \{ \partial_3 E(x_1, x_2, x_3)\} = \mp\delta(x_1)/2$ in full symbolic function sense

In terms of E, results are exactly the same as n=3 except the $\delta(x_2)$ factor is missing on the second line.

Exercise 6.20 More 2D results (121)

The results here are as in the n=3 case where k is defined with $2\pi R$ instead of $4\pi R^2$ in its denominator as shown in 6.53:

$\lim_{x \to s} u = \int_C dS_\xi \; a(\xi) * \ln(1/|s-\xi|)/2\pi$ $\quad$ // 6.50

$\lim_{x \to s} \partial_\nu u = \mp a(s)/2 + \int_C dS_\xi \; a(\xi) \, k(\xi,s)$ $\quad$ // 6.51

$\lim_{x \to s} v = \pm b(s)/2 + \int_C dS_\xi \; b(\xi) \, k(s,\xi)$ $\quad$ // 6.52

6.5 Integral Equations of Potential Theory (122)

So far in this chapter, we have solved a few no-source Laplace problems with simple geometry, such as unit disk, sphere, wedge, and so on, with the solution specified on the boundary. Now we are going to look at *arbitrary* bounding surfaces on which the solution is specified.

A. The double-layer integral equation method for solving Dirichlet (122 3D)

The raw notes are just fine and concise for this section, go read them if you want. The logic is very interesting: *Just suppose* you had some dipole density b(ξ) on your surface σ. You would write the Laplace solution as $v(\mathbf{x}) = \int_\sigma d\xi\, b(\xi)\cos(\mathbf{R},\mathbf{n})/4\pi R^2$ (*) . This result is just a superposition of dipoles as we know, and we got this result in a framework where there were no boundary conditions, just free space. But now take the limit of v(**x**) as $\mathbf{x} \to \mathbf{s}$ on σ. Call this limit f(**s**) and assume this is a known imposed boundary value on σ. Then using all the work we did in Section 6.4 above, we can say that

$$\lim_{\mathbf{x}\to\mathbf{s}} v(\mathbf{x}) \quad = -\, b(\mathbf{s})/2 + \int_\sigma dS_\xi\, b(\xi)\, k(\mathbf{s},\xi) \quad = f(\mathbf{s})$$

Notice that this is an n-1 dimensional integral equation for b(ξ) of the **Fred inhomo 2** variety (I,195). As just stated, we regard f(**s**) as our imposed boundary value for our solution v on the surface σ, just as we did for the simple problems like the unit disk where we had f(φ). And suppose we could solve this integral equation for a unique solution b(ξ) on the entire boundary σ. We could then insert this b(ξ) into (*) and obtain the solution to our problem v(**x**).

What does this really say? It says that two different "situations" have the exact some Laplace solution:

(1) situation where we impose f(**s**) on boundary σ
(2) situation where we construct a dipole layer b(**s**) on σ which causes f(**s**) on boundary σ

In the exercises below we are going to solve the unit sphere and disk using this method. For the unit disk, say, we impose some f(φ) on the boundary and solve the problem.

Question about the reality of the dipole layer

Question: Is there *in fact* a real actual dipole layer on the boundary?? We do know that, were we to *put* such a dipole layer on the boundary, we would create f(φ) on the boundary and we would create the right solution inside.

I think the answer is that such a dipole layer is *merely one way* to create f(**s**) but that there are other ways as well. Here are two arguments:

(1) For example, I think you could find a third "situation" where the same f(**s**) and internal solution u is created by a single layer charge density a(**s**). For example, it seems to me that we could just repeat the above integral equation discussion of this section and say this:

$u(\mathbf{x}) = \int_\sigma d\xi\ a(\xi) 1/4\pi R \qquad (**)$

$\lim_{\mathbf{x}\to\mathbf{s}} u(\mathbf{x}) = \int_\sigma dS_\xi\ a(\xi) 1/4\pi R = f(\mathbf{s})$

We regard the second equation as a **Fred inhomo 1** variety. We solve if for a(ξ) and insert that into (**) to get u. So I would extend the list above to say

(1) situation where we impose f(**s**) on boundary σ
(2) situation where we construct a dipole layer b(**s**) on σ
(3) situation where we construct a simple layer a(**s**) on σ

Obviously, we don't expect a(s) = b(s).

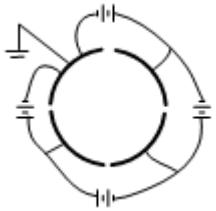
(2) My second argument is this. We know that for the disk or sphere we can say $u = \int PK\ f$ where PK is the appropriate Poisson Kernel and f is the boundary value. We could rewrite this two ways

$u = \int PK\ f = \int (f\ PK/E)\ E = \int aE \qquad a = f\ PK/E \qquad E$ = monopole fundy

$u = \int PK\ f = \int (f\ PK/D)\ D = \int bG \qquad b = f\ PK/D \qquad D$ = dipole fundy

So here are explicit formulas for the equivalent simple and double layer densities a and b that would produce the same solution u and which have the same f(s) on the boundary. Of course this only applies to a spherical boundary, not a general boundary σ.

So imagine a real sphere and we panel it with a million little plates and hook each to a battery in a network like this



So in this method we are really creating a surface charge density a with our million little capacitors and we produce some f(θ,φ) in this manner on the surface. There will of course be some u inside the sphere given by $u = \int PK\ f$.

Alternatively, we could actually place a million little electric dipoles on the spherical surface, each made of two charges and a little stick. Each dipole can have a different p value by varying the charge or the stick length (let's say charge). We arrange this pattern to be b = f PK/D and we obtain the same solution inside as the previous paragraph.

So we wonder why Stakgold did not talk about this a(ξ) integral equation method? Probably there is a good reason. [It comes up in the very next section!]

B. A way to approach the combined interior/exterior Dirichlet problem. (123 nD)

In this section, Stak cleanly defines an interior and an exterior problem relative to an arbitrary closed surface σ. For the exterior problem, he requires that the solution vanish at ∞. In both problems, u = f on σ. The logic is pretty clean, and I outline it in (a) and (b) of the raw notes and we end up with 6.65 where we have a single equation which gives both the interior and exterior solutions called u_i and u_e. Which you get depends on where you put x. So 6.65 really says

$$\int_\sigma 1/4\pi R \; I \; dS_\xi = u(\mathbf{x}) \quad \text{for all } \mathbf{x}\text{, inside or out.}$$

Now recall my conjecture above about a simple-layer approach

$$u(\mathbf{x}) = \int_\sigma d\xi \; a(\xi) 1/4\pi R \qquad (**)$$

$$\lim_{\mathbf{x}\to\mathbf{s}} u \quad = \int_\sigma dS_\xi \; a(\xi) 1/4\pi R \quad = f(\mathbf{s})$$

We see that my simple layer density $a(\xi) = I(\xi) = \partial_n u_i - \partial_n u_e$ where ξ is on σ. So this I(ξ) *is* the surface charge density that would replicate the solution to our problem! In fact, it gives the solution both inside and outside the boundary, which is pretty impressive.

He goes right on to form my integral equation and says yes, it is in fact useful.

Example (125) : Simultaneous computation of u in and out for the unit sphere (3D)

Things have gotten into a strange order here in Stak's presentation. We are going to do the unit sphere using the double-layer integral equation method in Exercise 6.21 below. But right here we are going to do the same problem using the simple-layer integral equation method.

The method is reviewed in the raw notes. We start with (**) above, $4\pi \; f(s) = \int d\Omega_\xi \; I(\xi) 1/R$. We install into this the double Y_{nm} expansion of 1/R where R = |**s**-**ξ**| (both on sphere), rearrange a bit, and discover that the projections I_{nm} of appear on the right. We then expand the LHS on its f_{nm} and find by comparing LHS=RHS that $I_{mn} = (4\pi/N_{0n}) \; f_{mn}$ where N_{on} is a certain constant. So we have "diagonalized" things and we now know the I_{mn} given f. The solution to our problem is therefore the expression after the first equals sign, where {..} = I ,

$$u(\Omega,r) = \int d\Omega_\xi \; (1/4\pi R) \; \{ \Sigma_{n'}\Sigma_{m'} \; I_{m'n'} \; Y_{n'm'}(\Omega_\xi) \; \} \quad = \Sigma_n\Sigma_m \; r_<^{\,n} \; r_>^{-n-1} Y_{nm}(\Omega) \; f_{mn}$$

To evaluate this expression, we install the double Y_{nm} expansion of 1/R where now R = |**x**-**ξ**| and only **ξ** is now on the sphere. This gives the expression after the second = sign, since we were able to use Y orthogonality one time to reduce from three Y to one Y, we installed $I_{mn} = (4\pi/N_{0n}) \; f_{mn}$, and then

installed the value of the constant N_{0n}. This is the final answer, Notice that, as advertised, we get a solution that works both inside and outside of the sphere. If r is inside, then $r_< = r$ and $r_> = 1$ and get $\Sigma_n\Sigma_m$ r^n $Y_{nm}(\Omega)$ f_{mn} and in the other case we get $\Sigma_n\Sigma_m$ r^{-n-1} $Y_{nm}(\Omega)$ f_{mn}. These results agree with the general forms we wrote earlier [p 109 (6.25) and p 110 B].

Question: Why did our integral equation diagonalize to such a simple form?

Notice how the Stak presentation in the above example seems very complicated and mysterious, though it is of course correct. The reader must wonder why we get such a simple result $I_{mn} = (4\pi/N_{0n})\, f_{mn}$ when we process this integral equation. I explain the general idea pretty well on p 45 of my addition theorem thesis where I look at an integral equation that is a "group measure convolution equation". My equation here is $4\pi\, f(\Omega) = \int d\Omega_1\, I(\Omega_1)\, [1/R](\Omega, \Omega_1)$ and the diagonalized equation according to my write-up should be $4\pi f^{\ell}_{mm'} = \Sigma_{m''}\, I^{\ell}_{mm''}\ [1/R]\, \ell_{m''m'}$, where we project things onto our group representation functions $D^{\ell}_{mm'}(\Omega)$. But my integral here is not the full group measure integration because the second azimuth (call it ψ) is missing. We could consider $f(\Omega) = f(\Omega,\psi) = f(g)$ and same for other functions, but there is no actual ψ dependence, so these functions are not general functions on the whole group. Then we have the following, where $d\Omega_1 d\psi_1 = dg_1$

$$4\pi\, f(\Omega,\psi) = \int d\Omega_1 d\psi_1\ I(\Omega_1,\psi_1)\ \{\ [1/R](\Omega, \psi, \Omega_1, \psi_1)\ \delta(\psi - \psi_1)\ \}$$

$$4\pi\, f^{\ell}_{mm'} \qquad = \Sigma_{m''} \qquad I^{\ell}_{mm''} \qquad\qquad [1/R]\, \ell_{m''m'}$$

(I am arm-waving here a bit, but am sure this is generally right). We for sure can now say that

$$f^{\ell}_{mm'} = f^{\ell}_{m}\, \delta_{m',0} \qquad\qquad I^{\ell}_{mm''} = I^{\ell}_{m0}\, \delta_{m'',0}$$

so we might as well set m' = 0 on the LHS and we then have

$$4\pi\, f^{\ell}_{m0} = I^{\ell}_{m0}\ [1/R]\, \ell_{00} \qquad\qquad \Rightarrow \qquad 4\pi\, f^{n}_{m0} = I^{n}_{m0}\ [1/R]\, n_{00}$$

where in the last step we use Stak's "n" for what we usually call $\boldsymbol{\ell}$, the group rep Casimir label. It must then be a fact that $[1/R]\, n_{00} = N_{0n}$. So the reason things diagonalize like this is that we have something that is a special case of a group convolution equation. No doubt I have this written up somewhere, but let's not go look now, we understand the main point.

The Neumann Problem in 3D and for general n ≠ 2 (126) verbatim

Same as Dirichlet but we specify $\partial_n u = f$ on our surface σ. The divergence theorem forces an extra condition which is that $f_{00} = 0$ as shown in 6.69 -- this is because $dS\, \partial_n u_i = d\mathbf{S}\bullet\nabla u_i$ is what appears in the surface integral side of the divergence theorem for $\nabla\bullet\nabla u_i = \nabla^2 u_i = 0$. We can carry over lots of our Dirichlet machinery including an integral equation with something like I in it, but Stak does not do that.

He claims that the $f_{00}= 0$ condition is not required for the *exterior* Neumann problem. If we look at the result p 127 B which would in fact apply in the exterior problem, the surface would be the unit sphere + the sphere at r=∞. If u ~ 1/r and $\partial_n u \sim 1/r^2$ and area ~ r^2, perhaps this r=∞ sphere makes a contribution to the surface integral, which relieves us of having to require that B be true on the inner unit sphere, thus allowing us to have $f_{00} \neq 0$. Whatever f_{00} we have, the solution will offset it on the r=∞ sphere. Check this in example below!

Example 1: Interior Neumann on unit sphere. Use our usual Y_{nm} expansion and meet the BC and you obtain result F which confirms that $f_{00} = 0$ as a precondition for there to be a solution. Plug that a_{mn} back into expansion and result is p 128 A (notice the 1/n), valid for ANY f defined on the sphere (with $f_{00}= 0$). Constant A is the n = 0 term in effect. General expansion p 127 D seems obvious to me, but he will do it as an exercise later.

Example 2: Exterior Neumann on unit sphere. Same starting expansion for u, but r^{-n-1} powers. Else solve in exactly the same manner and result is p 128 F (notice the –1/(n+1) and no constant A). There is no constant A because of the extra imposed condition u(∞) = 0 which exactly knocks out that A. Also of course we don't allow divergence at ∞. Now check our conjecture above. For large r, our solution is dominated by the n = 0 term which is $-f_{00}/(1)\ r^{-1}$, so $\partial_n u \sim +f_{00}/r^2$. The r=∞ surface integral is then going to be $-4\pi f_{00}$, the - sign because its normal is inward for this surface term. On the inner surface of course $\partial_n u = f$, so its contribution is $4\pi f_{00}$ for the n = 0 term and yes, they cancel.

The Dirichlet and Neumann Problems in 2D (128) verbatim

I agree with expression G for the general form of the 2D problem interior or exterior.

Dirichlet 2D

For **interior Dirichlet**, the log and neg power terms go away, and we set A and b_n from our BC on the unit circle. For **exterior Dirichlet**, the log and A and positive power terms go away, and we set the a_n from our BC. If you require u(∞) = 0, then you must have A = 0 which in turn means condition $f_0 = 0$. Alternatively, you could allow u(∞) = A ("bounded") and then any f_0 is allowed. On page 129 he comments that some physical problems that look like 2D Dirichlet don't fit this mold, and example being the cross section of a charged cylinder which is a sort of hybrid 2D/3D problem. I'm sure we shall see this eventually.

Neumann 2D

Now what about the 2D **exterior Neumann** problem? On page 129 he puts a circle C_r outside our given closed curve C and shows that the surface integral over C_r is fixed by f and in general won't be 0 and it is independent of r, so must be true for r = ∞, which then gives us our exterior problem. But then he shows that if you don't include the log(r) term in your solution form (ruling it out, say, because at r = ∞ it diverges), you always have the C_r integral be 0 for r=∞, which contradicts the fact that it is determined by f and is in general not 0. So you need the log term! So we **have to accept** u = const * ln(r) + A as an allowed form for r → ∞, which we state as u/ln(r) → constant (bounded). I guess the constant could

depend on θ, so that is why he says "bounded". All very good. [reminds me of Coulomb scattering with an inverse square potential, those extra log things appear.]

Exercises (130)

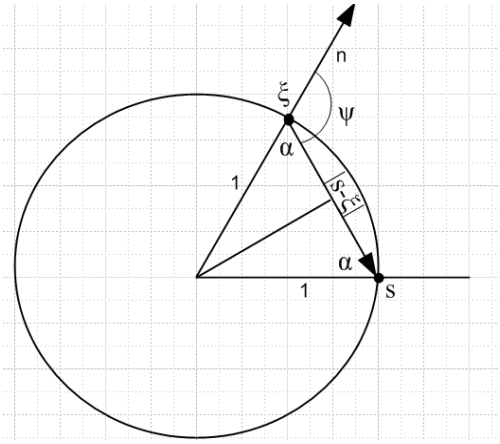
Exercise 6.21. Interior Dirichlet unit sphere using double-layer integral equation method.

This is the exercise I promised earlier we would get to. It turns out to be a long and painful problem with several parts which in total consume six notes pages (of raw notes).

Recall from p 122 (subsection A above) that our "double layer method" integral equation is this:

$$\lim_{\mathbf{x}\to\mathbf{s}} v(\mathbf{x}) \quad = -\, b(\mathbf{s})/2 + \int_\sigma dS_\xi \; b(\xi)\, k(\mathbf{s},\xi) \quad = f(\mathbf{s}) \qquad k(\mathbf{s},\xi) = \cos(s-\xi,n)/4\pi R^2$$

where **s** and ξ are two points on the sphere's surface. I write cos(s-ξ,n) = cosψ and show this angle ψ in a Visio drawing, a(s)/2



Luckily, the raw notes have a summary of the calculation which I now paste here verbatim:

Part A. I verified that cosψ = - |**s**-ξ|/2 by geometry (**s** and ξ both on unit sphere) and therefore 6.57 says this [" double layer b causes f(s) at point s on sphere "]

$$f(\mathbf{s}) = -b(\mathbf{s})/2 - 1/8\pi \int b(\xi) dS_\xi / |\mathbf{s}-\xi|$$

which I prefer to write this way,

$$f(\mathbf{r}) = -b(\mathbf{r})/2 - 1/8\pi \int b(\mathbf{r}') d\Omega' / R \qquad R = |\mathbf{r}-\mathbf{r}'|$$

Part B. By projecting onto Y_{nm} harmonics and using $1/R = \Sigma_n P_n(\cos\gamma)$ = addition theorem, this equation became (our same diagonalization idea noted earlier)

$$f_{mn} = -b_{mn}/2 - (1/8\pi)\, N_{0n}\; b_{mn} \;=\; -\, b_{mn}\,(n+1)/(2n+1)$$

which we then solved trivially to get

$$b_{mn} = -\, f_{mn}\,(2n+1)/(n+1)$$

which result we were able to verify on the web. [Stak only gives a few hints, not these details.]

Part C. We then wrote our solution u(**r**) according to 6.55 where **r** lies inside the unit sphere. This is the potential due to our double-layer b function,

$$u(\mathbf{r}) = (1/4\pi) \int d\Omega'\; b(\mathbf{r}')\;\; \{\, (rz - 1) / R^3 \;\} \qquad\qquad z = \cos\gamma \quad \gamma = \text{angle}\,(\mathbf{r},\mathbf{r}')$$

where we installed our computed value that $\cos\psi = (rz - 1)/R$. At this point, I had to derive for myself this fancy sum rule

$$(rz - 1) / R^3 \;= -\,\Sigma_{n=0}\;\; r^n\,(n+1)\,P_n(z) \qquad\qquad z = \cos\gamma$$

I installed this into the above for u(r) and then expanded

$$P_n(\cos\gamma) = N_{0n}\,\Sigma_m\,(1/N_{mn})\;Y_{nm}(\Omega)\;Y^*_{nm}(\Omega')$$

and I ended up with this final result

$$v(\mathbf{r}) = \;\; \Sigma_{nm}\;\; r^n\, f_{mn}\, Y_{nm}(\Omega)$$

which agreed with p 109 6.25 and A, and hence agrees implicitly with 6.26.

Exercise 6.22. Interior Dirichlet unit disk using double-layer integral equation method. (130)

Our double-layer integral equation in 3D was this (from above)

$$\lim_{\mathbf{x}\to\mathbf{s}} v(\mathbf{x}) \qquad = -\, b(\mathbf{s})/2 + \int_\sigma\; dS_\xi\; b(\xi)\; k(\mathbf{s},\xi) \qquad\qquad = f(\mathbf{s})$$

where $k(\mathbf{s},\xi) \;= \cos(s\text{-}\xi,n)/4\pi R^2$. In 2D, we get this same equation, but $k(\mathbf{s},\xi) \;= \cos(s\text{-}\xi,n)/2\pi R$. This follows from Exercise 6.18 (B1) above, where we superposed some 2D dipoles.

If we now specialize to the unit circle, then $k = \cos\psi/2\pi R$ with the same geometry as our 3D picture above, and we still have that the Part A result that $\cos\psi = -\;|\mathbf{s}\text{-}\xi|/2 = -R/2$. Thus, $k = -1/4\pi$ = a constant, one of the things we are supposed to show here. Our integral equation is then

$$f(\mathbf{s}) = \pm b(\mathbf{s})/2 + \int d\ell_\xi\; b(\xi)\; k(\mathbf{s},\xi) \;= \pm b(\mathbf{s})/2 - 1/4\pi \int d\ell_\xi\; b(\xi) \qquad\qquad 6.52$$

which in polar coordinates becomes

$$f(1,\varphi) = \pm b(1,\varphi)/2 - 1/4\pi \int_{-\pi}^{\pi} d\varphi'\, b(1,\varphi') \qquad \text{or}$$

$$f(\varphi) = \pm b(\varphi)/2 - (1/2)(1/2\pi) \int_{-\pi}^{\pi} d\varphi'\, b(\varphi') = \pm b(\varphi)/2 - (1/2)b_0 \qquad b_0 = (1/2\pi) \int_{-\pi}^{\pi} d\varphi'\, b(\varphi')$$

A *lot* of detail then ensues, but again I can quote my raw notes summary verbatim:

We show first that the "kernel" is a constant, as we were requested to do:

$$\text{2D: } k(s,\xi) = \cos\psi/2\pi R = -R/4\pi R = -1/4\pi$$

The "integral equation for double layer density b in 2D" is then this:

$$f(\varphi) = \pm b(\varphi)/2 - (1/2)(1/2\pi) \int_{-\pi}^{\pi} d\varphi'\, b(\varphi') = \pm b(\varphi)/2 - (1/2)b_0$$

The "solution" for b(φ) is thus (for the lower sign which is for interior)

$$b(\varphi) = f_0 - 2f(\varphi) \qquad \text{// interior} \qquad \text{and } b_0 = -f_0$$

So we have successfully "solved our integral equation for b". The next step is to install this solution into our w(r) formula which, for the 2D case, is this [using w instead of u here]

$$w(r,\varphi) = \int d\varphi'\, b(\varphi')(r\cos\gamma - 1)/R * 1/2\pi R$$

where we use the same 3D result that $\cos\psi = (r\cos\gamma - 1)/R$. We can rewrite this as

$$w(r,\varphi) = -(1/4\pi)\int d\varphi'\, b(\varphi')(-2r\cos\gamma + 2)/R^2$$

then we can make the trivial replacement $(-2r\cos\gamma + 2) = 1 - r^2 + R^2$ to get

$$w(r,\varphi) = -(1/4\pi)\int d\varphi'\, b(\varphi')(1 - r^2 + R^2)/R^2$$

$$= -(1/4\pi)(1 - r^2) \int d\varphi'\, b(\varphi')/R^2 - (1/4\pi) \int d\varphi'\, b(\varphi')$$

$$= -(1/4\pi)(1 - r^2) \int d\varphi'\, b(\varphi')/R^2 - (1/2)\, b_0$$

We then insert $b(\varphi) = f_0 - 2f(\varphi)$ and $b_0 = -f_0$ to get,

$$= (1/2\pi)(1 - r^2) \int d\varphi'\, f(\varphi)1/R^2 - (1/4\pi)(1 - r^2)f_0 \int d\varphi'\, 1/R^2 + (1/2)f_0$$

The first term is our desired result 6.11, so the last two terms must cancel. We must then have

$$(1/2\pi)\,(\,1 - r^2)\int d\varphi'\; 1/R^2 \;= 1 \qquad \text{or} \qquad \int_{-\pi}^{\pi} d\varphi'\; 1/R^2 \;= 2\pi / (1-r^2)$$

and this last result is what you get from GR p 366 3.613.2 with n = 0. QED.

What about the exterior solution? Retracing the above steps we have

$$f(\varphi) = +b(\varphi)/2 - (1/2)(1/2\pi) \int_{-\pi}^{\pi} d\varphi'\; b(\varphi') \;= +b(\varphi)/2 - (1/2)b_0$$

$$b(\varphi) = 2f(\varphi) - b_0 \qquad \text{and} \qquad f_0 = 0 \qquad \text{// recall Stak saying this is required!}$$

and b_0 is an undetermined constant which I will just set to 0. Then we have $b(\varphi) = 2f(\varphi)$. Now going to the next step above, we had

$$w(r,\varphi) \;= -(1/4\pi)\,(\,1 - r^2) \int d\varphi'\; b(\varphi')/R^2 - (1/2)\;\; b_0 \;= -(1/4\pi)\,(\,1 - r^2) \int d\varphi'\; b(\varphi')/R^2$$

and if we install $b(\varphi) = 2f(\varphi)$, we get

$$w(r,\varphi) \;= -(1/2\pi)\,(\,1 - r^2) \int d\varphi'\; f(\varphi')/R^2$$

which is what I know is the right answer: *same as interior but with a minus sign!* If we take the large r limit of this thing, $R \rightarrow r$ so we get

$$w(r,\varphi) \;\rightarrow\; -\,(\,1 - r^2)/r^2\;(1/2\pi) \int d\varphi'\; f(\varphi') \;= f_0$$

which agrees with p 129 A which just says the solution is "bounded" at $r = \infty$. We find that for the unit circle at least, the value at infinity is the same as at the center, which is the MV of the interior solution.

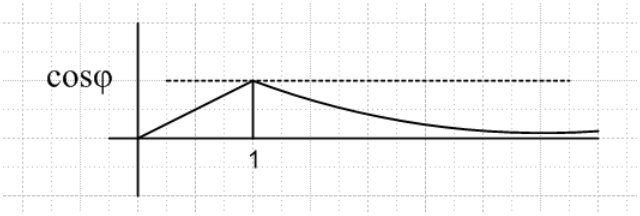
Example (PhL) of 2D Dirichlet unit circle: Use $f(\varphi) = \cos\varphi$. Then our interior solution is this

$$w(r,\varphi) \;= +(1/2\pi)\,(\,1 - r^2) \int d\varphi'\; \cos\varphi' \,/[\,1 + r^2 - 2r\cos(\varphi'-\varphi)]$$

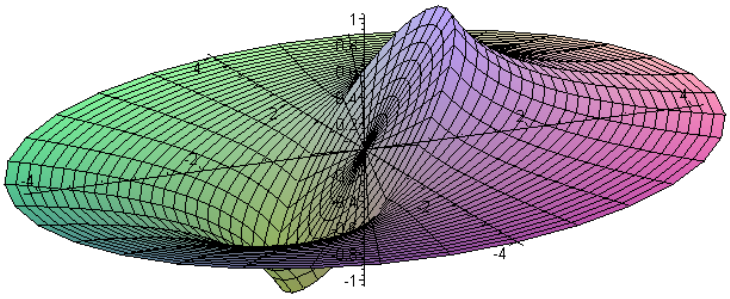
I do the integrations involved here (see raw notes) and find these results:

$w(r,\varphi) = r\cos\varphi$ // interior
$w(r,\varphi) = r^{-1}\cos\varphi$ // exterior

Here then is $w(r,\varphi)$ as a function of r for a given value of φ,



and here is a full plot of the solution w(r,φ):



Recall that $\hat{\mathbf{n}} = \nabla_{3d}(g) / |\nabla_{3d}(g)|$ is the normal to surface g(x,y,z) = 0. Our interior surface is z = r cosφ = x, so we have g(x,y,z) = z - x. So we compute

$$\mathbf{n} = \nabla (z - x) = \hat{\mathbf{z}} - \hat{\mathbf{x}} \qquad n^2 = 2 \qquad \text{so} \quad \hat{\mathbf{n}} = (\hat{\mathbf{z}} - \hat{\mathbf{x}})/\sqrt{2}$$

which is in a constant direction, which explains why the interior solution lies on a plane as shown.

Exercise 6.23 Neumann on 2D unit circle (130)

Just above I did the complete 2D Dirichlet on the unit circle, interior and exterior, for any f(θ). Now Stak asks us to repeat this for Neumann instead of Dirichlet, so $\partial_r u = f(\varphi)$ at r = 1. Fine. But this time we get to use the simple expansion method (not the integral equation method), and here we go:

A. Interior. General form of solution is $u(r,\varphi) = \Sigma_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\varphi}$ including the n = 0 term. We compute $\partial_r u(r,\varphi)|^{r=1} = \Sigma_{n=-\infty}^{\infty} a_n |n| e^{in\varphi}$ and expand $f(\varphi) = \Sigma_{n=-\infty}^{\infty} f_n e^{in\varphi}$ so that $f_n = a_n|n|$, which requires $f_0 = 0$ as he points out in the text. Our interior solution is then

$$u(r,\varphi) = \Sigma_{n\neq 0} \; (f_n/|n|)\, r^{|n|}\, e^{in\varphi}$$

This says u(0,φ) = 0. The MVT says this must be the average around the circle, but that average is f_0 which we just showed was 0.

A. Exterior. General form of solution is $u(r,\varphi) = \Sigma_{n\neq 0} a_n r^{-|n|-1} e^{in\varphi} + A + B \ln(r)$ where we must include the log term! We compute $\partial_r u(r,\varphi)|^{r=1} = \Sigma_{n\neq 0} a_n (-|n|-1) e^{in\varphi} + B$, expand f(φ) as before, so that now $(-|n|-1)\, a_n = f_n$ for $n \neq 0$ and $B = f_0$. Our exterior solution is then

$$u(r,\varphi) = -\Sigma_{n\neq 0}\, f_n/(|n|+1)\, r^{-|n|-1}\, e^{in\varphi} + A + f_0 \ln(r)$$

Review of Chapter 6 up to this point.

In the first 3 sections of this chapter, we solved the Laplace equation with no sources at all, but with the solution prescribed on certain bounding surfaces. This subject is known as "the Dirichlet problem" (without sources). For simple geometries, we could exactly solve the problem. We started with the 2D unit disk, did separation of polar variables, found general forms of solutions, then determined the constants in those forms from the prescribed boundary values. We learned the properties of harmonic functions. In later sections, we learned that these solutions and boundary values could be equivalently generated by certain charge or dipole distributions, but not in these 3 sections.

Then in Sections 6.4 and 6.5 we started looking at sources, but only sources distributed on a closed boundary σ, not δ function sources within the volume surrounded by the boundary or on it. These distributed sources are called layers; the monopole and dipole layers are called simple and double layers. By simple superposition of known point-source expressions (fundies), we obtained Laplace equation solutions as simple integrals of the form $u(x) = \int_\sigma dS_\xi\, c(\xi)\, K(x,\xi)$ where c is the layer density and K some kernel function. For the simple layer, the kernel $K(x,\xi)$ is really $E(x|\xi)$ such as $1/4\pi R$ which is called a "fundamental solution". In the double-later case, $K(x,\xi)$ is really $D(x|\xi)$ which is the fundy solution for a point dipole source. These integral solution expressions apply in the absence of boundary value impositions on *other* surfaces σ'. For example, there were no nearby conductors σ' which might develop charge distributions and thus affect the problem solution. We learned in detail how to take the limit of our integral expressions as a point x in the interior approaches a point s on the source-bearing surface σ. In some cases, such as the double layer, you cannot simply take this limit under the integral sign due to the singular nature of $K(x,\xi)$, and in fact a certain "extra term" appears when you properly take this limit. Once we knew how to take the limit, we could then start thinking of u(s) as f(s), a boundary value on σ itself, and we then obtained integral equations relating boundary values f(s) to the surface density c(ξ). In simple geometries, we could solve this integral equation and obtain the same results we got earlier using simpler methods, but the integral equation method applies to all geometries. We understood the notion of "equivalent problems" that are not physically the same: (1) some f is imposed on σ by fiat; (2) a simple or a double layer can be found which actually causes this same f to occur on σ. We obtained a different integral equation for simple versus double layer, though both were driven by f(s) as the inhomogeneous term. The double layer equation has the "extra term" which is that eigenvalue type term appearing in a Fred inhomo 2 integral equation as shown page 195 of Stak I. The simple layer integral equation did not have this extra term and so was a Fred inhomo 1 equation.

At this point, for the first time, we dealt with "the Neumann problem" (p 126) which is identical to "the Dirichlet problem" but the boundary value f(s) is for $\partial u/\partial n$ instead of u on the surface σ. We found, however, that one must have $f_{00} = 0$ for the 3D case, otherwise the integral equation has no solution. For Neumann, you take the integral solution expression $u(x) = \int_\sigma dS_\xi\, c(\xi)\, K(x,\xi)$ and you have to differentiate it once ∂_n before taking the limit $x \to s$ for the boundary, and that is why we did a lot of work on not just u and v (simple and double layer integrals), but also on $\partial_n u$ and $\partial_n v$. That is to say, these latter two are needed for Neumann problems. In both these cases, "extra terms" appear.

Our initial work was done in 3D and we considered both interior and exterior problems (to the surface σ). We imposed a requirement that u be bounded at $r = \infty$ and of course finite at r=0. We found on page

125 that, in the integral expression for u(x) in which c(ξ) is a single layer density a(ξ), a(ξ) = I(ξ) which is the (nameless) difference of $\partial_n u$ across the boundary for the internal and external solutions. We used this "method" to find the solution of the unit sphere and we computed I(ξ) as part of the solution. We were happy to see a single integral expression provide both the interior and exterior solution. We then looked at 2D Dirichlet and Neumann problems and a few new features appeared. One was that we needed $f_0 = 0$ for the Neumann, in analogy to the $f_{00} = 0$ in 3D. But more interesting is that we must maintain the ln(r) solution in 2D which means u is no longer bounded as r→∞. The correct fact is that u/ln(r) should be bounded.

NOW for the first time in this chapter, starting in Section 6.6, we are going to consider the Laplace solution in the presence of both a boundary σ and a δ-function point source inside the boundary. As we did with the string in 1D, we require that the solution vanish on the boundary σ. Such a solution is called the Green's Function and is denoted g(x|ξ) as in Volume I. So that is what we are now about to do. But first some comments. Remember that a "fundamental solution" is any solution of LE=δ(x-ξ) in the absence of any boundary condition, while a "Green's Function" must also vanish on the boundary, and we usually write LG=δ(x-ξ) or Lg=δ(x-ξ) in this case.

Comment on Integral Equations. In the 1D world of ODE's, in volume I of Stak, we often dealt at the same time with an ODE and with its corresponding integral equation (IE) which always involved a certain function k(x,ξ) called "the kernel" and implied a linear operator called K. Usually the ODE was easier to solve for simple cases, but the IE had nicer "properties" since the integral operator was often bounded or Hilbert-Schmidt. Most of our theorems were proved based on the IE since things are more "controlled" in that framework. For example, we found many theorems relating to the location of the eigenvalues μ or λ of an integral operator. For both the ODE and the IE, there is an eigenvalue problem and the eigenvalues are determined by the imposed boundary values of the solution. Also, the IE world allowed us to prove that the eigenfunctions of the eigenvalue problem formed a complete set in certain common situations, something that was hard to show in the corresponding ODE world.

In volume II we are in the world of multiple variables and PDE's. We might have wondered right at the start, had we been curious, if there were an integral equation associated with each PDE. In the above sections, we discovered that there are in fact several "associated integral equations" related to a given PDE. Of course we have specialized in Chapter 6 to only a single PDE, the Laplace equation. In the associated integral equations, the variable in the equations is not the solution u (as it is in the ODE world), but is an ancillary function which we interpret as either a single or double layer density. So at least we have something in the way of an integral equation for a PDE. For the ODE, the associated IE was integrated over the interval (a,b) which is a region R. For the PDE, the associated IE's are integrals not over R, but over the boundary σ or R.

Perhaps we will learn more about this subject in upcoming sections. Maybe there is an integral equation directly for the solution u whose kernel then would be the inverse of the PDE operator, that is to say, $K = L^{-1}$. This subject did not occur in Chapter 5, our only other Stakgold chapter involving multiple variables (volume I was all single variable). We certainly know that scattering theory uses integral equations which are associated with a certain potential V(r). The PDE involved here is the classical wave equation or the Schrodinger equation of some sort. In either case, we use Hamiltonian mechanics as our calculational tool and that is how V(r) gets into the picture: K + V(r) = H = E. Perhaps this is only a PDE in QM, not in classical mechanics. Just wandering here. [In fact, integral equations are going to come up right away in the next section: the variable will be not u, but g, the special Green's Function solution.]

One final note: When I was reading this chapter, I thought that every k(x,y) integral equation had an associated ODE, but I now realize that is only true for a very narrow range of k(x,y) kernels, namely, those kernels which are Green's Functions of an ODE problem. I wrote a little doc on this and talked a bit to Richard Price who did not seem too interested. For 1D the conclusion seems clear, less so in 3D.

6.6 Green's Function for L = $-\nabla^2$ (130)

First, here is a definition: The **"Green's Function" g(x|ξ)** on an *open* and *bounded* region $\mathcal{R}$ is the solution not of Laplace, but of Laplace with a δ(x-ξ) source term, AND with g = 0 on a boundary σ fully enclosing the region $\mathcal{R}$. This is like the string thing, where the two ends were tied down to y = 0, but now we are in $\mathcal{R}_n$. So we have $-\nabla^2 g(x|\xi) = \delta(x-\xi)$.

In contrast, the **"fundamental solution" E(x|ξ)** is a certain r-symmetric solution of Laplace with a δ source where we have no boundary values (thus justifying the symmetric nature of the solution). We know that we can add any Laplace solution to E and get another fundy solution, as we add a homogeneous solution to any particular solution of an ODE. Back on p 48, we found the fundy solutions for a variety of differential operators L. We usually go for the simplest form, setting extra constants to 0, or requiring that the fundy → 0 at ∞ (if possible), and we require an r-symmetric form. In any event, as listed in that Section 5.8 entitled Fundamental Solutions (my "fundies"), we have in mind a very specific function E for each operator L, and of course here we only care about L = Laplace in which case E = 1/4πR. In general, the r-symmetry implies that E is symmetric in its two arguments, and in our special case R = |**x**-ξ| which shows this symmetry explicitly.

Now comes the basic idea: Define function v as v ≡ g - E. Then v is a solution of Laplace (ie, no source) which has the boundary value f = -E. [v is harmonic in $\mathcal{R}$] We have spent this *entire chapter* learning how to solve just such a problem, and we know among other things that the solution exists and is unique. For example, 6.11 gives the solution v for the unit disk given some f(φ).

We defer until Section 6.7 the actual calculation of specific Green's Functions g(x|ξ).

Stak then reels off some theorems which are "general properties of Green's Functions"

Theorem 1: g exists and is unique.

Theorem 2: The function g(x|ξ) is symmetric.

Theorem 3: g is positive throughout R.

Theorem 4: For n ≥ 3, g < E throughout R

Theorem 5: The integral operator G implied by g is completely continuous.

Theorem 6: For n = 2 and n=3, G is a H-S operator.

I did not study the complete continuity proof too much but accept it as true.

Comment: We know that, given any function f(x,ξ), we can "think of it" as the kernel of an integral operator F. Thus, he associates g(x|ξ) with an integral operator G and shows above that it is in fact Hilbert Schmidt in 2 and 3 dimensions (recall $\mathcal{R}$ must be bounded). However, we have not yet discovered any integral equations in which G appears. No reason we cannot talk about G, nevertheless.

Solution of the Dirichlet Problem. (135)

As noted above, we defer until Section 6.7 below the task of computing actual Green's Functions. Here we assume that we *know* what g(x|ξ) is for a particular problem, and we show how knowledge of g then lets us write down a completely solution of the full Dirichlet Problem.

So, on the one hand, we have our Green's PDE for g with g=0 on σ. On the other hand, we now write the Poisson equation for u with a volume source distribution q(x) *and* with the imposition that u = f on σ. We "process" these two PDEs and we end up with this result:

$$u(x) = \int_R d\xi\, g(x|\xi)\, q(\xi) - \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n}\, g(x|\xi) \qquad = T1 + T2 \text{ (two terms)} \qquad (*)$$

There is much to be said about this result.

(0) (Note added 10.24.09) If σ is a grounded metal surface, f(ξ) = 0. If q(ξ) = point charge at ξ, then u(x) = g(x|ξ). Put a point charge at the enclosed origin, then u(x) = g(x|0). For the exterior problem (capacitor against great sphere), I conjecture that we have a point charge at r = ∞ and somehow solution must be u(x) = g(x|∞). Stak never mentions this idea, maybe it is wrong. It would be a way to find the surface charge on the exterior of a metal object that forms a capacitor with r = ∞ -- an alternative to the integral equations methods discussed later in this chapter.

(1) first, assuming we can compute g(x|ξ) for our problem (ie, given σ of whatever shape), (*) gives a *complete solution* to our problem for arbitrary volume source density q inside σ, and for arbitrary boundary value f on σ. You give me σ and q and f, and I will give you the solution u. The solution is unique by construction. This is the "general Dirichlet problem" (ie, with sources).

(2) If f = 0 on σ, we have only term T1 which we can think of as a *particular* solution of Lu = q. Since g = 0 on σ, it is obvious that the superposition integral T1 will also be 0 on σ, ie, that we will have f = 0. Ie, the first term T1 is just a superposition of little Green's Functions one for each point source q(x), and each of which vanishes on the boundary. If we apply $-\nabla^2$ to T1, we will get q(x).

(3) If q = 0 inside σ, then we have only term T2 which we can think of as a solution of the homogeneous equation Lu=0. We know that we can always add homo solutions to a particular solution to get a new solution of Lu=q. But the homo solution shown here is the right one to add if you want to have f on σ. To show it is a homo solution, we apply $-\nabla^2$ to T2, we can let it hit g(x|ξ) in there to make δ(x-ξ), but since x is inside and ξ is on the boundary, we get no hit so that $-\nabla^2 T2 = 0$, QED.

Comment: Looking at 6.11, we conclude that the Poisson Kernel = $-\partial_{\xi n}\, g(x|\xi)$ where g is the Green's Function for the unit disk (which we have not yet computed, hopefully later!) Similarly for 6.26 in 3D. In 6.11, x = (r,φ) and ξ = (1,ψ).

(4) We know that the quantity $\Sigma_{\mathbf{x}}(\xi) = \partial^{(\xi)}{}_{\mathbf{n}}\, g(x|\xi) \equiv - I(x|\xi)$ happens to appear in a *different* Laplace problem. That problem is the problem where you put a point charge at x on the interior, and have u = 0 on the boundary. That is, this different problem is the "Green's Function problem for a point charge". Physically you could enforce u = 0 by pouring metal on the outside of the boundary. In this different problem $\Sigma_{\mathbf{x}}(\xi) = \partial^{(\xi)}{}_{\mathbf{n}}\, g(x|\xi) \equiv - I(x|\xi)$ is the surface charge density one finds on the boundary, "induced" by the point charge in the interior. [See raw notes "Redo the surface charge density argument."] Now, we can if we like rewrite our above result this way:

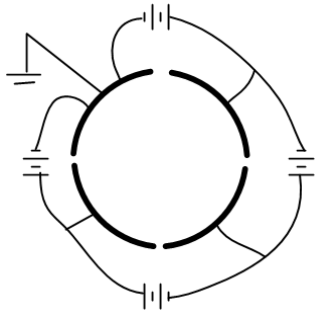
$$u(x) = \int_{\mathbf{R}} d\xi\, g(x|\xi)\, q(\xi) + \int_{\sigma} dS_{\xi}\, f(\xi)\, I(x|\xi)$$

Now, in the unit disk problem with 6.11 as the answer, q = 0 and the complete answer is this

$$u(x) = \int_{\sigma} dS_{\xi}\, f(\xi)\, I(x|\xi) \qquad I(x|\xi) = \text{Poisson Kernel}$$

This is a no-sources Dirichlet problem. It does not apply in electrostatics to a problem where σ is surrounded by metal, because in such a problem you would have $f(\xi) \equiv 0$. If you take away the metal, there can be no induced charge on σ.

How then would you apply f(ξ) on a surface σ? In electrostatics, the only way I can think of is this: have a million little isolated metal plates lining σ, hooked up to a network of batteries. Here is an example showing 4 such plates:



This picture implies some f(ψ) and if you throw that into our formula, you get a u(r,θ) inside if this is a picture of a unit disk, say. You can compute $E_{\mathbf{r}}(1,\theta) = -\partial_{\mathbf{r}} u(r,\theta)|^{\mathbf{r=1}}$ and then a pillbox argument says that this must be (proportional to) a surface density of charge on the inner surface of these plates!

In our example with $f(\varphi) = \cos\varphi$, we found $u(r,\varphi) = r\cos\varphi$ on the inside, so $\partial_{\mathbf{r}} u = \cos\varphi$, so there must be a surface charge $\Sigma(\varphi) \sim \cos\varphi$.

The conclusion is this: if you create f(ξ) in the manner just described, there will exist some surface charge density Σ(ξ) on the inner surface, and this will NOT be I(x|ξ), which is the surface charge in another problem that also involves a sphere. To be specific with the unit disk, we have

$$u(r,\varphi) = \int d\psi\, f(\psi)\, k(r,\varphi-\psi) \qquad \text{where k is the Poisson kernel} = I(\, r,\varphi;\, 1,\psi)$$

The inside charge density in the million metal plates situation will be this:

$\Sigma(\varphi) \sim \int d\psi\, f(\psi)\, \partial_{\mathbf{r}}\, I(\, r,\varphi;\, 1,\psi)$ // assuming order interchange is allowed

so only in this sense is the thing I(x|ξ), which is a charge density in a certain Green's problem, connected with our charge density in the plates problem.

Comment: Therefore, the Poisson Kernel in 6.11 $= -\partial_{\xi \mathbf{n}}\, g(x|\xi) = pk(x=r,\varphi;\, \xi=1,\psi) = -\Sigma_{\mathbf{x}}(\xi)$. We can then interpret the negated Poisson Kernel as the inside-metal surface charge distribution you would get as a function of ψ around the circle, if you placed a unit point charge inside the disk at location x = (r,φ).

(4A). I don't have a proof yet, but I would like to imagine that the following is also true, where we put all our charges "on the same footing"

$$u(x) = \int_{\mathbf{R}} d\xi\, E(x|\xi)\, q(\xi) + \int_{\sigma} dS_{\xi}\, E(x|\xi)\, \Sigma_{\mathbf{x}}(\xi) = \int_{\mathbf{all}} \rho(\xi)\, E(x|\xi)$$

I thought we used this idea in Jackson to compute things, but I am unsure of it right now.

(5) If in case (3) above (where q=0) we also set f = C, we know that u = C everywhere, so we get the result from (**) that $1 = -\int_{\sigma} dS_{\xi}\, \Sigma_{\mathbf{x}}(\xi)$ which says the total induced charge is -1, which balances our inside point charge of +1. Stak has not really brought up the notion of free charge and bound charge, so the notion of Gauss's law where $\mathbf{E} = -\nabla g$ is not quite clear here, but we know it all works. If we simply define ρ such that $-\nabla^2 g = \rho$ in all space for our problem, then certainly $\int dV\, \nabla \bullet \nabla g = -Q_{\mathbf{enclosed}}$ for a large volume enclosing the boundary σ, the surface of that volume being entirely inside the metal. That integral would then be 0 by the divergence theorem, so we get $Q_{\mathbf{enclosed}} = 0$ which is then consistent with our just found result that 1 + (-1) = 0.

Comment on Integral Equations: In the case f = 0 we have found that $u(x) = \int_{\mathbf{R}} d\xi\, g(x|\xi)\, q(\xi)$

or u = Gq. We can regard *this* as the "integral equation" that we associate with the PDE $-\nabla^2 u = q$. Thus. the kernel of the integral equation is the Green's Function of $-\nabla^2 G = \delta$. And yes, $G = L^{-1}$. Notice how the boundary conditions of the PDE must be u = 0 on σ for this to work.

If the PDE boundary conditions are u = f, then we know our "integral equation" is this:

$$u(x) = \int_{\mathbf{R}} d\xi\, g(x|\xi)\, q(\xi) - \int_{\sigma} dS_{\xi}\, f(\xi)\, \partial_{\xi \mathbf{n}}\, g(x|\xi) = G\, q + I\, f$$

but this is not a very NICE integral equation. Maybe we could do parts somehow in the second term to get this into standard integral equation form. Stak has not yet done anything like this.

Eigenvalue Problem for $L = -\nabla^2$ (136) *

Preliminary ignorant comment: What are the eigenvalues of L? Knowing pretty much nothing, I might think vaguely that $e^{\pm ik.x}$ is an eigenfunction where k^2 is the eigenvalue and is thus positive and real. We shall see. We are perhaps asking what is the spectrum of L. [Here I am forgetting that the BC's are part of the EV problem, and in fact the critical part.]

The EV problem is to solve $-\nabla^2 u = \lambda u$ where u=0 on boundary σ, similar to what we did for the string in 1D. Just from the PDE and various Green theorems, we can quickly show that

- eigenvalues λ are all real and positive (hence we can talk about $\mu = 1/\lambda$ and μ are all positive)
- eigenfunctions of different eigenvalues are orthogonal

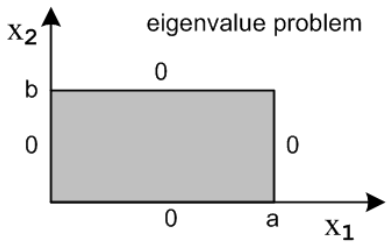
- we can cast the EV PDE as an integral equation $u = \lambda Gu$ with λ as EV (2nd kind homo), integral over R (6.86 - 1). We know that when the kernel is symmetric and CC, the eigenfunctions form a complete set, hence that is true here.

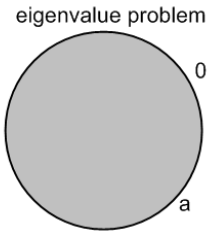
For the first two items, the full **complex inner product on L_2 functions** is "reintroduced". Inner products have been completely "idle" in this chapter so far. Why is that? Let's just look back again. In 6.2 we solve Laplace on the Dirichlet disk and get 6.11 with f(φ). In 6.3 it is MVT and min/max theorem and then a long list of sample calculations like the wedge. We do use functions there like sin(..x) which are orthogonal and such, but not so important. In 6.4 we are on to surface layers and the tricky issue of approaching a surface, and dipoles are introduced. In 6.5 we do the integral equations for the surface densities with more sample calculations. In 6.6 we opened with some g(x|ξ) work, and only now are we going to require the inner product! An EV problem Ax = λx involves operators A in a Hilbert Space. This begins in E^n where we have a notion of **x•y** for vectors and operators are square matrices, and then for integral equations involving things like $\int_R d\xi\ g(x|\xi)\ q(\xi)$ the matrix is g(x|ξ), called G, of infinite dimensions, and so on, so this takes us to our ∞ dim L_2 Hilbert Space with its usual complex inner product which in turn provides a norm.

As for the third item above, the u(σ) = 0 part of the EV problem lets us set f = 0 in our main result above so we have $u(x) = \int_R d\xi\ g(x|\xi)\ q(\xi)$ but of course $q = \lambda u$ for the EV problem, so we have at once the **integral EV equation** $u(x) = \lambda \int_R d\xi\ g(x|\xi)\ u(\xi)$ which we write as $u = \lambda Gu$ or $Gu = \mu u$, and we have now arrived at 6.86 on page 138.. Yes, g(x|ξ) is a symmetric kernel, and is CC, so EF's are a complete set!

Comment on integral equations. Lest the point be missed, we have a nice integral equation $u = \lambda Gu$ which we can associate with our PDE $Lu=\lambda u$. For both, we must have u=0 on σ, part of the problem definition. Earlier I was wondering if there is an IE associated with a PDE. In the case the PDE includes u(σ)=0, here is such an IE. In the case that Lu=q, still with the requirement that u=0 on σ, we show below that we have another associated "integral equation" which is simply u = Gq.

Two Examples: Rectangle and Disk (139)





I would call these the "canonical examples" of Laplace EV problems, the rectangle and the disk. In the first case we get $\sin(m\pi x_1/a)\sin(n\pi x_2/b)$ and $\lambda_{m,n} = (m\pi/a)^2 + (n\pi/b)^2$ which is the TE waveguide type solution. Only for these magic values of λ do we have solutions other than u = 0 on the entire rectangle. In the second case we get $e^{in\varphi} J_n(\beta^{(n)}_m r)$ where n = any integer and zeros $\beta^{(n)}_m$ are the such that $J_n(\beta^{(n)}_m) = 0$, enumerated by m, and we have that $\lambda_{m,n} = [\beta^{(n)}_m]^2$. The raw notes for these two examples are very good, go read them for details on the mechanics of these calculated solutions.

Green's Function for Unbounded Regions (142)

We suddenly go back to page 135 and I quote my notes above:

"OK, here we go. On the one hand, we have our Green's PDE for g with g=0 on σ. On the other hand, we now write the Poisson equation for u with a volume source distribution q(x) *and* with the imposition that u = f on σ. We "process" these two PDEs and we end up with this result:

$$u(x) = \int_R d\xi\, g(x|\xi)\, q(\xi) - \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n}\, g(x|\xi) \quad = T1 + T2 \text{ (two terms)} \qquad (*) \quad "$$

This was the solution to the "interior" Dirichlet problem since sources q are inside R and the solution is then inside $R \equiv R_i$. We can ask about the solution to the corresponding "exterior" Dirichlet problem on region R_e outside σ. In specifying this problem, we say $-\nabla^2 u = q$ with u = f on σ (as before), but we add a new condition that u = 0 on the great sphere which is part of the boundary of R_e. This makes the great sphere part of R_e's boundary make no contribution to the dS_ξ integration, and we only have the σ part. But if we want n to remain pointing outward, we get a sign change and the result is then

$$u(x) = \int_{Re} d\xi\, g(x|\xi)\, q(\xi) + \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n}\, g(x|\xi) \qquad \text{// which is 6.92 (exterior)}$$

(this part verbatim) So the point here is that for such an R_e problem, we have a complete closed form solution to our Poisson equation in terms of q(x) charge and f(σ) prescribed on σ (and being 0 at ∞). So this is no harder than the interior problem.

What about "the EV problem" on R_e ? In Chapter 4 we saw how, when we took (a,b) to (a,∞), the discrete eigenvalues λ_i moved closer together and formed a continuum (branch cut in $g(x|\xi; \lambda)$) so that the possibility of a continuous spectrum appeared, related to the operator B^{-1} being unbounded (see meta notes there). So that is what happens here since we have r = ∞. He says that operator G is no longer completely continuous (compact, since acts on an infinite domain eg).

He points out that you can have R_e be external to a *set* of R_i regions and things are the same. But you can also have an unbounded region that cannot be thought of as R_e (external to a bounded region). An example would be the infinite wedge. { spend 2.25 hours in AM session today 8.5.09, * to here}

Comment: Today is 9.20.09, so my first reading of the current section was 6.5 weeks ago. This is the time gap between the original raw notes and these meta notes, it is too large a gap! In the gap I had a Cape Cod trip of 11 days and also studied up on the first 3 chapters of Ahlfors to prep for the conformal mapping stuff, see below. Ahlfors was much harder than I thought it would be, I was expecting just a quick diversion.

Comment on Integral Equations and PDE's.

By now we have seen LOTS of integral equations related to our PDE. Here is a little list of the types:

(1) The "associated" integral equations for surface densities a and b from which we can obtain a solution:

a. $\lim_{\mathbf{x}\to\mathbf{s}} v(\mathbf{x}) = -b(\mathbf{s})/2 + \int_\sigma dS_\xi\, b(\xi)\, k(\mathbf{s},\xi) = f(\mathbf{s})$ inhomo Fred 2 dipole layer

$v(\mathbf{x}) = \int_\sigma d\xi\, b(\xi)\cos(\mathbf{R},\mathbf{n})/4\pi R^2$

b. $\lim_{\mathbf{x}\to\mathbf{s}} u(\mathbf{x}) = \int_\sigma dS_\xi\, a(\xi) 1/4\pi R = f(\mathbf{s})$ inhomo Fred 1 monopole layer

$u(\mathbf{x}) = \int_\sigma d\xi\, a(\xi) 1/4\pi R$

In these equations, we have some kind of kernel called k related to the fundy solution. The x→s limits take us to boundary values which are presumably known, so in these integral equations we would like to solve for b(ξ) or a(ξ). Then we insert that result into the non-surface-limit equation to get u(**x**) or v(**x**). The region of integration in these equations is not the volume R, it is the boundary σ.

(2) The "integral equation" in connection with the Green's Function Problem for PDE using "L"

$-\nabla^2 g(x|\xi) = \delta(x-\xi)$ $\qquad g(\sigma) = 0$

$-\nabla^2 u = q(x) \quad \Rightarrow \quad u(x) = \int_R d\xi\, g(x|\xi)\, q(\xi) - \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n}\, g(x|\xi)$

particular homo sol for BC's

$Lu = q$ $\qquad u = Gq$

Here u = Gq is not really an integral equation in the sense that we are going to solve it for some unknown function, but it is an equation that contains an integral. It is the solution the PDE, and we see that $G = L^{-1}$.

(3) The integral equation in connection with the eigenvalue problem

$Lu = \lambda u$ $\qquad$ $u = \lambda \int_R d\xi\, g(x|\xi)\, u(\xi) = \lambda Gu$ $\qquad$ homo Fred 2

So, if someone asks you whether Stakgold discusses the integral equation version of a PDE, the answer is yes, but not in the full sense of Volume I where he talked about $L_\lambda = L - \lambda I$ and discussed spectra.

(4) The integral equation in connection with finding the Green's function, a subject treated below. It turns out this is just an application of (1)b above where u = v, which is the function of g = E + v, and where then f(s) = v(s) = -E(s) since g(s) = 0. I just copy from (1)b above, changing the name of the integration variable to t,

$\lim_{x\to s} v(x) = \int_\sigma dS_t\, a(t) 1/4\pi R = f(s) = -E(s)$ $\qquad$ inhomo Fred 1 $\qquad$ monopole layer

$v(x) = \int_\sigma dS_t\, a(t) 1/4\pi R \qquad\qquad 1/4\pi R = E(x,t)$

The surface density a(t) in this case goes by the name -I(t), so rewrite the above again as

$\lim_{x\to s} v(x) = -\int_\sigma dS_t\, I(t)\, E(x,t) = -E(s)$ $\qquad$ inhomo Fred 1 $\qquad$ monopole layer

$v(x) = -\int_\sigma dS_t\, I(t)\, E(x,t) \qquad\qquad 1/4\pi R = E(x,t)$

But in the Green's Function world, everything depends parametrically on the location ξ at which you place your point charge, so we can write the above once again in this way

$\lim_{x\to s} v_\xi(\mathbf{x}) = -\int_\sigma dS_t\, I_\xi(\mathbf{t})\, E(x,t) = -E_\xi(s)$ $\qquad$ inhomo Fred 1 $\qquad$ monopole layer

$v_\xi(\mathbf{x}) = -\int_\sigma dS_t\, I_\xi(\mathbf{t})\, E(x,t) \qquad\qquad 1/4\pi R = E(x,t)$

And then one more time, where we put this parameter in a different location and use bar notation

$-\lim_{x\to s} v(x|\xi) = \int_\sigma dS_t\, I(\xi|t)\, E(x|t) = E(s|\xi)$ $\qquad$ inhomo Fred 1 $\qquad$ monopole layer

$v(x|\xi) = -\int_\sigma dS_t\, E(x|t)\, I(\xi|t) \qquad\qquad 1/4\pi R = E(x|t)$

So the integral equation plan is to solve the first integral equation for the layer density $I(\xi|t)$ and then insert that into the second equation to get v, and then the last step is $g(x|\xi) = E(x|\xi) + v(x|\xi)$. The ordering of the two arguments of function $I(\xi|t)$ seems backwards, but I guess you do that any way you want. It turns out that $I(\xi|t) \equiv -\partial_{nt} g(\xi|t)$ as shown below, but that fact is not relevant right here.

Exercises (143) -- eleven problems here

These are meat and potatoes problems one would not want to skip in a meta review. Most are Dirichlet eigenvalue problems for particular situations: for given σ, solve for EF's and EV's.

When I started this reading, I was thinking vaguely that $e^{\pm ik.x}$ were the eigenfunctions of the Laplace equation since $-\nabla^2 u = k^2 u$ with $\lambda = k^2$ and so $\lambda > 0$. I now see that these might be sort of pseudo-eigenfunctions for σ = infinite surface, perhaps $e^{\pm ikr}/r$ in sphericals. In the examples below with finite σ, we get discrete eigenvalues with names like $\lambda_{n,k,m}$. But yes, $\sigma = \infty$ is the unbounded situation, just as discussed above, and we do expect the poles of g (not really discussed here as they were in Volume I in 1D) to coalesce into the continuous positive real axis.

As an aside, I don't think Stak has generalized his spectral theory of operators work to multiple variable differential operators, though he did generalize the theory of distributions and Fourier analysis in Chapter 5. Just as a reminder, here are some 1D results that I imagine are generalizable:

4.1 Introduction to the Regular Boundary Value Problem (259)

Stak opens this chapter with a long example of a "regular BV problem",

$L_\lambda = (L - \lambda)$ where $L = -D^2$, interval is $(0,\ell)$, BCs are $u(0)=0$ and $u(\ell)=0$

He finds the EV's λ_k and EF's φ_k of $L_\lambda u=0$, and the φ_k form a complete set. He then solves the inhomo $L_\lambda u=f$ to get u(x) below, and then some Big Gun results are developed: [s(x) = 1 in this example]

$u(x) = \Sigma_k u_k \varphi_k = \Sigma_k f_k \varphi_k/(\lambda_k - \lambda)$ (00) $= \int d\xi\, g(x|\xi;\lambda) f(\xi)$

$-(1/2\pi i) \oint d\lambda\, u(x;\lambda) = f$ (0) $f_k \equiv \int d\xi\, \varphi_k(\xi) f(\xi)$

$-(1/2\pi i) \oint d\lambda\, g(x|\xi;\lambda) = \delta(x-\xi)$ (1) // special case of (0)

$g(x|\xi;\lambda) = \Sigma_k \varphi_k(\xi)\varphi_k(x)/(\lambda_k - \lambda)$ (2) // the bilinear expansion

$\delta(x-\xi) = \Sigma_k \varphi_k(x)\varphi_k(\xi)$ (3) // insert (2) into (1)

$g(x|\xi;\lambda) = [\,1/\sqrt{\lambda}\sin(\sqrt{\lambda}\,\ell)\,]\sin(\sqrt{\lambda}\,x_<)\sin(\sqrt{\lambda}\,(\ell - x_>))$ // poles at λ_k

$g(x|\xi;0) = [\,1/\ell\,]\,x_<\,(\ell - x_>)$ // $\lambda \to 0$ limit of above

Now on to the eleven exercises!

Exercise 6.24. State Theorem 4 for n = 2 dimensions.

Recall from above:

Theorem 3: g is positive throughout R. // this is true for $n \geq 2$, by the way
Theorem 4: For $n \geq 3$, $g < E$ throughout R

For n = 2, the modified Theorem 4 is this: $g \leq E + (1/2\pi)\ln(h)$ where h is the diameter of the smallest disk which encloses R.

Exercise 6.25. Show that G is Hilbert-Schmidt in n = 2 dimensions.

I prove this by mimicking the n=3 proof given in the text, and using result 6.24 above.

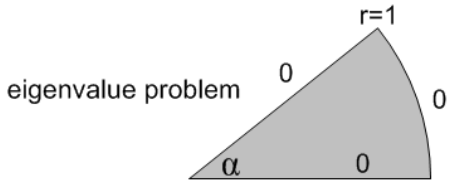
Exercise 6.26. Show that the kernel $k = a(x,\xi)/R^m$ (m < n) generates a CC operator G.

where a is some reasonable continuous function. I skipped this problem, don't know if used later. It is true that many of our "kernels" of interest in this chapter seem to have this general form.

Exercise 6.27. Show that $g(x|\xi)$ in 6.90 is symmetric (ie, for the R_e external problem).

This was pretty easy, a slight modification of the R_i proof given in the text.

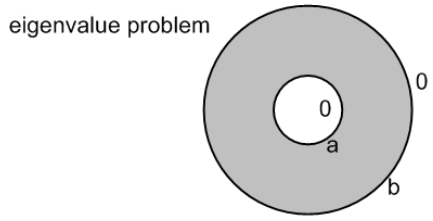
Exercise 6.28. Find the Laplace eigenfunctions for a 2D disk wedge r = 1



I reuse pieces of previous work and find these results: $\lambda_{n',m} = [\beta^{(n')}_m]^2$ and

$\Phi_{n'}(\varphi)\, J_{n'}(\beta^{(n')}_m r) = \sin(n'\varphi)\, J_{n'}(\beta^{(n')}_m r)$ $\quad n' = (n\pi/\alpha)$ with n = 1,2,3...

Exercise 6.29. Find the Laplace eigenfunctions for a 2D annular ring between r = a and r = b.



We are back to $\mu = n^2$ for the φ side of things, and our Bessel solution is now more general, something like this:

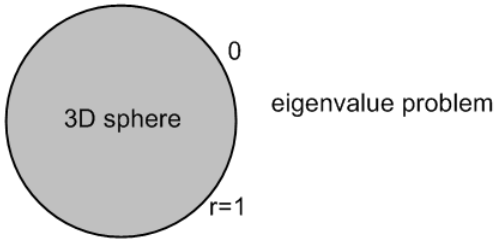
$$f_{mn}(r) = A_n\, J_n((\beta^{(n)}_m r) + B_n\, N_n((\beta^{(n)}_m r)$$

and we then have these two BC's

$$f_{mn}(b) = A_n\, J_n((\beta^{(n)}_m b) + B_n\, N_n((\beta^{(n)}_m b) = 0$$
$$f_{mn}(a) = A_n\, J_n((\beta^{(n)}_m a) + B_n\, N_n((\beta^{(n)}_m a) = 0$$

which we can trivially solve for A_n and B_n. Our solutions are then like p 141 C, with the Bessel J replaced by the above linear combination with A_n and B_n figured out and installed. In the raw notes, I have some history on Karl Neumann and date his functions roughly 1867. Jackson and GR use N, everyone else uses Y.

Exercise 6.30. Eigenfunctions of Laplace for the unit sphere in 3D.

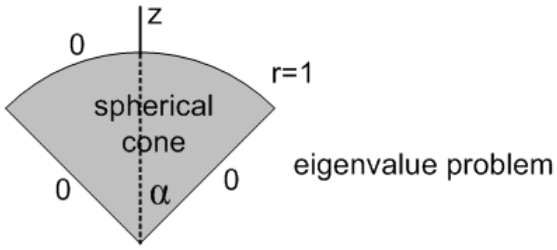


Stak does most of the work for us and our results are these: $\lambda_{k,n,m} = [\beta^{(n+1/2)}{}_k]^2$

$$(1/r)\, J_{n+1/2}(\beta^{(n+1/2)}{}_k\, r)\;\; Y_{n,m}(\theta,\varphi) \;\sim\; (1/\sqrt{r}\,)\;\; j_n(\beta^{(n+1/2)}{}_k\, r)\;\; Y_{n,m}(\theta,\varphi)$$

so the eigenvalues have the usual (2m+1) degeneracy. Note "spherical" Bessel function here.

Exercise 6.31. Eigenfunctions of Laplace for a *cone* of the unit sphere in 3D.



Here I am all on my own! This is the 2D "wedge" problem extended to 3D, sort of. After false starts with Plans A thru E, I finally got success with Plan F.

Plan F. Suppose we set $\lambda = \nu(\nu+1)$ just as a change of variable. Then the ODE without regard to BC's is going to have solutions $P_\nu^m(x)$ and $Q_\nu^m(x)$, we know that for sure. We would select the $P_\nu^m(x)$ presuming that the Q functions blow up somewhere, not sure where. Then our problem is to choose ν such that $P_\nu^m(\cos\alpha) = 0$ so that things vanish on the sides of the cone (different from what we did many times above with Bessel functions which used zeros in argument not order). So define these zeros:

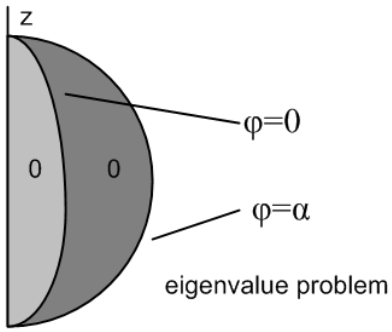
$$P_\nu^m(\cos\alpha) = 0 \qquad \nu_{m,i} = \nu(mi) \text{ = the zeros } i = 1,2.3\ldots \text{ index} \qquad \lambda_{m,i} = \nu_{m,i}(\nu_{m,i}+1)$$

The solution eigenfunctions are then these

$$u_{m,i,k}(r,\theta,\varphi) = (1/r)\, J_{\nu(mi)+1/2}\,[\,\beta^{(\nu(mi)+1/2}{}_k\, r]\; e^{im\varphi}\, P^m{}_{\nu(mi)}(\cos\theta)$$

where m = 0, ±1, ±2 ...., and i = 1,2,3..... and k = 1,2,3.... The $\beta^{(\nu(mi)+1/2}{}_k$ are the zeros of $J_{\nu(mi)+1/2}(x)$ in x, but first you have to find the **ν(mi)** which are the zeros in ν of $P_\nu^m(\cos\alpha)$. No doubt I could make Maple find the first hundred zeros here, and for each of those, a hundred zeros of J. What a complete and utter mess! Could not find any confirmation on the web. A good Stak problem.

Exercise 6.32. Eigenfunctions of Laplace for an *orange slice* of the unit sphere in 3D.



This is the more obvious 2D wedge problem extended to 3D. As with previous problems, I cobble together a solution. The angular part is this: [see correction made in raw notes!! 3/25/10]

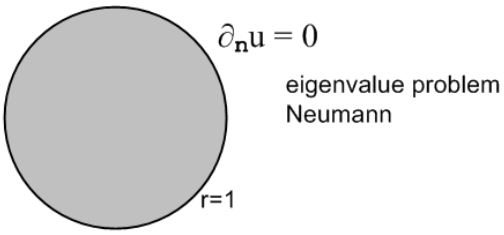
$\Theta_n(\varphi)\,\Phi_m(\varphi) = P_n^{m'}(\cos\theta)\,\sin(m'\varphi)$ n = 0,1,2,3.... m' = mπ/α with m = ±1, ±2 etc

where the key difference (compared to the sphere case) is that m' must be as shown to get vanishing on the orange slice sides. The radial part is the same as for the full sphere. Thus our answer is this:

$\lambda_{k,n,m'} = [\beta^{(n+1/2)}_k]^2$ n = 0,1,2,3.... m' = mπ/α with m = ±1, ±2 etc

$(1/r)\,J_{n+1/2}(\beta^{(n+1/2)}_k\, r)\;P_n^{m'}(\cos\theta)\,\sin(m'\varphi) \;\sim\; (1/\sqrt{r}\,)\;j_n(\beta^{(n+1/2)}_k\, r)\;P_n^{m'}(\cos\theta)\,\sin(m'\varphi)$

Exercise 6.33. Laplace on R where $\partial_n u = 0$ (Neumann) on σ; apply to disk. (145)



First we have to rework our general Dirichlet conclusions to get general Neumann conclusions. As before, eigenvalues are real and positive. You can think of λ = 0 as and EF as well with u = constant, not very interesting. The eigenfunctions are still orthogonal as in Dirichlet, if you trace through the page 137 E derivation of this fact.

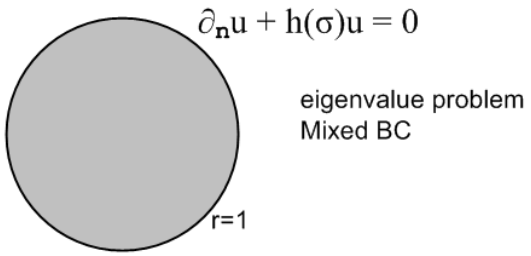
Now we turn to the disk problem with Neumann BC on the circle. We have

$u_n(r) = J_n(\sqrt{\lambda}\, r)$
$u_n'(r) = J_n'(\sqrt{\lambda}\, r)\,\sqrt{\lambda}$
$u_n'(1) = J_n'(\sqrt{\lambda}\,)\,\sqrt{\lambda} \;= 0$

so what we need here are the zeros not of J_n but of J_n' and I call these $\gamma^{(n)}_k$. The problem solution is then this:

$u_{n,k}(r,\varphi) = J_n(\gamma^{(n)}_k r)\, e^{in\varphi}$ n = 0,±1,±2 ... k = 1,2,3... $\lambda_{n,k} = [\gamma^{(n)}_k]^2$

Exercise 6.34. Repeat last but with "mixed" BC; apply to the disk (145)



I think this is the first mention of any kinds of "mixed" boundary value condition in this book.

(a) I first verify for this "mixed" BC situation that $\lambda > 0$ and eigenfunctions are orthogonal.

(b) Now assume h(σ) = a constant h on the circle. Angular is same as previous, $e^{in\varphi}$, but now to meet the mixed BC at r = 1 we get this messy condition: $J_n'(\sqrt{\lambda})\sqrt{\lambda} / J_n(\sqrt{\lambda}) = -h$. This has some weird solutions we write as $\sqrt{\lambda} = \rho^{(n)}_k$ with k = 1,2,3 and we get this solution

$u_{n,k}(r,\varphi) = J_n(\rho^{(n)}_k r)\, e^{in\varphi}$ n = 0,±1,±2 ... k = 1,2,3... $\lambda = [\rho^{(n)}_k]^2$ = EV's

---

6.7 Methods for finding the Green's Function (146)

In the previous section we learned how to *use* the Green's Function to solve Lu = q Poisson with BC, both Dirichlet and Neumann and some mixed. Here finally we find out how to *compute* this Green's Function for a given σ.

The various methods are numbered (1), (2) and so on.

(1) The Integral Equation Method

Our starting ingredients are our two PDE's

$-\nabla^2 g(x\|\xi) = \delta(x-\xi)$	g = 0 on σ	g = Green's
$-\nabla^2 E(x\|t) = \delta(x-t)$	no BC, "free space Green's"	E = fundy

If we "process these equations in the usual way", as shown in the raw notes in complete detail, we end up with the following equation which gives g(x|ξ) as an integral of $\partial_{nt} g(\xi|t)$,

$g(x|\xi) = E(x|\xi) - \int_\sigma dS_t\, E(x|t)\, I(\xi|t)$ // **which is 6.96** x and ξ are inside R

$I(\xi|t) \equiv -\partial_{nt} g(\xi|t)$ t is on σ

or

$g(x|\xi) = E(x|\xi) + v(x|\xi)$ where $v(x|\xi) = -\int_\sigma dS_t\, E(x|t)\, I(\xi|t)$

in terms of our earlier writing of v as $v \equiv g - E$. Following our limit-taking rules of earlier sections, we set $x \to s$ (s on σ) and then we know that $v(s|\xi) = -E(s|\xi)$ since $g = 0$ on σ. [do not wrongly confuse the "v" function here with the dipole "v" function of the $x \to s$ limit discussion!] The last equation above is then:

$E(s|\xi) = \int_\sigma dS_t\, E(s|t)\, I(\xi|t)$ s on σ t integrate on σ, ξ inside R // **which is p 147 A.**

Now change variable names $s,\xi,t \to x,t,\xi$: [I include this because this form is used in the example .]

$E(x|t) = \int_\sigma dS_\xi\, E(x|\xi)\, I(t|\xi)$ x on σ ξ integrate on σ, t inside R

So here is the "integral equation method":

A. Solve the above integral equation for I. We know E, and we know σ, so that ought to be possible,.

B. Compute $v(x|\xi) = -\int_\sigma dS_t\, E(x|t)\, I(\xi|t)$

C. Then we have $g(x|\xi) = E(x|\xi) + v(x|\xi)$, and we have our Green's Function!

Comment #1: This is not really new! First, here is a quote from a few pages above:

" Now comes the basic idea: Define function v as $v \equiv g - E$. Then v is a solution of Laplace (ie, no source) which has the boundary value $f = -E$. [v is harmonic in $\mathcal{R}$] We have spent this *entire chapter* learning how to solve just such a problem, and we know among other things that the solution exists and is unique. For example, 6.11 gives the solution v for the unit disk given some f(φ). "

Now, one method we learned for solving Laplace with f on some boundary σ was what I called "the single layer integral equation method". Again I quote from above:

$u(\mathbf{x}) = \int_\sigma dS_\xi\, a(\xi) 1/4\pi R$ (**) // monopole superposition

$\lim_{\mathbf{x}\to\mathbf{s}} u(\mathbf{x}) = \int_\sigma dS_\xi\, a(\xi) 1/4\pi R = f(\mathbf{s})$ // Fred inhomo 1 (surface limit)

If we translate this into our current variable names, we get

$v(\mathbf{x}) = \int_\sigma dS_t\, a(\mathbf{t}) E(x|t)$ (**)

$\lim_{\mathbf{x}\to\mathbf{s}} v(\mathbf{x}) = \int_\sigma dS_t\, a(\mathbf{t})\, E(s|t) = f(\mathbf{s})$

Now we talk about $v(x|\xi)$ where ξ is a *parameter*, namely it is the point ξ in $-\nabla^2 g(x|\xi) = \delta(x-\xi)$. So let's rewrite the above once again exposing this parameter,

$$v(\mathbf{x}|\xi) = \int_{\sigma} dS_{\mathbf{t}}\, a(\mathbf{t}|\xi) E(x|t) \qquad (**)$$

$$\lim_{\mathbf{x}\to\mathbf{s}} v(\mathbf{x}|\xi) = \int_{\sigma} dS_{\mathbf{t}}\, a(\mathbf{t}|\xi)\, E(s|t) = f(\mathbf{s}|\xi)$$

So this is still my "simple layer integral equation method". In our current application, we know that on σ, our boundary value is $f(\mathbf{s}|\xi) = -E(\mathbf{s}|\xi)$ so the last step above gives this:

$$E(\mathbf{s}|\xi) = \int_{\sigma} dS_{\mathbf{t}}\, E(s|t)\, [-a(\mathbf{t}|\xi)]$$

Comparing to our current Section 6.7 discussion, we conclude that my $[-a(\mathbf{t}|\xi)]$ is Stak's $I(\xi|t)$. In my method, I have no idea that the function a is somehow related to $\partial_{\mathbf{nt}} g(\xi|t)$, and I don't need this knowledge. Suppose I just agree to give my unknown surface density a new *name*: $a(t|\xi) = -I(\xi|t)$. Then my method of solution is this:

A. First, solve this integral equation for $I(\xi|t)$: $E(\mathbf{s}|\xi) = \int_{\sigma} dS_{\mathbf{t}}\, E(s|t)\, I(\xi|t)$ // 147 A

B. Insert that result into this equation: $v(\mathbf{x}|\xi) = -\int_{\sigma} dS_{\mathbf{t}}\, I(\xi|t) E(x|t)$

C. The Green's function is then: $g(x|\xi) = E(x|\xi) + v(x|\xi)$

This is precisely the method being presented in this section! The only new fact is obtained by doing the "processing" noted above, and this reveals that $I(\xi|t) \equiv -\partial_{\mathbf{nt}} g(\xi|t)$. Obviously this information is not very useful if you are trying to compute g in the first place!

Comment #2: I have long been trying to tie this all in with page 125 where the following claim is made, put in our current variables,

$$\partial_{\mathbf{nx}} v(x)|^{+} - \partial_{\mathbf{nx}} v(x)|^{-} = -J(x) \qquad v(x) = -\int_{\sigma} dS_{\mathbf{t}} E(x|t)[-J(t)]$$

We could then add ξ as a parameter everywhere, as we did above, to get

$$\partial_{\mathbf{nx}} v(x|\xi)|^{+} - \partial_{\mathbf{nx}} v(x|\xi)|^{-} = -J(x|\xi) \qquad v(x) = -\int_{\sigma} dS_{\mathbf{t}} E(x|t)[-J(t|\xi)]$$

Since E won't have a discontinuity over the boundary σ, I think we can restate this as

$$\partial_{\mathbf{nx}} g(x|\xi)|^{+} - \partial_{\mathbf{nx}} g(x|\xi)|^{-} = -J(x|\xi) \qquad v(x) = -\int_{\sigma} dS_{\mathbf{t}} E(x|t)[-J(t|\xi)]$$

Now we know that $g = 0$ on σ. Maybe I can somehow claim $g \equiv 0$ everywhere outside of σ, so that would say $\partial_{\mathbf{nx}} g(x|\xi)|^{+} = 0$. Then I would have (dropping the - label for the interior gradient)

$$-\partial_{\mathbf{nx}} g(x|\xi) = -\ J(x|\xi) \qquad\qquad v(x) = -\int_{\sigma} dS_{\mathbf{t}} E(x|t)[-J(t|\xi)]$$

To make this integral equation look like the current presentation, I need to claim $[-J(t|\xi)] = I(\xi|t)$.
Then my left equation would say this:

$$-\partial_{\mathbf{nx}} g(x|\xi) = -\ J(x|\xi) = I(\xi|x)$$

which says $I(\xi|x) = -\partial_{\mathbf{nx}} g(x|\xi) = -\partial_{\mathbf{nx}} g(\xi|x)$ so that $I(\xi|t) = -\partial_{\mathbf{nt}} g(\xi|t)$. This final result does in fact agree with what we are seeing in the current presentation. Thus, the "integral equation method" work that Stakgold did back on page 122-125 gives the same results we are getting in the current section.

Comment #3: In the 1D string Green's function, we often "worked with" the jump in the first derivative at the point $x = \xi$ for $g(x|\xi)$. This activity was far from the endpoints (a,b) which here is boundary σ.
Stak has said nothing (that I noticed) concerning what the multi-dimensional analog of this idea is.

Consider that $\nabla^2 g(x|\xi) = -\delta(x-\xi)$. Suppose we integrate this over a tiny sphere around point ξ. On the RHS we get -1. The LHS from the divergence theorem is $\int d\mathbf{S}_{\mathbf{x}} \bullet \nabla_{\mathbf{x}} g(x|\xi)$. So here is the general rule:

$$\int d\mathbf{S}_{\mathbf{x}} \bullet \nabla_{\mathbf{x}} g(x|\xi) = -1 \qquad\qquad \text{" jump condition"}$$

In the 1D case, $a(x) = 1$ for $L = \nabla^2$ and $\nabla_{\mathbf{x}} g(x|\xi) = \partial_{\mathbf{x}}\ g(x|\xi)$ so $\int_{\xi-\varepsilon}^{\xi+\varepsilon} dx\ \partial_{\mathbf{x}}\ g(x|\xi) = -1$. Perhaps this goes unmentioned because in $n > 1$ dimension it is not very helpful in doing calculations as it is in 1D.

[Well, we used exactly this idea to fund the fundy solutions in the first place.]

Only Example of the Integral Equation Method: the unit disk. (147)

This is an exceedingly complicated example, despite the simplicity of the geometry! It has several spinoff connections to other things we have done before, and each of these connections needs to be clarified. It serves also to introduce us to the method of image charges.

Step A: Solve $E(s|\xi) = \int_{\sigma} dS_{\mathbf{t}}\ E(s|t)\ I(\xi|t)$.

Setup. In order to match the picture on page 147, we have to change names in this equation. First, change ξ to x to get as shown on the left, then change t to ξ on that result to get as shown on the right:

$$E(s|x) = \int_{\sigma} dS_{\mathbf{t}}\ E(s|t)\ I(x|t) \qquad\qquad E(s|x) = \int_{\sigma} dS_{\xi}\ E(s|\xi)\ I(x|\xi) \quad (*)$$

We know that s and ξ are on the circle and x is inside. So let $s = (1,\theta)$ and $\xi = (1,\psi)$ and $dS_{\xi} = d\psi$. Put x on the real axis (for the moment) so $x = (r,0)$. We then use the general law of cosines to find that

$|s-x|^2 = 1^2 + r^2 - 2r\cos\theta$ $\qquad$ $|s-\xi|^2 = 1^2 + 1^2 - 2\cos(\psi-\theta)$.

In general we know that $E = -(1/2\pi)\ln R = -(1/4\pi)\ln R^2$ in our 2D world. Therefore

$E(s|x) = -(1/4\pi)\ln(1 + r^2 - 2r\cos\theta)$ $\qquad$ $E(s|\xi) = -(1/4\pi)\ln(2 - 2\cos[\psi-\theta])$

And of course $I(x|\xi)$ can only be a function of $x = (r,0)$ and $\xi = (1,\psi)$ so call it $I(r,\psi)$. Install the above E's into the integral equation (*), cancel the $-(1/4\pi)$ and finally we have **p 147 D, our starting position!**

Lemma : $v(r,-\theta) = v(r,\theta)$ Recall that $v(r,\theta)$ is the Laplace solution in the disk with BC that $v(1,\theta) = -E(s|x) = -E(r,\theta)$. Since this boundary condition is symmetric under $\theta \to -\theta$, the solution $v(r,\theta)$ must also have this symmetry. Here is how we prove this last statement (which is intuitively obvious) [P means the reflection in x axis op]

$Lv = 0$	$v(\sigma) = -E(r,\theta) = -E(r,-\theta)$	since $E = E(R)$ and $R(r,\theta) = R(r,-\theta)$
$PLP^{-1}Pv = 0$	$Pv(\sigma) = -PE(r,-\theta) = -E(r,\theta)$	
$L(Pv) = 0$	$(Pv)(\sigma) = -E(r,\theta)$	

This shows that functions v and Pv solve the same "PDE system". Since the solution is unique, we conclude that $Pv = v$. Therefore, $v(r,-\theta) = v(r,\theta)$.

Claim that I is symmetric. I tried various proofs, and kept ended up with things like $0 = \int_{-\pi}^{\pi} d\psi\, [I(r,\psi)-I(r,-\psi)]E(\psi-\theta)$. I could not conclude that I is symmetric because the E shown has both signs in the integration. That led me to this different proof. We use the fact that we know that $I(x|\xi) = -\partial_{\mathbf{n}_\xi} g(x|\xi)$. Let $\xi' = R\xi$ be the reflection of point ξ in the x-axis, R = reflection operator. Then we have

$$I(x|\xi') = -\partial_{\mathbf{n}_{\xi'}} g(x|\xi') = -\mathbf{n}_{\xi'} \bullet \nabla_{\xi'}\, g(x|\xi')$$

Now $\mathbf{n}_{\xi'} = R\,\mathbf{n}_\xi$ and $\nabla_{\xi'} = R\nabla_\xi$. For example, $\partial_{\mathbf{x}'} = \partial_{\mathbf{x}}$ and $\partial_{\mathbf{y}'} = -\partial_{\mathbf{y}}$. This tells us that

$$\mathbf{n}_{\xi'} \bullet \nabla_{\xi'} = R\,\mathbf{n}_\xi \bullet R\nabla_\xi = \mathbf{n}_\xi \bullet \nabla_\xi$$

But we also know that $g(x|\xi') = g(x|\xi)$ because this is true for E and above in our Lemma we showed it to be true for v, which is to say, $v(r,-\psi) = v(r,\psi)$. Therefore

$$I(x|\xi') = -\mathbf{n}_{\xi'} \bullet \nabla_{\xi'}\, g(x|\xi') = \mathbf{n}_\xi \bullet \nabla_\xi\, g(x|\xi) = I(x|\xi) \quad \Rightarrow \quad I(r,-\psi) = I(r,\psi)$$

Summary of Further Processing. We start with p 147 D obtained in our setup above. Using the fact that I is symmetric in ψ, we expand $I(r,\psi)$ in a Fourier $\cos\psi$ series with coefficients $I_n(r)$ $n = 0,1,2...$ At the same time, we write each log as a series sum using a tricky fact we proved earlier in the book. At this point, we have this intermediate result:

$$-2\sum_{n=1} (r^n/n)\cos(n\theta) = -2\sum_{m=0} I_m(r) \sum_{n=1}(1/n) \int_{-\pi}^{\pi} d\psi\, \cos(n(\psi-\theta))\cos(m\psi)$$

We look up the integral in Schaum and find it to be $\delta_{n,m}\,\pi$ which tells us that

$$-2\,\Sigma_{n=1}\,(r^n/n)\cos(n\theta) = -2\pi\Sigma_{n=1}(1/n)\,I_n(r)\cos(n\theta)$$

where the m=0 term got killed off. From this we conclude that

$$I_n(r) = r^n/\pi \qquad n = 1,2,3...$$

To get $I_0(r)$ we have to use 6.84 which says $\int I(x|\xi)dS_\xi = 1$ and we find $I_0(r) = 1/(2\pi)$. We then conclude that

$$I(r,\psi) = 1/2\pi + 1/\pi\,\Sigma_{n=1}^{\infty}\,r^n\cos(n\psi)$$

As per Stak's plan, we computed this sum as well earlier in the book, and when we insert that into the above, we get that

$$I(r,\psi) = 1/2\pi * [\ 1 - r^2\ ] / (1 + r^2\text{-}2r\cos\psi) = I(x|\xi)$$

Looking at our picture, we know that if we let x move to angle φ off the real axis, we would have

$$I(r,\varphi,\psi) = 1/2\pi * [\ 1 - r^2\ ] / (1 + r^2\text{-}2r\cos(\psi\text{-}\varphi)) = I(x|\xi)$$

So we have now successfully concluded Step A: we have an explicit expression for I(x|ξ).

Step A1: The point here is to show that our solution agrees with 6.11 derived earlier. We go back to our earlier general result,

$$u(x) = \int_R d\xi\, g(x|\xi)\, q(\xi) - \int_\sigma dS_\xi\, f(\xi)\,\partial_{\xi n}\, g(x|\xi)$$

In our problem, there are no inside sources so q(ξ) = 0, and we know also $-\partial_{\xi n}\, g(x|\xi) = I(x|\xi)$, so

$$u(r,\varphi) = \int_\sigma dS_\xi\, f(\xi)\, I(x|\xi) = \int_{-\pi}^{\pi} d\psi\, f(\psi)\ \{\ 1/2\pi * [\ 1 - r^2\ ] / (1 + r^2\text{-}2r\cos(\psi\text{-}\varphi))\ \}$$

and this replicates our famous 6.11.

Step B and C. We are now to compute $g(x|\xi) = E(x|\xi) + v(x|\xi)$. Well, not so fast, Batman. Here we have another *elaborate* set of steps to get the result into a form we want. We first know that

$$v(x|\xi) = -\int_\sigma dS_t\, E(x|t)\, I(\xi|t) \qquad \Rightarrow \qquad g(x|\xi) = E(x|\xi) - \int_\sigma dS_t\, E(x|t)\, I(\xi|t)$$

Here we are sort of doing Step B as well as Step C. Now replace ξ with x_0,

$$g(x|x_0) = E(x|x_0) - \int_\sigma dS_t\, E(x|t)\, I(x_0|t)$$

Next we describe x = (r,φ) and x_0 = (r_0,φ_0) and t = (1,ψ) . Then the above becomes 6.98. We then insert our known series for E(x|t) and I(x_0|t) to get page 149 A and B. Insertion of these gives C, and we do the sum there to get D. The log term there must be a new way to write v(x|x_0). This is reprocessed into the form shown in G which shows that v(x|x_0) is the potential caused by an image charge at x_o* plus a constant that is a function of r_0. The final Green's Function form is then **p 149 H** which says this: the g(x|x_0) for our disk problem can be represented as the sum of two fundy solutions and a constant. One fundy solution is for a unit charge at ξ, the other for a unit charge at an image point ξ*. This seems extremely mysterious at this point!

Finally, let's place the following two results side by side, where each integral is over σ and has I(x|ξ) as a weighting function,

$$u(x) = \int_\sigma dS_\xi\, I(x|\xi)\, f(\xi) = \int_\sigma dS_\xi\, I(x|\xi)\, f(\xi)$$

$$v(y|x) = -\int_\sigma dS_\xi\, I(x|\xi) E(y|\xi) = \int_\sigma dS_\xi\, [\, \Sigma_{\mathbf{x}}(\xi)\,]\, E(y|\xi) = v_{\mathbf{x}}(y)$$

Although both integrals involve $\int_\sigma dS_\xi\, I(x|\xi)$, these two lines are solutions to *two different problems*.

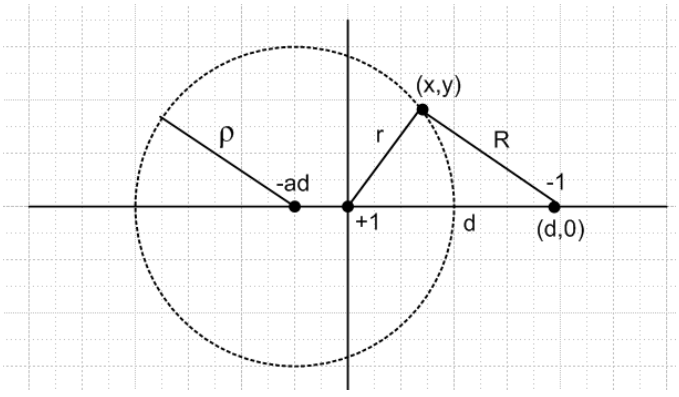
(1) The first tells us u(x) inside a surface σ on which f(ξ) is imposed.

(2) The second tells us $v_{\mathbf{x}}(y)$ inside the same surface σ if that surface is plastered with a charge distribution – I(x|ξ) .

(3) In problem 2, if we were to add a point charge at x, then our solution would be in fact v(y|x) + E(y|x) which would be just g(y|x) which vanishes on σ.

(1.5) Digression to show that two 2D point charges generate circles of constant potential

In 2D, we have E = -(1/2π) ln R. Consider two point charges as shown in this picture:



where both are on the x axis, and the +1 charge is at x=0, the -1 charge is at x=d, and the corresponding distances are r and R to the general point (x,y). The total potential is this:

$u(x,y) = (+1)\,[\,-(1/2\pi)\ln(r)] + (-1)\,[\,-(1/2\pi)\ln(R)] = -(1/2\pi)\ln(r/R).$

We seek the form of a locus such that u(x,y) = constant. But that means r/R = constant. But this is just the condition of the Circles of Apollonius as in our geometry document. So right away, from that document, we know that the loci are circles, and we know these facts, where r/R = α and so $u = -(1/2\pi)\ln(\alpha)$. In our picture we show the case $\alpha = r/R < 1$. So

$c_x = -(\alpha^2 d)/(1-\alpha^2)$ // location of circle center
$c_y = 0$

$\rho = \{\,\alpha / |1-\alpha^2|\}\ |\,d\,|$ // radius

$|c_x| = (\alpha^2|d|)/|1-\alpha^2| = \alpha\rho$

As noted, $\alpha = r/R > 0$, the distance ratio.

Another dimensionless constant of interest would be "a" where

$a = -c_x/d$

If we put the center at $c_x = -ad$ as shown, then we must have

$a = \alpha^2/(1-\alpha^2) \quad \Rightarrow \quad \alpha^2 = a/(a+1) \quad \Rightarrow \quad (1-\alpha^2) = 1/(a+1)$

Then we get

$\rho^2 = (\alpha^2)\,d^2/(1-\alpha^2)^2 = a\,d^2/(1-\alpha^2) = a(a+1)d^2$

Here is one more ratio of interest:

(distance from circle center to +1 charge)/ (distance from circle center to -1 charge)

$= D/D^*$

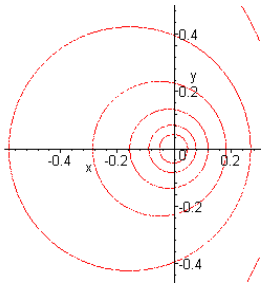
$= ad/(ad+d) = a/(a+1) = \alpha^2$

As noted, the potential is constant on our circle with value $u = -(1/2\pi)\ln(\alpha)$. If we want the potential to be *zero* on our circle, we should add this amount to u. Then we have

$u(x,y) = (1/2\pi)\ln(R/r) + (1/2\pi)\ln(\alpha)$

I asked Maple to plot v = -ln(r/R) for the above geometry with d = 1 and for v = 1/2, 1, 3/2...3 so that the loci plotted are equally spaced in terms of the value of the potential. I did this by writing out r and R in terms of Cartesian x and y. The implicitplot function runs x and y over a grid and whenever -ln(r/R) is found to be within some small ε of the half integer specified, it makes a red dot. In this way, for each half integer of potential value we get a circle!

```
r := ( x^2 + y^2)^(1/2):   R := ( (x-1)^2 + y^2)^(1/2):
v := -ln(r/R);
with(plots):implicitplot({seq(v=N/2,N=1..6)},x=-0.6..0.3, y=-0.5..0.5, grid = [200,200], scaling=CONSTRAINED);
```



(1.6) How does this circle stuff relate to p 149H ?

Here we have to move our coordinate system origin of Section (1.5) above to the center of the circle. In this "primed" coordinate system, our point on the circle has coordinates (x',y') or in polar (r',φ'). Our positive unit charge is at (ad,0) which is called (r_0,0). Our negative unit charge is located at the image point (ad+d,0) which I show below is the same as (ρ^2/r_0,0) .

Now,

$$r_0 \equiv ad = \alpha^2 d/(1-\alpha^2) = \alpha\rho$$

So we in fact have $r_0 = \alpha\rho$. Furthermore, we can compute that

$$ad+d = ad/\alpha^2 = r_0/\alpha^2 = \rho^2/r_0$$

So our potential solution which is 0 on the circle of radius ρ is this: (R and r means distances here)

$$u(x',y') = (1/2\pi)\ln(R/r) + (1/2\pi)\ln(r_0)$$

$$= [-(1/2\pi)\ln(r)] - [-(1/2\pi)\ln(R)] + (1/2\pi)\ln(r_0)$$

$$= E(x',y' \mid r_0,0) - E(x',y' \mid [\rho^2/r_0,0]) + (1/2\pi)\ln(r_0)$$

= (fundy of +1 charge) – (fundy of -1 image charge) + (constant)

and in polar coordinates this is

$$u(r',\varphi') = E(r',\varphi' \mid r_0,0) - E(r',\varphi' \mid 1/r_0,0) + (1/2\pi)\ln(r_0) \quad \text{// polar}$$

If we now get rid of the primes and assume we are in this new coordinate system, get

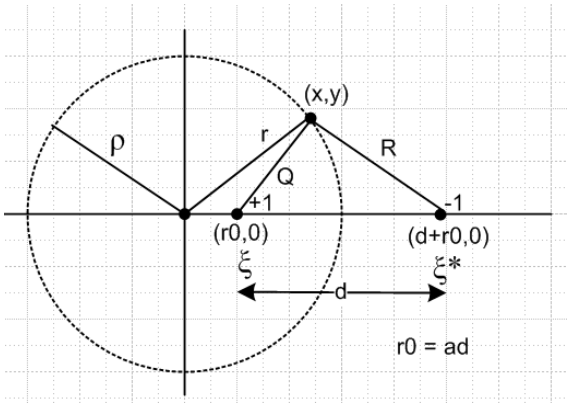
$$u(r,\varphi) = E(\, r,\varphi \mid r_0,0) \;-\; E(r,\varphi \mid \rho^2/r_0,0\,) \;+\; (1/2\pi)\ln(r_0)$$

$$= g(r,\varphi \mid r_0,0)$$

and this is what p 149 H is saying. He uses $x = (r, \varphi)$, $\xi = (r_0, 0)$, $\xi^* = (\rho^2/r_0, 0)$ so he says

$$g(x \mid \xi) = E(x \mid \xi) \;-\; E(x \mid \xi^*) \;+\; (1/2\pi)\ln(r_0)$$

Here is a picture of this new primed coordinate system situation, but with primes removed



To avoid ambiguity, we have replaced the "distance r" of the above analysis with symbol Q, since "r" now has a new meaning.

Summary of 1.6: (in the new coordinate system). I have shown that if you put a positive unit point charge at $\xi = (r_0,0)$ and a negative image charge at $\xi^* = (\rho^2/r_0,0)$, the sum of the fundy potentials of these charges, when added to $(1/2\pi)\ln(r_0)$, will result in zero potential on a circle about the origin which has radius ρ. But of course all three terms satisfy 2D Laplace inside this circle. Therefore, the solution I have found here must be $g(x|\xi)$ for the radius-ρ disk, since we know it is unique. So basically I have found the solution by a *much* simpler method that all the fancy stuff involved above in our "example of applying the integral equation method to finding the Green's Function for the unit circle".

Comment on the 3D dipole equipotential lines.

I wondered what a similar 3D analysis would yield for two point charges in 3D. I started then with this, where I restricted my interest to the z=0 plane, and where I am back to the unprimed coordinate system:

$$4\pi V = 1/r - 1/R \qquad \text{where } r^2 = (x^2 + y^2) \qquad R^2 = [(x-d)^2 + y^2]$$

so that

$$(4\pi V)^2 = 1/r^2 + 1/R^2 - 2/(rR) \qquad \text{// } \pm V \text{ are now both included due to squaring}$$

$(4\pi V)^2 - 1/r^2 - 1/R^2 = -2/(rR)$
$r^2R^2(4\pi V)^2 - R^2 - r^2 = -2rR$

$[r^2R^2(4\pi V)^2 - R^2 - r^2]^2 = 4r^2R^2$ // ±r are now both included due to squaring
$[r^2R^2v^2 - R^2 - r^2]^2 - 4r^2R^2 = 0$ $v = 4\pi V$

Let's have Maple write this out in terms of x and y. I defined symbols a,b,c,e as shown below, and then the equation following is a restatement of the last equation above, and I define the expression to be f:

$a \equiv r^2$ $b \equiv R^2$ $c \equiv v^2$ $e \equiv abc$
$f \equiv [e - b - a]^2 - 4ab = 0$

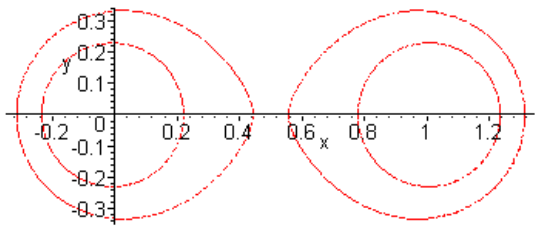
Maple code is then as follows. I set up all the symbols and then asked Maple to expand everything so I could see the equation of the curve in Cartesian coordinates.

```
restart;
a := x^2 + y^2;
b := (x-d)^2 + y^2;
c := v^2;
e := a*b*c;
f := (e - b - a)^2 - 4*a*b;
g := expand(f);
h := sort(g,[x,y]);
```

We then find that the LHS or our equation LHS = 0 is this function 0 = h(x,y; v,d)

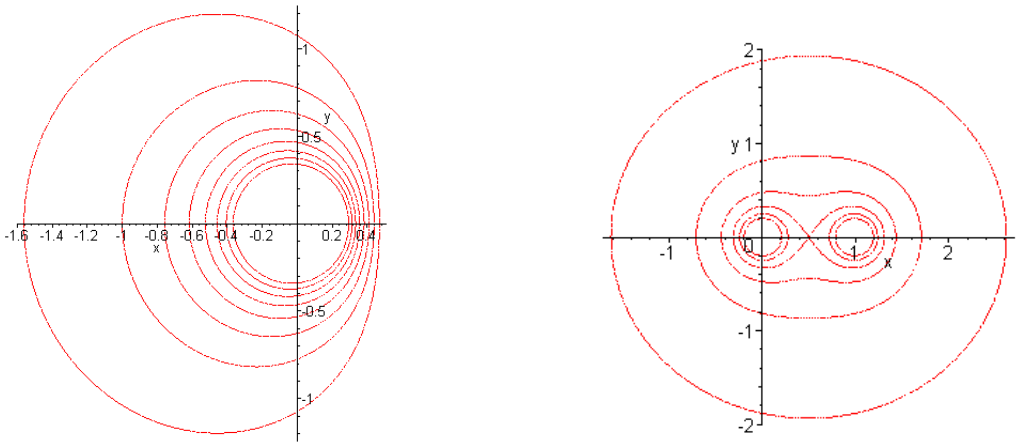
$$h := v^4x^8 + 4v^4x^6y^2 + 6v^4x^4y^4 + 4v^4x^2y^6 + v^4y^8 - 4v^4dx^7 - 12v^4dx^5y^2 - 12v^4dx^3y^4 - 4v^4dxy^6 + 6v^4d^2x^6$$
$$- 4v^2x^6 + 14v^4d^2x^4y^2 - 12v^2x^4y^2 + 10v^4d^2x^2y^4 - 12v^2x^2y^4 + 2v^4d^2y^6 - 4v^2y^6 + 12v^2dx^5 - 4v^4d^3x^5$$
$$+ 24v^2dx^3y^2 - 8v^4d^3x^3y^2 - 4v^4d^3xy^4 + 12v^2dxy^4 - 14v^2d^2x^4 + v^4d^4x^4 + 2v^4d^4x^2y^2 - 20v^2d^2x^2y^2$$
$$+ v^4d^4y^4 - 6v^2d^2y^4 + 8v^2d^3x^3 + 8v^2d^3xy^2 + 4d^2x^2 - 2v^2d^4x^2 - 2v^2d^4y^2 - 4d^3x + d^4$$

which gives us a very horrible **8th degree polynomial in x and y for our equipotential surface**. The only variables are v and d. I then used the same implicitplot idea to see what these curves looked like. Due to the fact that I had to square an equation twice, the above curve is actually the locus of four different situations -- each charge being ± 1. For d=1 and v=3.75 I got this: (don't have a trace on this)



where you see four curves instead of 1. For + on the left and - on the right, the correct curve is the inner one on the left.

Once I knew about things being 8^{th} degree, I dispensed with the messy form above and asked Maple for a more direct implicitplot of v = 1/r - 1/R and that produced the following nice set of loci, where innermost has v = 2 and outermost has v = 1/4 [and v = 0 will be the vertical plane between the two charges.] On the right I plot v = 1/r+ 1/R for fun.



This was for equal and opposite charges. You could argue that there *is* a true sphere on which there is a constant potential, but that sphere here has an infinite radius and corresponds to the dividing plane between the charges. As we see below, if you make the two charges have different *magnitudes*, then this infinite radius sphere becomes a finite radius sphere. In our image method below, we position the two charges such that V = 0 on this true sphere. This one surface is a sphere, and all other values of V continue to give 8th degree surfaces.

(2) The Method of Images

The idea is to find one or perhaps more "image charges" to add to your point charge at ξ such that the sum of the fundies of all these charges achieves g = 0 on the surface you are interested in. In the 2D example above, we were able to do this with a single image charge of -1 at the usual image location. In that example, the equipotential surface for *any* value of v was a circle, and in particular, for v = 0 we got a circle, and this circle was the one we wanted. In the first example below, the v = 0 surface will be a sphere, but for $v \neq 0$ the equipotential surfaces are messy 8^{th} degree polynomials.

Example 1: Green's function for the unit sphere (3D) (150)

(a) Stakgold's method (150)

We have a charge inside our sphere at position ξ, and we guess an image charge of size A sitting outside at ξ* where $|\xi^*| = \rho^2/|\xi|$ ($\rho = 1$). Then our v(x,ξ) is as shown in C. We then quickly arrive at F which is our candidate solution to this problem: the potential at x due to a point charge at ξ and vanishing on surface σ. We made it vanish by construction in D for any point s on σ, so we know that our guess was right.

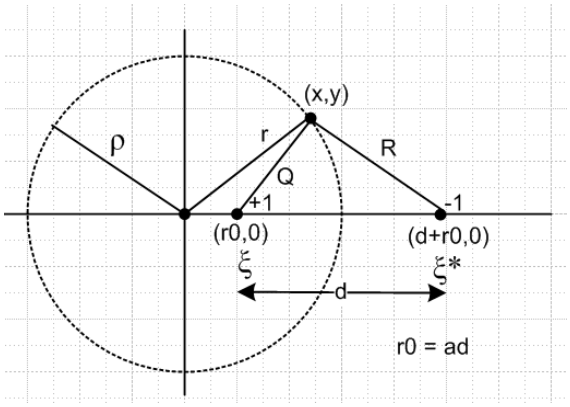
Now just to convince us even more, he computes our friend $\partial_{n_\xi} g(x|\xi)$ appears in our general equation

$$u(x) = \int_R d\xi\, g(x|\xi)\, q(\xi) - \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n}\, g(x|\xi) \qquad \text{// } q = 0 \text{ in source-less Dirichlet}$$

so we suspect $\partial_{n\xi} g(x|\xi)$ [$\equiv -I(x|\xi)$] will be minus the Poisson kernel as shown in our 3D result way back in 6.26. He *laboriously* computes this derivative on page 151 and in 6.103 it all falls out that the resulting $I(x|\xi)$ is exactly the Poisson kernel in 6.26. I am skipping the math detail here because I don't think it contains anything new for me.

(b) My method

I use the primed coordinate system shown above and replicated here, but this is a z=0 slice now:



To adapt this to our 3D case, imagine the +1 charge point to be $\xi = (r_0,0)$, the point on the sphere slice to be **r**, and the negative charge out at $\xi^* = (d+r_0,0)$. In order to get 0 potential on the spherical surface, we need to make two choices (1) set d so that $d+r_0 = \rho^2/r_0$ (2) scale the negative charge by factor ρ/r_0. This means that our image charge has size $-\rho/r_0$ and is located at $(\rho^2/r_0, 0)$. Since we want V = 0 on the sphere, our equation is simply 1/Q = k/R where $k = \rho/r_0$. We don't need Maple to do the algebra which shows that the V=0 surface is in fact the sphere of radius ρ shown. Here is the math, which we first do for the z = 0 slice:

$$1/Q = 1/R' \quad => \quad Q = R' = (r_0/\rho)\, R \quad => \quad Q^2 = (r_0/\rho)^2\, R^2$$

which says

$$[(x-r_0)^2 + y^2] = (r_0/\rho)^2\, [(x - \rho^2/r_0)^2 + y^2]$$

so it is not hard to imagine we might get a circle. We don't need Maple for this:

$$r_o^2[(x- \rho^2/r_0)^2 + y^2] = \rho^2[(x-r_0)^2 + y^2]$$
$$r_o^2[x^2 + \rho^4/r_0^2 - 2x\rho^2/r_0 + y^2] = (\rho^2x^2 + \rho^2r_0^2 - \rho^2 2xr_0 + \rho^2y^2)$$
$$r_o^2x^2 + \rho^4 - 2x\,\rho^2r_0 + r_0^2y^2 = \rho^2x^2 + \rho^2r_0^2 - \rho^2 2xr_0 + \rho^2y^2$$
$$r_o^2x^2 + \rho^4 + r_0^2y^2 = \rho^2x^2 + \rho^2r_0^2 + \rho^2y^2$$

$x^2(\rho^2 - r_o^2) + y^2(\rho^2 - r_o^2) = \rho^4 - \rho^2 r_0^2 = \rho^2(\rho^2 - r_0^2)$
$x^2 + y^2 = \rho^2$

Now, suppose we throw in z^2 in all the right places. That means we add $+z^2$ in both opening []. This means that z^2 carries through just as does y^2 and we will then get this result

$x^2 + y^2 + z^2 = \rho^2$ QED!

Conclusion: Draw a sphere of radius ρ centered at the origin. A distance $r_0 < \rho$ out on the x axis put a charge +1. A distance ρ^2/r_0 out on the x axis put an image charge $-(\rho/r_0)$. Let v be the sum of the potentials of these two charges. One will then find that v = 0 on the sphere! Therefore, the sum of these two potentials is the Green's Function for the sphere. This is the content of equation 6.102 where the ratio $1/|\xi|$ is our ρ/r_0, since he has $\rho = 1$.

Example 2: Green's function for a half space (3D)

We know the answer to this one: image charge of -1 on the opposite side equidistant. This makes g = 0 on the plane and so gives you the Green's Function for the half-space, as shown in 6.104. We can then compute $\partial_{n\xi} g(x|\xi) = - I(x|\xi) = \Sigma_x(\xi)$ to find the charge distribution that would be present in the associated problem of the left half plane filled with metal, and a point charge placed at ξ . Having done this, we then know I(x|ξ) so we know the solution to our no-source Dirichlet problem, which solution is $u(x) = \int_\sigma dS_\xi \, f(\xi) \, I(x|\xi)$. And if we take the limit using our double-δ rule, we verify u→f on the surface.

Example 3: Neumann's function for a half space (3D)

Stak calls the Green's Function here h, Neumann condition is $\partial_n h = 0$ at the dividing plane. The image is the same as in the previous example but has +1 charge instead of -1.

The equipotential lines in the z=0 plane for Examples 2 and 3 are shown above just before heading "The Method of Images".

Example 4: Line charge parallel to a plane (3D)

This is really a 2D problem that I have already done above, if you look at a cross section. So put an image charge of -1 on the opposite side and that gives the answer. The equipotentials are the family of circles I showed above.

(3) The Method of Full Eigenfunction Expansion (153) (nD)

First, a little volume 1 review. On page 259 of Chap 4 we looked at the driven equation with $L = -\partial_x^2 - \lambda I$, and of course we were in a 1D world. We had the EV problem Lφ = 0 and we found some φ_n and λ_n. Then on page 261 we arrived at a certain "bilinear series" for the Green's function 4.8 with $\lambda_n - \lambda$ in the denominator, namely, [functions were real back there I think so no overbar on second φ]

$g_\lambda(x|\xi) = \Sigma_n \varphi_n(x)\varphi_n(\xi)/ (\lambda_n-\lambda)$

We regarded λ as a parameter of g and obtained various fancy results. We even thought about analyticity in λ of g, for example.

Now fast forward to the current context where now $L = -\nabla^2$ so in effect we have $\lambda = 0$. We expect to get our bilinear series with $\lambda = 0$, and that is what 6.108 says, but now we are in a 3D world (or nD)

$$g(\mathbf{x}|\xi) = \Sigma_n u_n(\mathbf{x})\bar{u}_n(\xi)/ \lambda_n \qquad (6.108)$$

and Stak derives this result for us in the text quickly and simply using Green and Fourier. One of those "processing" things.

So this immediately provides a "method" of computing the Green's Function for a problem! You first solve the eigenvalue problem, and then at once you know g, all done. In a problem with R = rectangle, we know that the eigenfunctions are sines which vanish on their respective boundaries, so it is pretty clear from the formula that g = 0 there as well. In fact, on page 154 Stak shows us the above expansion for both the rectangle and for the unit sphere, using results we developed earlier in the text. The only bad news is that these are *double* sums. In terms of (6.108) we have n = (n,m) for the rectangle.

(4) The Method of Partial Eigenfunction Expansion (154) (2D only)

This is a somewhat tricky method, but the reward is that you get a single sum instead of a double sum for your resulting Green's Function g. Our *double* sum form was this,

$g(\mathbf{r} \mid \mathbf{r}') = \Sigma_{nm} \varphi_{nm}(\mathbf{r}) \bar{\varphi}_{nm}(\mathbf{r}')/\lambda_{nm}$ $\qquad \varphi_{nm}(\mathbf{r}) = p_n(x)\, q_m(y)$ // see for example p 154 6.109

which involved both sets of eigenfunctions p_n and q_m, one set for each "direction". But now we will have instead a *single* sum form, where we expand only on the eigenfunctions in *one* of our two directions (either one, I will call these eigenfunctions $f_m(x)$). Here is that single sum form: [here $\mathbf{r} = (x,y)$]

$g(\mathbf{r} \mid \mathbf{r}') = \Sigma_m g_m(y; \mathbf{r}')\, f_m(x)$. // 6.113 and p 155 C

where $g_m(y; \mathbf{r}')$ is merely the projection of $g(\mathbf{r} \mid \mathbf{r}')$ onto $f_m(x)$, as for example in 6.112,

$$g_m(y; \mathbf{r}') \equiv \int dx\, g(\mathbf{r} \mid \mathbf{r}')\, f_m(x)$$

If we then take our original PDE $-\nabla^2 g(\mathbf{r}|\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$ and process it as shown in the raw notes (mult by f_m and integrate over x; do parts twice on the first term and use $-\nabla^2 f_m = \lambda_m f_m$ in that term) and that processed PDE becomes this:

$$-\lambda_m g_m(y; \mathbf{r}') - \partial_y^2 g_m(y; \mathbf{r}') = f_m(x') \delta(y-y') \qquad \text{// which is p 156 A}$$

This is just a 1D Green's equation in variable y with an extra multiplier function $f_m(x')$ on the right which we know (in theory) how to solve from our Volume I efforts. So the idea is that we solve this equation for $g_m(y; \mathbf{r}')$ and then we have our single sum solution, all done!

So the basic idea here is that you pick one of the two 2D directions for your basic eigenfunctions $f_m(x)$, and you never compute or use the eigenfunctions in the other direction. This is why he is calling this a "*partial* eigenfunction expansion".

I comment that when I was back doing those 0,0,0,f type 2D problems, I was able to get away with a single sum form like this (quantized "n" created by the 0,0 direction, no other quantum numbers)

$$u(x, y) = \Sigma_n f_n(x)h_n(y) \qquad \text{eg:} \qquad f_n = \sin(\sigma_n x) \qquad h_n = \text{sh}(\sigma_n y)$$

and that "form" was general enough to find the unique solutions I found. But for a 0,0,0,0 type Green's Function problem (the four quadrilateral boundaries) this form is *not* general enough and we need the weighted double sum form I show above, $g(\mathbf{x}_1|\mathbf{x}_2) = \Sigma_{nm}\, p_n(\mathbf{x}_1)\, q_m(\mathbf{x}_2)/\lambda_{nm}$. Here we are quantized in both directions.

On page 157 Stak talks about **another method** where we do as above to get $g_m(y; \mathbf{r}')$, and then we expand that thing on $q_n(x)$ as well to get

$$g(\mathbf{r} \mid \mathbf{r}') = \Sigma_{mn}\, g_{mn}(\mathbf{r}')\, p_m(x)\, q_n(y)$$

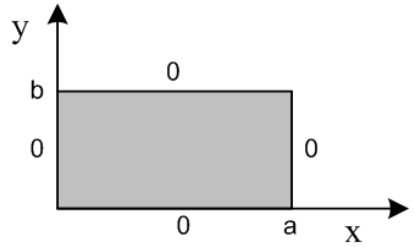
This is just a double Fourier projection of our problem. Comparison with our original double expansion then shows that

$$g_{mn}(\mathbf{r}') = p_n(x')\, q_m(y') / \lambda_{nm}$$

which is exactly what we see in p 157 C. If I have understood this correctly, I do not understand why Stak says "which is not typical" because it would seem that this method always gives the same double sum as the generic double expansion.

We are now going to have **three examples** where we use the *partial* EF method to find Green's g.
The examples are: rectangle, unit circle, and infinite horizontal strip. All three of these can be thought of as cross sections of 3D problems.

Example 1. Green's function for the rectangle. (155)



This is the "canonical" example and follows along as outlined above. Stak does so many different things here, I have to list them with letters.

(a) Find $g_m(y; \mathbf{r}')$. The 1D Green's is shown p 156 A, and we solve this thing for $g_m(y; \mathbf{r}')$ with result as in C where $y_<$ = min(y,y') in my notation :

$g_m(y; \mathbf{r}') = (2/m\pi) \sin(m\pi x'/a) / \sh(m\pi b/a) * \sh(m\pi y_</a) \sh(m\pi[b-y_>]/a)$

(b) Fast Convergence. We then look at a particular situation where we have a...y.....y'...b in which case

$g_m(y; \mathbf{r}') = (2/m\pi) \sin(m\pi x'/a) / \sh(m\pi b/a) * \sh(m\pi y/a) \sh(m\pi[b-y']/a)$

If we now assume all four numbers are widely spaced from each other, so we can roughly say for example y >> a and y' >> y, etc, then you find that

$g_m(y) \sim e^{-m\pi|y'-y|/a}$ as I show in the raw notes.

This means that probably only the leading term in the sum $g(\mathbf{r} \mid \mathbf{r}') = \Sigma_{m=1} g_m(y; \mathbf{r}') f_m(x)$ is needed to get an accurate approximation for $g(\mathbf{r} \mid \mathbf{r}')$, and we shall see an example of this in a moment.

(c) Other way. In p 157 A Stak states $g_m(x; \mathbf{r}')$ where we pick the other set of eigenfunctions (ie, we use the other direction as our basis set).

(d) The g_{mn} method. It is at this point that he makes his only comments on the $g_{mn}(r')$ method and gets p 157 C, which I have already commented on above.

(e) Dirichlet application. We are now at the lower section of p 157 where Stak then starts talking about how we might use our newly found Green's g to solve the standard Dirichlet problem. We always go back to our general result (as I have done many times now),

$$u(x) = \int_R d\xi\, g(x|\xi)\, q(\xi) - \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n}\, g(x|\xi) \qquad \Sigma_{\mathbf{x}}(\xi) = \partial^{(\xi)}{}_n\, g(x|\xi) \equiv - I(x|\xi)$$

and if we are doing sourceless Dirichlet this is just

$$u(\mathbf{r}) = - \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n}\, g(\mathbf{r}|\xi) \qquad \Sigma_r(\xi) = \partial^{(\xi)}{}_n\, g(\mathbf{r}|\xi) \equiv - I(\mathbf{r}|\xi)$$

where dS_ξ runs around the rectangle perimeter. We consider a special case where we have $f(x) \neq 0$ only along the top of the rectangle, so this becomes

$$u(\mathbf{r}) = - \int_0^a dx\, f(x)\, \partial_{\mathbf{y}'} g(\mathbf{r}|\mathbf{r}') \qquad (*) \qquad \Sigma_{\mathbf{r}}(\mathbf{r}') = \partial_{\mathbf{y}'} g(\mathbf{r}|\,\mathbf{r}') \equiv - I(\mathbf{r}|\,\mathbf{r}')$$

On top of page 159 Stak takes our result

$g(\mathbf{r} \mid \mathbf{r}') = \Sigma_m\, g_m(y; \mathbf{r}')\, f_m(x)$. // 6.113 and p 155 C

$g_m(y; \mathbf{r}') = (2/m\pi) \sin(m\pi x'/a) / \sh(m\pi b/a) * \sh(m\pi y_</a) \sh(m\pi[b-y_>]/a)$

and computes $\partial_{\mathbf{y}'} g(\mathbf{r}|\,\mathbf{r}')$ in p 158B (where he omitted a factor I have filled in) , and then he inserts this into our integral to get $u(\mathbf{r}) = - \int_0^a dx\, f(x)\, \partial_{\mathbf{y}'} g(\mathbf{r}|\mathbf{r}')$ as shown in p 158 C. We still have of course the

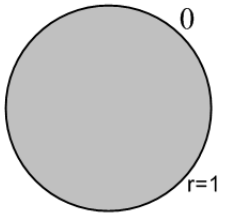
Σ_m sitting there.

(f) Fast Convergence example. Now at bottom of page 158 we get an **example of the fast series convergence** noted earlier. If we specialize and set f(ξ) = 1 along the top edge (0 elsewhere), then only the odd terms in the integral shown at the right side of 6.117 survive, so we get Σ_m with m = 1,3,5.. and so on. If we keep just the first two terms in this sum, we can compute u(**r**) in the middle of the rectangle from 6.117 with result p 158 E. Stak claims that the correct answer is 0.25, and here is what Maple says about our first two terms:

```
evalf(  (4/Pi)*(  sinh(Pi/2)/sinh(Pi) - 1*(1/3)*sinh(3*Pi/2)/sinh(3*Pi)    ) );
```

first term only = .2537164166
first two terms = .2499040972

Example 2. Green's function for the unit circle. (159)

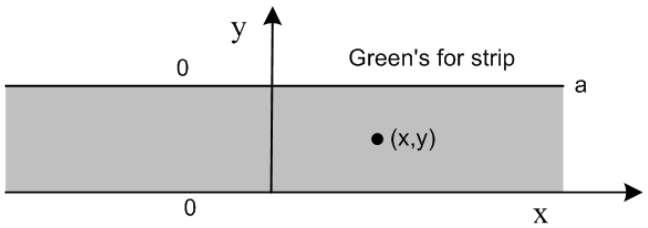


This is just turning the crank on a similar problem, though here variables r and φ are not on an equal footing as they are in the rectangle. He chooses the $f_n(\varphi) = e^{in\varphi}$ as the basis functions. Here is a little tour: [g(**r**|**r'**) = g(r.φ; r',φ')]

6.118	original PDR
B	projection of g onto $e^{in\varphi}$ giving $g_n(r; \mathbf{r}')$
D	the 1D Green's ODE for $g_n(r; \mathbf{r}')$
B,C (160)	solution for $g_n(r; \mathbf{r}')$ where $r_<$ = min(r,r') etc [he calls r' = r_0]
F	final result for g with φ' = 0
6.119	final result for g, our single sum
I	computing I = $-\partial_{r'}$ g(r.φ; r',φ') then setting r'=1
J	insert I into the usual Dirichlet sourceless integral
	(if we then did the sum, we would have 6.11, but Stak does not do this

He does not retreat this problem starting with the $h_n(r)$ functions, I think that is an exercise.

Example 3. Green's function for two horizontal plates with a line charge (2D strip). (161)



(a) The new feature here is that are strip is infinitely long. We select the pseudo-eigenfunctions $e^{i\alpha x}$ in this "direction" to work with, spectrum is α = any real number. [$g(\mathbf{r}|\mathbf{r}') = g(x.y; x'=\xi,y'=\eta)$]

6.120	original PDE with variables x and y
6.121	projection of g onto $e^{i\alpha x}$ giving $g_n(y; \mathbf{r}')$ [he calls this $(1/2\pi)$ g^(y;**r**')]
D	the 1D Green's ODE for g^(y;**r**')
A (162)	solution for g^(y;**r**') where $y_<$ = min(y,y') // he calls y' = η
C	final result for $g(\mathbf{r}|\mathbf{r}') = g(x,y; x'=\xi,y'=\eta)$: it is an integral instead of a single sum
6.125	compute $I = -\partial_{y'} g(x.y; x',y')$ then setting y'= a; get result as integral
6.126	how we would use this I in a sourceless Dirichlet problem

(b) Now in this example, Stak DOES redo the entire problem with the OTHER set of eigenfunctions

F	projection of g onto $f_n(y)$ giving $g_n(x; \mathbf{r}')$
A (163)	the 1D Green's ODE for $g_n(x; \mathbf{r}')$
B	solution for $g_n(x; \mathbf{r}')$, depends only on y' because I think x' = 0 at this point
C	final result for $g(\mathbf{r}|\mathbf{r}')$ with x'=0 still
6.127	final result for $g(\mathbf{r}|\mathbf{r}')$
6.128	compute $I = -\partial_{y'} g(x.y; x',y')$ then setting y'= a; get result as a sum
F	another way to write this I
6.129	yet another way to write this I, this is the best way he claims

(5) Complex Variable Method for 2D (164)

(a) Conformal mapping of a region R containing z_0 to the unit disk containing 0.

The **Riemann Mapping Theorem** which is mentioned at the start of Ahlfors Chapter 6 (which I have not read yet, having read only Ch 1 thru 3), says this: in the complex plane, given any simply-connected region R, and given an arbitrary point z_0 lying *inside* R, there exists an analytic mapping f(z) (1-1) which will map R into the unit disk, and will map z_0 into the origin.

It does not say how to find this mapping f. However, we are invited to think of the mapping as being in two parts that you concatenate. Assume your original point is z_0 inside R. You first find a mapping f_1 that takes R into the unit disk and takes z_o in R to some location $z_0' = f_1(z_0)$ inside the unit disk. Once you have done that, there is an obvious second mapping f_2 which maps the unit disk into itself *and* brings z_0' to the origin, so that $f_2[z_0'] = 0$. That mapping f_2 is claimed to be this:

$$f_2(z) = e^{i\alpha} (z-z_0') /(1-z\bar{z}_0') \qquad \bar{z}_0' \equiv 1/z_0'$$

I show in the raw notes that this mapping does what we want. So the concatenated mapping is ($f_1 \equiv s$)

$$f(z,z_0) = f_2[f_1(z)] = e^{i\alpha} (f_1(z)- f_1(z_0)) /(1- f_1(z) \bar{f}_1(z_0) = e^{i\alpha} (s(z)- s(z_0)) /(1- s(z) \bar{s}(z_0))$$

(b) If we find the mapping f(z) which does as required above, then the amazing result is that we then know the Green's Function for region R!!! It is just this

$$g(\mathbf{r}|\mathbf{r}_0) = -(1/2\pi)\ln|f(z,z_0)| \qquad \mathbf{r} = (x,y) \qquad z = x+iy \qquad \text{etc}$$

In the raw notes, I shows that g is harmonic within R (being the real part of an analytic function ln[f(z)]), that g = 0 on the boundary ($f(z_\sigma, z_0) = e^{i\psi}$ so |f|=1 and ln1 = 0) , and that g satisfies the Green's equation which is $-\nabla^2 g(\mathbf{r}|\mathbf{r}') = \delta^2(\mathbf{r}-\mathbf{r}')$.

This really is an impressive result, I wonder who first discovered it? A quick scan shows Green's Functions existed around 1830, but the mapping theorem was only stated in 1851, so perhaps soon after 1851.

As the most basic example, the Green's Function for a unit circle containing point z_0 has s(z) = 1 so is given by

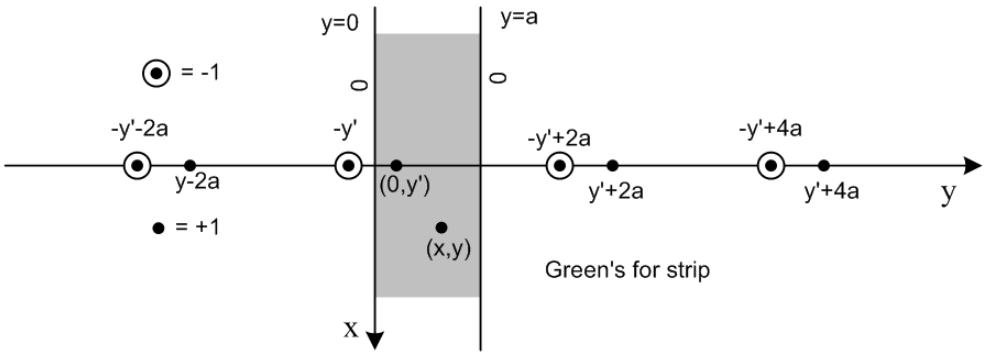
$$g(\mathbf{r}|\mathbf{r}_0) = -(1/2\pi)\ln|f(z,z_0)| = -(1/2\pi)\ln|f_2(z,z_0)| = -(1/2\pi)\ln|(z-z_0)/(1-z\bar{z}_0)|$$
$$= -(1/2\pi)\ln|z-z_0|/|1-z\bar{z}_0| = -(1/2\pi)\ln|z-z_0|/(|z-1/\bar{z}_0||\bar{z}_0|)$$
$$= -(1/2\pi)\ln|z-z_0|/(|z-z_0^*||z_0|)$$
$$= -(1/2\pi)\ln|z-z_0| + (1/2\pi)\ln|z-z_0^*| + (1/2\pi)\ln|z_0|$$

which replicates our image method solution of page 149 H.

At this point, I switched over to a new raw notes document "raw 2" since the first was getting very long. This new document opens with comments on the following 12 problems. I read and thought about each of the problems, but I only really did about half of them.

Exercises (166)

Exercise 6.35 : Method of images for a strip (2D) (166)



This problem consumed a lot of time, and I wrote up a whole separate doc on it ("exercise 6_35 v2.doc") and here I just quote some of the results. For the strip as shown above -- which is really a horizontal strip y=0 to y=a, when axes are correctly oriented -- the Green's Function is given by each of these forms:

$$g(x,y\,|0,y') = -(1/2\pi)\Sigma_n \ln\left[x^2 + (y - y'+2na)^2\right] + (1/2\pi)\Sigma_n \ln\left[x^2 + (y+y'-2na)^2\right]$$
$$= -(1/2\pi)\Sigma_n \ln\left\{\left[x^2 + (y - y'+2na)^2\right] / \left[x^2 + (y+y'-2na)^2\right]\right\}$$

$$= -(1/2\pi) \ln \Pi_n \{ [x^2 + (y - y'+2na)^2] / [x^2 + (y +y'-2na)^2] \}$$

$$g(x,y\,|0,y') = -(1/4\pi) \ln \{ [ch(\pi x/a) - \cos(\pi(y-y')/a))] / [ch(\pi x/a) - \cos(\pi(y+y')/a)] \}$$

$$g(x,y\,|0,y') = \Sigma_{n=1}^{\infty} (1/n\pi)\, e^{-(n\pi/a)|x|} \sin(n\pi y/a) \sin(n\pi y'/a) \qquad // = 6.127$$

The first is due to the method of images which forces g=0 on the two strip edges. I completely understand this form of the solution, it makes 100% sense, see raw notes which show charge pairings. It is trivially rewritten in the second form shown, and then a sum of logs becomes the log of a product to get the third form. It turns out this infinite product is of a standard form which allows it to be written in a nice closed form which is shown in the fourth line above. I wrote up details in "theory of infinite products.doc" which derives lots of product formulas, such as

$\sin(\pi z) = z\,\pi\,\Pi_{n=1}^{\infty} (1-z^2/n^2)$ " Euler's product for the sine "
$\cos(\pi z) = \Pi_{n=1}^{\infty}(1- z^2/a_n^2)$ $a_n = (2n-1)/2$

$\sin(\pi x)/\sin(\pi y) = (x/y)\,\Pi_{n=1}^{\infty}[n^2-x^2]/[n^2-y^2]$ two-sine
$[\sin(\pi x_1) \sin(\pi x_2)] / [\sin(\pi y_1)\sin(\pi y_2)] = \Pi_n [(n-x_1) (n-x_2)] / [(n-y_1) (n-y_2))]$ four-sine

$$\Pi_n [(n-a)^2 + b^2] / [(n-c)^2 + d^2] = [\sin(\pi(a+ib)) \sin(\pi(a-ib))] / [\sin(\pi(c+id)) \sin(\pi(c-id))]$$

The last line here is the one which applies to our current case, and it is just a special case of the four-sine formula shown directly above it. The names Euler, Hadamard and Weierstrass are associated with this work.

The last line above shows the solution written as an expansion on the transverse eigenfunctions of the strip. Yet another form is shown p 162 as 6.123 where we expand instead on the longitudinal eigenfunctions. In this direction, the interval is infinite and we get a continuous real eigenvalue spectrum.

Finally, in document "greens functions and change of coordinates.doc" I write the above g formulas for a more standard positioning of the strip

$$g(x,y|\,x',y') = (1/4\pi)\Sigma_n \ln \{ [(y-y')^2 + (x - \{x'+2na\})^2] / [(y-y')^2 + (x + a - \{-x'+2na\})^2]$$

$$g(x,y|\,x',y') = -(1/4\pi) \ln \{ [ch(\pi(y-y')/a) - \cos(\pi(x-x')/a))] / [ch(\pi(y-y')/a) + \cos(\pi(x+x')/a)] \}$$

$$g(x,y|x',y') = \Sigma_{n=1}^{\infty} (1/n\pi)\, e^{-(n\pi/a)|y-y'|} \sin(n\pi[x+a/2]/a) \sin(n\pi[x'+a/2]/a)$$

where the strip is vertical running from x = -a/2 to +a/2, and the Green's charge is at (x',y').

Exercise 6.36 : Show surface charge same in both strip solutions. (166)

This is just a technical exercise giving us some experience working with the function I instead of g. We compute this "surface charge" function I in two ways and show we get the same thing either way. It is of course nice to see an expression for the surface charge along metallic strip edges in the presence of an in-strip Green's charge.

Exercise 6.37 : Integral Equation method on the strip. (167)

The "integral equation method" has us solve a certain integral equation for a function called "I", and then we insert the solution "I" into 6.96 to get our Green's function g. The integral equation for "I" has this general form, where E is the usual 2D fundamental solution,

$$E(x|t) = \int_{\sigma} dS_{\xi}\, E(x|\xi)\, I(t|\xi) \qquad x \text{ on } \sigma \quad \xi \text{ integrate on } \sigma, \ t \text{ inside } R$$

Since the strip has two edges, we end up with two coupled integral equations for I

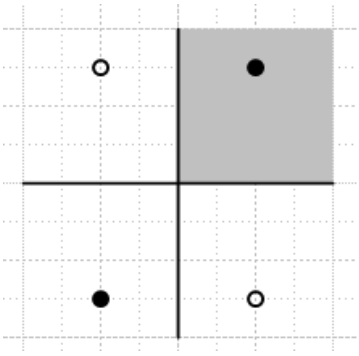
$$E(x,0|t) = \int dx'\, E(x,0|x',a)\, I(t|x',a) \; - \int dx'\, E(x,0|x',0)\, I(t|x',0)$$

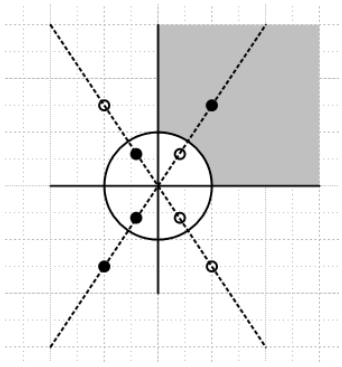
$$E(x,a|t) = \int dx'\, E(x,a|x',a)\, I(t|x',a) \; - \int dx'\, E(x,a|x',0)\, I(t|x',0)$$

where the left integral is on the y=a edge and the second on the y=0 edge. We are told to diagonalize these convolution-form equations using the Fourier Transform, solve the problem in Fourier space, then come back and we have a result for I and from that we get g. I did not "do" this problem.

Exercise 6.38 : Find some image charges to match a certain 2D region. (167)

These were interesting cases, I just show some pictures:





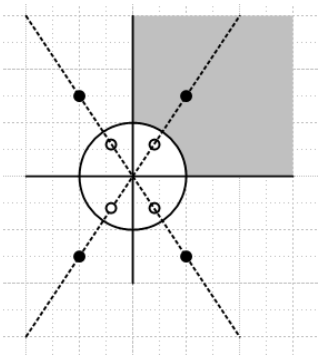
In both cases, we want g = 0 on the boundary σ of our gray region.

Exercise 6.39 : Partial Expansion Solution of Unit Disk using Radial Eigenfunctions (167)

Stakgold comments on this particular ODE (2D Laplace radial equation) at many places in his two volumes, and I first review what we learn there. We then show that for this problem the radial eigenfunctions are (sin[νln(r)]) and the final Green's g can be written as 6.132, where we "expand" on these eigenfunctions. As with any infinite interval [that is, r=0,∞], the eigenvalue ν is continuous. I did not do the problem, but followed all his text on it.

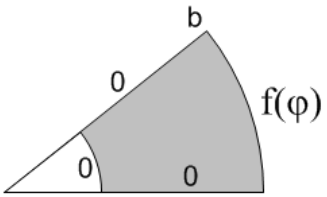
Exercise 6.40 : A Mixed BC Green's Function example : Dirichlet and Neumann (168)

The image charge solution is this picture:



where now we want $\partial_{\mathbf{n}} g = 0$ on the straight boundaries, and $g = 0$ on the circular one, so this is a "mixed" BC problem. We are then asked to solve this with a φ basis function expansion, which I did not do.

Exercise 6.41 : Curved Quadrilateral Green's "both ways", r and φ expansion (168)



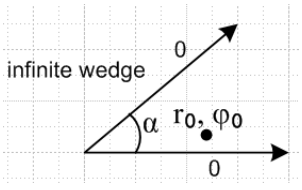
$f(\varphi) = 0$ for the Green's Problem

This is a very well defined problem and I should do it some day.

Exercise 6.42 : Unit Sphere Green's Function by the Y_{lm} expansion method. (169)

We did the unit sphere Green's on page 150 using the image method and quickly got a solution there as 6.102. Here we are asked to do the problem "the standard way", another one for my rainy day collection.

Exercise 6.43 : Infinite Wedge, do it both ways (2D) (169)



Using the φ eigenfunctions functions we have a Fourier Sine Series job, and using the radial eigenfunctions, we have a Mellin Transform job. I understand the problem, but I did not do it.

Exercise 6.44 : Half-plane as starting region or as an intermediate region (170)

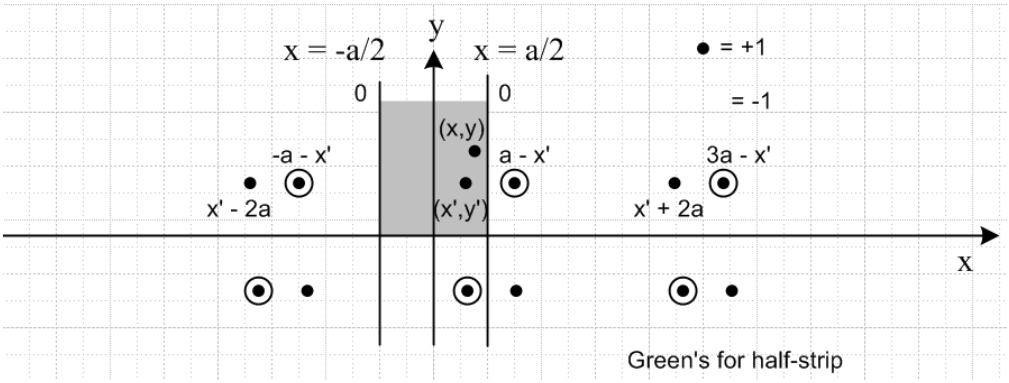
We are shown how to map the upper half plane into the unit disk. If we can find a transform s(z) to take some arbitrary region R to the upper half plane, then we can concatenate to get a map from R to disk. The

concatenated formula is $f_3(z) = e^{i\alpha}\ (s(z)- s(z_0))\ /(\ s(z)- \bar{s}(z_0))$. This of course implies a certain form for the Green's Function on R.

$$g(\mathbf{r}|\mathbf{r}_0) = -\ (1/2\pi)\ \ln |\ f_3(z,z_0)\ |\ = -\ (1/2\pi)\ \ln |\ \ (s(z)- s(z_0))\ /(\ s(z)- \bar{s}(z_0))\ |$$

Exercise 6.45 : Vertical half-strip of width π, done 4 ways (170)

In (a) we are supposed to do this by the image method, which is quite easy,



Using appropriate adjustments of results previously obtained (coordinate transforms on Green's) this gives the following forms for g:

$$g(x,y|\ x',y'\ ) = (1/4\pi)\Sigma_n\ \ln\ \{\ [\ (y-y')^2 + (x - \{x'+2n\pi\})^2\ ]\ /\ \ [(y-y')^2 + (x + \pi - \{-x'+2n\pi\})^2\ ]$$
$$*\ \ [(y+y')^2 + (x + \pi - \{-x'+2n\pi\})^2\ ]\ /\ \ [\ (y+y')^2 + (x - \{x'+2n\pi\})^2\ ]\ \ \}$$

$$g(x,y|\ x',y'\ ) = -(1/4\pi)\ \ln\ \{\ \ [ch(y-y') - \cos(x-x')]\ /\ [ch(y-y') + \cos(x+x')]$$
$$*\ \ [ch(y+y') + \cos(x+x')]\ /\ \ [ch(y+y') - \cos(x-x')]\ \}$$

$$g(x,y|x',y'\ )\ = \Sigma_{n=1}^{\infty}\ (1/n\pi)\ \{\ e^{-n|y-y'|} -\ e^{-n|y+y'|}\ \}\ \sin(n[x+\pi/2])\ \sin(n[x'+\pi/2])$$

In (b) we are supposed to solve this doing a transverse sine eigenfunction expansion. Since I already knew the answer (last shown above), I did not bother to do this.

In (c) we are supposed to do it with the longitudinal eigenfunctions which are sin(νy) where ν is a continuous eigenvalue. I did this application of the "partial eigenfunction expansion" in full detail and I got this answer, which I don't know whether is correct or not:

$$g(x,y\ |x',y') =\ \ (2/\pi)\ \ \int_0^{\infty} d\nu\ \ \sin(\nu y)\ \sin(\nu y')\ \ \sin(\nu(x_<+\pi/2)\ )\ \sin(\nu(x_>-\pi/2)\ )\ /\ \ (\ \nu\ \sin(\pi\nu)\ )$$

In (d) we are supposed to find g using conformal mapping. We find that f(z) = sin(z) maps our strip (with a = π) into the upper half plane, so we do the concatenation trick just shown in 6.44 above to get the Green's g. I did this in detail, and got a result which matches my middle result above.

Exercise 6.46 : Modified Green's function for Neumann Boundary Conditions (171)

The Neumann BC forces a consistency condition which in turn requires that one work with the a certain "modified" Green's Function equation and solution -- something we did way back in Section 1.6 of Vol I. we are told to use this method to find Green's for the Neumann-BC unit disk. The answer is given in 6.137, and no, I did not do the problem.

6.8 Some Physical Applications of Potential Theory (171)

A. Charged Conductor and Electrostatic Capacity (3D) (171)

If you place some charge Q>0 on a metal object and ask how it distributes itself, there are **several paths** you could take. This is what I call a "Problem Two" in the meta-meta notes.

{wrong!} **"One approach** is to find the **Green's Function** for the exterior Dirichlet problem (using methods discussed in the last section), appropriate since the boundary is metal. Recall that $\Sigma(\xi) = \partial_{\mathbf{n}} u|^{\mathbf{in}} - \partial_{\mathbf{n}} u|^{\mathbf{out}}$ from Section 6.5 above (n points out). For the exterior problem, $u^{\mathbf{in}} = 0$ so we have $\Sigma(\xi) = -\partial_{\mathbf{n}} u|^{\mathbf{out}} = -\partial_{\boldsymbol{\xi}\mathbf{n}}\, g(x|\xi) \equiv I(x|\xi)$ by Stak. So we can summarize this approach as follows: (1) find the exterior Green's Function any way you can; (2) $\Sigma(\xi) = -\partial_{\boldsymbol{\xi}\mathbf{n}}\, g(x|\xi)$, all done. Our surface σ is regarded as **closed** here. If it is open, then I would think of both sides as being part of an exterior problem and the object itself is then a closed boundary with a very thin interior. "

Correction: The above paragraph is **wrong** because it describes finding the charge on the exterior of a metal object which is induced by the presence of a Green's point charge at exterior location x. We really want the charge on the metal object thought of as a capacitor relative to a great sphere of uniformly distributed charge! These are totally different situations!!! So maybe this integral equation method is the only method for finding the answer! It is the only method Stakgold provides!

Another approach to finding the surface charge distribution is to use our "**integral equation method** of potential theory" where we postulate that we can model the potential of a Problem One with an *equivalent* surface charge I like to call $\Sigma(\boldsymbol{\xi})$. For Problem One, this is just some monopole layer function that generates a specified $f(\mathbf{s})$ on σ. This is a Dirichlet problem and we have from above

$$\lim_{\mathbf{x}\to\mathbf{s}} u(\mathbf{x}) = \int_{\sigma} dS_{\boldsymbol{\xi}}\, \Sigma(\boldsymbol{\xi})\, E(s|\xi) = f(\mathbf{s}) \qquad \text{inhomo Fred 1} \qquad \text{monopole layer}$$

$$u(\mathbf{x}) = \int_{\sigma} dS_{\boldsymbol{\xi}}\, \Sigma(\boldsymbol{\xi})\, E(x|\xi) \qquad 1/4\pi R(x,\xi) = E(x|\xi)$$

In Problem One, $\Sigma(\boldsymbol{\xi})$ is just "a function", it is not a real surface charge. It is an intermediate tool such that we solve the first equation above for $\Sigma(\boldsymbol{\xi})$, then insert that into the second and obtain $u(\mathbf{x})$ that meets the requirement $u(\mathbf{s})$ = specified $f(\mathbf{s})$. We know that there is a related Problem 2 in which $\Sigma(\boldsymbol{\xi})$ is the *actual* sticky surface. For this problem, we recognize the second line above as just a superposition of point charge potentials.

Now Stak suggests we use this exact integral equation method to solve our **Problem Two** by setting **f(s) = constant.** After all, Σ(s) is what we are after, so why not solve for it directly, then we don't need to know u(**x**), though it is available from the second equation above.

When we talk about "adding charge" to a metal object, we are really talking about forming a **capacitor** with that object as one electrode and the great sphere as the other. We will end up with the metal object sitting at some +V/2 and the great sphere (from which we obtained the charge to put on the object) sitting at some -V/2. For this capacitor, we have as usual that Q = CV. In our problem Q is specified, there is some C, and in the end there will be some potential (1/2) Q/C on the metal object. Another way to think of this is that we know $\int_\sigma dS_\xi\ \Sigma(\xi) = Q$. So if we can find Σ(ξ) as a functional form, we can scale it later to get the right Q, and then we will have the right Σ(ξ). For this reason, we can just **set f(s) = 1 on the metal object.**

So our entire problem is now just to solve this integral equation for Σ :

$$\int_\sigma dS_\xi\ \Sigma(\xi)\, E(s|\xi) = 1\ (= f(s)) \qquad E\,\Sigma = 1 \qquad \text{first kind Fred}$$

Stak then goes on to develop *another* integral equation for Σ(ξ). Recall our surface boundary rules from earlier (replacing a by Σ)

$$u(\mathbf{x}) = \int_\sigma dS_\xi\ \Sigma(\xi)\, E(x|\xi)$$

$$\lim_{\mathbf{x}\to\mathbf{s}} \partial_\nu u = \mp\,\Sigma(s)/2 + \int_\sigma dS_\xi\ \Sigma(\xi)\, k(\xi,s) \qquad k(\xi,s) = \cos(\xi\text{-}s,\nu)/\,[4\pi|\mathbf{x}\text{-}\xi|^2]$$

Recall also that ν is the outward normal at s, so the LHS here is in fact - Σ(s) on the outer surface of our object. And we are approaching from the outside so we have the upper sign. Thus,

$$-\Sigma(\mathbf{s}) = -\Sigma(s)/2 + \int_\sigma dS_\xi\ \Sigma(\xi)\, k(\xi,s)$$

$$\int_\sigma dS_\xi\ \Sigma(\xi)\, k(\xi,s) = (-1/2)\,\Sigma(\mathbf{s}) \qquad K\Sigma = \mu\Sigma \qquad \mu = -1/2 \qquad \text{second kind Fred homo}$$

At this point Stak discusses doing the above process on an **open surface**. We obtain the first kind Fred shown above as our integral equation, where Σ(ξ) is the sum of the charge densities on the two sides of the surface. The second kind Fred does not work out because we get one equation in two unknowns. This section on open surfaces does not do much for me.

Having set us up with integral equations for the 3D problem, Stakgold never does any examples. He then switches to the 2D situation, and then presents three 2D examples!

B. Charged Conductor in 2D (174)

Specifically, we are dealing here with a 2D ring of charge Q = 1 radius 1. $\int_C d\boldsymbol{\ell}\,(-\partial_r u) = 1$ just says the charge adds up to +1 since $-\partial_r u = \Sigma$. But this is true for any larger circle as well, just saying charge enclosed is 1. If you apply this rule to the general 2D "form" for u, you discover that there ***must* be a log(r) term**, and it *must* have coefficient $-\pi/2$. This all applies not just to a 2D ring of course, but to a 2D σ boundary of any bounded shape we shall call C, so let's assume the general case from now on.

Now we consider the **region R bounded by C and some large circle**. We take u = 0 on C since metal (our choice of the f(s) = constant for this problem), and for ξ in this region R we find

$\int dV\,[\,u\nabla^2 E - E\nabla^2 u] = -u(\xi)$ since $\nabla^2 u = 0$ and $\nabla^2 E = -\delta$. This is the LHS of a Green #2 whose RHS is an outer-ring integral plus an integral over C of $(u\partial_n E - E\partial_n u)$. where of course the first term vanishes since u = 0 on C. Stack shows that the outer-ring integral as $r\to\infty$ is exactly equal to c_0 , the constant term we assume in our general "form" for u. We thus get:

$u(\mathbf{x}) = c_0 + \int_C d\boldsymbol{\ell}_\xi\; E(x|\xi)\,\Sigma(\xi)$ // result here for 2D problem 6.146

$u(\mathbf{x}) = \int_\sigma dS_\xi\, E(x|\xi)\,\Sigma(\xi)$ // previous 3D result shown above (ref?)

The issue here is that E ~ ln(R) in 2D which is a "long range" effect, the great circle has a Green #2 integral c_0 which is not arbitrary but is part of the solution to the problem. If you insist that u(s) = K on your conductor, then c_0 will have to be the right value to make this be true.

If we take $x\to s$ as usual, the first line above becomes

$0 = c_0 + \int_C d\boldsymbol{\ell}_\xi\; E(s|\xi)\,\Sigma(\xi)$ $\qquad E\,\Sigma = -c_0$ first kind Fred

The second kind Fred has the exact same form we got before since $\partial_\nu c_0 = 0$, I just copy it down

$\int_\sigma dS_\xi\,\Sigma(\xi)\,k(\xi,s) = (-1/2)\,\Sigma(\mathbf{s})$ $\qquad K\Sigma = \mu\Sigma$ $\qquad \mu = -1/2$ second kind Fred homo

but of course in the 2D case we have $k(\xi,s) = \cos(\xi\text{-}s,\nu)/\,[2\pi|\mathbf{x}\text{-}\xi|]$ instead of the 3D kernel shown above.

Example 1: Charged circular cylinder (2D circle)

He is solving for $\Sigma(\mathbf{s})$ using the first kind Fred integral equation shown above. With acrobatics, he diagonalizes the thing and "discovers" that $\Sigma(\mathbf{s})$ (= I(s) which is the symbol he uses) is a constant. In fact he finds that $\Sigma(s) = c_0/\,[a\ln(a)]$ $(a \neq 1)$ where a is the radius of our cylinder. If we normalize Σ to Q total, we then find that $c_0 = Q\ln(a)/2\pi$ (but he uses Q = 1). He then computes u(x) and gets $u(x) = (Q/2\pi)\ln(r/a)$. So u = 0 on the ring, and u is log divergent as $r\to\infty$ as noted above. You cannot have a "finite capacitor" consisting of this ring and the great circle.

However (my idea), suppose we superpose this problem with a problem which has -Q on a larger ring at radius b. Then u = 0 on the inner ring, but $u = (Q/2\pi)\ln(b/a)$ as you approach the outer ring from inside. Since Q = CV, you would say the capacitance was $2\pi/\ln(b/a)$. [we are in units where $\varepsilon_0 = 1$]

At the end of the raw2 notes on this Example, I comment on how this problem relates to HS kernel and SO(2) convolution diagonalization. In this case, the notion of a separable kernel diagonalization and SO(2) group diagonalization align with each other. The chosen basis functions are sines and cosines.

Example 2: Strip of width 2a (2D line segment on x axis -a to a) (178)

As in the previous example, we use $0 = c_0 + \int_C d\ell_\xi \; E(x|\xi)\,\Sigma(\xi)$ (the first kind Fred). I might have chosen C as a contour around the thin wire, but he uses his "open boundary" method so $\Sigma(\xi)$ will be the sum of charges on both side (of course we know each side will have the same density).

Now, this first-kind Fred integral equation has to be diagonalized. I might NOW approach this in a more group-theoretic manner, but I invested lots of time doing it his way so that is fine. The integral equation is shown p 179A with a huge constant expression on the LHS I call k. This form matches 6.190 of a later exercise, and that exercise shows that the solution to this integral equation is 6.195 which he quotes in p 179 B, and bang, out pops the important shape of the charge density $1/\sqrt{1-\tau^2}$ which is infinite at the ends but is integrable, and I plot this in my raw2 notes. Result normalized to Q=1 in 6.153. He finds in this problem that $c_0 = (Q/2\pi)\ln(a/2)$.

I took a **huge digression** ("independent study") at this point and tried to compute the potential u(x) for this problem. This led to multiple external documents and lots of burned time, and here is the answer:

$$u(x,y) = -(1/4\pi)\ln(2) - (1/4\pi)\ln\left(z + \sqrt{z^2-1}\right) + \text{c.c.} \qquad z = x + iy$$

$$= -(1/2\pi)\ln(2) - (1/8\pi)\ln\left|z + \sqrt{z^2-1}\right|$$

where I often just used a = 1 as here. I showed that this expression is constant on the wire, and I did a lot of other checks on it as summarized in the raw2 notes section on page 35 with this heading

Summary of external notes. "potential of extruded strip v2.doc"

I then made Maple plot the contours of constant u, and they sure looked like ellipses to me but I could not prove this for the longest time! I had to go write " find 2D wire equipotential curves in x y.doc " before I learned that $|z + \sqrt{z^2-1}| = K$ is an ellipse! I showed this was an ellipse by *brute force* in this document. I was puzzled since I knew $|z-1| + |z+1| = C$ was an ellipse with a simple geometric interpretation, but I could not construct any geometry where $z + \sqrt{z^2-1}$ was useful in that geometry. But later in math/ellipses.doc I show that you can consider the transformation

$$s = z + \sqrt{z^2-1} \qquad \text{with inverse} \qquad z = \text{ch}[\ln(s)] = (1/2)(s + s^{-1})$$

I showed that in s-space these two equations both describe the same circle

$$|s| = K$$

$$|s + s^{-1} + 2| + |s + s^{-1} - 2| = 2(K + K^{-1})$$

and when you map these into z-space, you get

$$| z + \sqrt{z^2-1} | = K$$

$$|z-1| + |z+1| = K + K^{-1}$$

Since the second it an ellipse, the first must be the same ellipse, QED.

Conjecture: I think if we map this entire problem into s-space, the locus of charge becomes the unit circle. The upper surface of the wire maps to the upper half circle, and the lower to the lower. The contours in s space are just circles, and they map into ellipses in z space. Much later I found a plot in Purcell showing the contours for this problem, so he must have "enjoyed" this strange problem as much as I did.

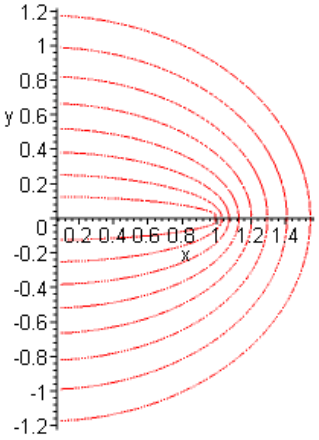
At the end of the raw2 notes on *this* Example, I comment on how this problem's integral equation gets diagonalized by a separable kernel form using sine and cosine functions, but I am unable to associate this diagonalization with any group convolution form. I was a little surprised by this fact.

Here are some plots of this problem. The first shows the ellipses of constant potential, while the second is a plot of $u(r,\theta) = | z + \sqrt{z^2-1} |$ (the actual solution has a minus sign. Thus, the plot below will be for negative charge on the wire)

```
restart;
f := Re(ln(z + sqrt(z^2-1))):
z := x + I*y:
f;
```

$$\ln\left(\left| x + Iy + \sqrt{(x+Iy)^2 - 1} \right|\right)$$

```
with(plots):implicitplot({seq(f=N/8,N=1..8)}, x=0.1..2, y=-2..2,grid = [100,100],
scaling=CONSTRAINED);
```



```
x := r*cos(theta);
```

$$x := r\cos(\theta)$$

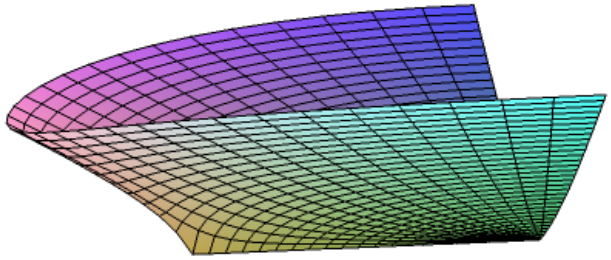
```
y := r*sin(theta);
```

$$y := r\sin(\theta)$$

```
f;
```

$$r\cos(\theta) + I\,r\sin(\theta) + \sqrt{(r\cos(\theta) + I\,r\sin(\theta))^2 - 1}$$

```
plot3d(Re(abs(f)), r=0..1.5, theta=-Pi/2+.001..Pi/2, coords=z_cylindrical,
numpoints=1000);
```



Example 3: Semi-infinite strip (now our 2D wire runs x=0,+∞) (178)

We think first of the full-infinite strip, meaning a full-infinite wire in 2D. We know this problem will have a uniform charge distribution that will not be integrable, and we know that this is really a 1D problem where E(y) = -(1/2)|y| so this is basically u(x,y) and things are not very interesting.

We then consider a situation with half this wire, starting at the origin. Whereas in the previous two examples Stak used the "integral equation method" to find the charge distribution (and potential), here he sets things up as a "form fit" problem in polar coordinates. One suspects that the finite-wire example could also have been done in this way, but he was trying to illustrate the integral equation method.

So when the dust cloud clears here, we find this solution for the half-wire

$u(r,\varphi) = Ar^{1/2}\sin(\varphi/2)$

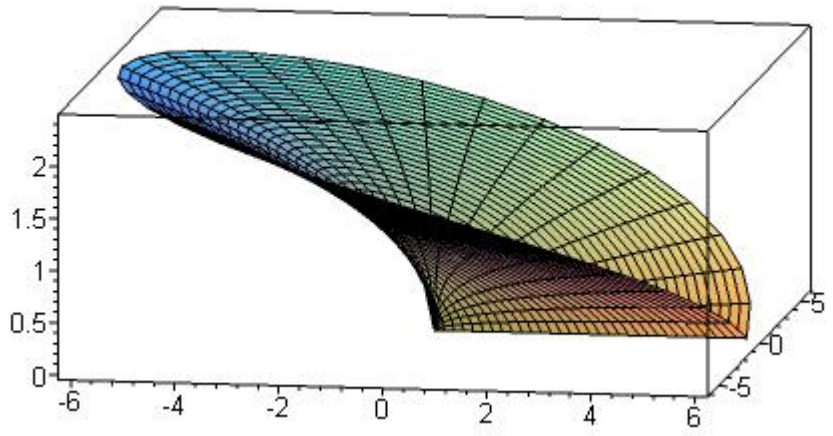
$\Sigma^{+}(x) = -\partial_n\, u(r,\varphi)|^{\varphi=0} = -1/r\,\partial_\varphi\,[Ar^{1/2}\sin(\varphi/2)]\,|^{\varphi=0} = -r^{-1/2}(A/2)\cos(\varphi/2)\,|^{\varphi=0} = -r^{-1/2}(A/2)$

and we get the same charge on the bottom somehow, so total charge is $\Sigma(x) = -Ax^{-1/2}$.

Here is a plot of $u(r,\varphi)$ with A = +1. Since this causes positive slopes coming onto the wire, this corresponds to a negative charge density on the wire. You can see how these slopes get very steep as we approach the end of the wire which is at the origin of this picture. The picture looks as shown no matter how far out you plot r. He comments that you cannot determine a value for A by integration over the half wire since $\int \Sigma(x)dx$ diverges at the far end. But we do see exactly the same behavior at the end of the wire that we saw in the finite wire situation.

```
>readlib(addcoords)(z_cylindrical,[z,r,theta],[r*cos(theta),r*sin(theta),z]):
```

>plot3d(r^(1/2)*sin(theta/2), r=0..6, theta=0..2*Pi, coords=z_cylindrical, numpoints=1000);



You can see the similarity between this plot and that of the finite wire shown above, onto which we have put some negative charge.

C. Grounded Conductor in an External Field 3D (174)

The raw notes are fine here, things are confusing only because there are so many symbols:

u_0	potential supplied by external source, could be anything
$u_s = u_e$	potential created *external* to object by induced charge on its surface
u_i	potential created *internal* to object by induced charge on its surface.
	Since $0 = u = u_0 + u_i$ inside object, we have $u_i = - u_0$
u	total potential when metal object is present, $u = u_0 + u_s$ (u=0 in and on object)

We can consider the problem as being one for u_s using our standard first kind integral equations:

$u_s(x) = \int dS_\xi E(x|\xi)\, I(\xi)$ // plug $I(\xi)$ in here to get $u_s(x)$ everywhere outside.

$-u_0(s) = \int dS_\xi E(s|\xi)\, I(\xi)$ // integral equation to find $I(\xi)$

where

$$I = \partial_n u_i - \partial_n u_e = - \partial_n u_0(x)^{x=s} - \partial_n u_s(x)^{x=s} = - \partial_n u(s) = \Sigma = \text{surface charge}$$

So solve the second equation for $I(\xi)$ since you know u_0, then plug that into the first equation to get u_s, then you know $u = u_s + u_0$, problem solved!

The second kind Fred version (for u_s) is this from above, take upper sign approaching from outside,

$\lim_{x\to s} \partial_\nu u_s = \mp \Sigma(s)/2 + \int_\sigma dS_\xi \, \Sigma(\xi)\, k(\xi,s)$ $\qquad k(\xi,s) = \cos(\xi-s,\nu)/[4\pi|s-\xi|^2]$

But $\partial_\nu u_s = \partial_\nu u - \partial_\nu u_0 = -\Sigma - \partial_\nu u_0$ on the surface, so have

$$-\Sigma(s) - \partial_\nu u_0(s) = -\Sigma(s)/2 + \int_\sigma dS_\xi \, \Sigma(\xi)\, k(\xi,s)$$

$$-\partial_\nu u_0(s) = \Sigma(s)/2 + \int_\sigma dS_\xi \, \Sigma(\xi)\, k(\xi,s) \qquad \text{// which is 6.160}$$

This is the Fred 2 inhomo, and the inhomo driving term is known since we know u_0.

Application to Sphere. Recall that in this case, $\cos\theta = -R/2a$ (I drew picture once), $k = -1/(8\pi aR)$. Then Fred 2 above reads:

$$-\partial_\nu u_0(s) = \Sigma(s)/2 + \int_\sigma dS_\xi \, \Sigma(\xi)\, \{-1/(8\pi aR)\} = \Sigma(s)/2 - (1/8\pi a) \int_\sigma dS_\xi \, \Sigma(\xi)/R$$

The Fred 1 meanwhile says

$$-u_0(s) = \int E(s|\xi)\, \Sigma(\xi) = (1/4\pi) \int dS_\xi \Sigma(\xi)/R$$

Since same integral appears in both, divide Fred 1 by 2a and add the two equations:

$$-\partial_\nu u_0(s) = \Sigma(s)/2 - (1/8\pi a) \int_\sigma dS_\xi \, \Sigma(\xi)/R$$

$$-u_0(s)/2a = (1/8\pi a) \int dS_\xi \Sigma(\xi)/R$$

$$-\partial_\nu u_0(s) - u_0(s)/2a = \Sigma(s)/2 \qquad \text{// which is p 183 A}$$

$$\Sigma(s) = -2\partial_\nu u_0(s) - u_0(s)/a$$

So we managed to not have to solve either integral equation! This is the charge distribution on a sphere for whatever external potential $u_0(x)$ you come up with! Of course you have to compute

$$\partial_\nu u_0 = \partial_n u_0 = \partial_r u_0 = (x/a)\, \partial_x u_0 + (y/a)\, \partial_y u_0$$

as I show in the raw1 notes.

(1) Suppose $u_0 = Ax$, so a **constant E field**. Then

$$\partial_\nu u_0 = (x/a)\, A$$

$\Sigma(s) = -2\ (x/a)\ A - Ax/a = -3\ Ax/a$

This problem appears in Jackson p 34 in 2.15 where he sets $A = -E_0$ and $\cos\theta = x/a$. Because Jackson uses things like $-\nabla^2 V = 4\pi\rho$, we always have $4\pi\rho_{jackson} = \rho_{stackgold}$ so we have to divide our result by 4π to get a Jackson result shown in 2.15. So our "3" here is correct.

(2) Suppose u_0 = **point charge** at r_0 from sphere center? The math is done in the raw notes and in this case we find that

$I = -(r_0^2 - a^2)/(4\pi a R^3)$

Jackson does this problem using images on page 29 where $r_0 = y$ for him, result in 2.5. The denominator he shows there must be R^3/y^3 so Jackson's result reads

$\sigma = -(q/4\pi)\ (a/y)\ (1 - a^2/y^2)\ (y^3/R^3) = -(q/4\pi)\ (a)\ (y^2 - a^2)\ (1/R^3) = -q(y^2 - a^2)/(4\pi a R^3)$

Since charge appears on both sides of this equation, there is no 4π difference. Notice that a "unit charge" has different meanings in these two units! Enough!

D. Steady Heat Conduction for "Composite Medium" (183)

These notes are verbatim from the raw2 notes:

Here "composite medium" is what I have drawn, k1 is thermal conductivity in uniform sub-region R_1 and so on. We want to solve for temperature u(x) everywhere. We assume perhaps some $u(\sigma)$ "matching conditions" : one is that the temperature is the same on both sides, the other is that the total heat flow out of the pillbox is 0, which is what 6.163 says. [we assume there are no heat sources in R] The problem is then to find u(x) everywhere inside R.

Another interesting way to set this problem up is to assume some $k(x,\varepsilon)$ where ε is a parameter and as $\varepsilon \to 0$, this $k(x,\varepsilon)$ gradually approaches the interface situation shown. We could then solve entire R without regard to the interface B, and then take the limit of that solution as $\varepsilon \to 0$. Of course solving for a general position-dependent k(x) probably requires more work, but it does seem a very reasonable method, and Stak seems to like it. Instead of Laplace, we have to solve 6.166 which I recall from his appendix on heat stuff.

Example: a one dimensional rod from along x = (-1,1). Here σ is the two end points and we set u = 0 at the left end and u = 1 at the right end. This of course sets up some steady heat flow through the rod which has some k(x). The entire problem is stated in p 184 A. He integrates the ODE once to get B, I agree. He then makes up a function $k(x,\varepsilon)$ which is smooth and which approaches a limit which makes our interface B be right at x = 0. So we have our general solution, and we now take the limit and somehow we get 6.167 which I did not do, and we have our solution on the two sides, and in fact he shows that the two matching conditions are in fact met. He then starts all over by the other method, we have Laplace then on both sides, we apply the BC's and get the same result 6.167. OK, I get it.

I think an electrostatic analog to this problem would be R = dielectric with ε_1 and ε_2 on the two sides. Probably V is continuous across the boundary, and so is D = εE, but not E, etc. The interface would have a kink in V, causing a field difference, and implying a bound charge layer on the interface.

E. Flow of an Ideal Fluid past an Obstacle (185)

Problem Setup. Stak has saved his fanciest most complicated example for his grand finale. The reader is supposed to understand that if you place on obstacle in a (very wide) laminar flowing river, you are somehow in the realm where the velocity vector is curl-free and so the **velocity vector v** can be represented as the gradient of a **velocity potential** u which satisfies Laplace, since our river contains an incompressible fluid. I got this all done in a separate document called "divergence J and V.doc" with confirmation from my Schaum fluid dynamics book. In this same document I cleared up my Torrey ideas on the analogy between current density and momentum density.

For the next step, one needs to put one's mind in a whole new physical application world. We are no longer doing electrostatics or heat flow. This river with obstacle has a solution u(**x**) and **v(x)** = ∇u at all points in the river. u is now the velocity potential, not the temperature, not the electrostatic potential. Since Laplace is linear, we can do superposition $u = u_0 + u_s$ just as we do in heat or electrostatics. For some reason, he refers to the no-obstacle solution as u_i instead of u_0 . He used u_0 in the related problem of a grounded conductor sitting in an initial "flow" of electrostatic potential u_0. But u_s is the "secondary solution", same name as before. If the river flows in the x direction (he does this since he is going to do a 2D example soon), and we know that $\mathbf{v}_i = \hat{\mathbf{x}}$, we know that $u_i = x$, pretty simple.

Now he states "formulations of the problem", one for u and one for u_s. A key idea is that at the surface of a boundary, **v** must be tangential only. Why? Why can't you have a little piece of fluid at a boundary with **v** pointing right at the boundary? If that were the case, that piece of fluid would flow right through the boundary at that point. That is what the velocity vector means. So we have $\partial_n u = 0$ on all boundaries (that are not sources or sinks). So boom, we have **Neumann boundary conditions**! Hurray.

So (6.169) is our PDE formulation for total velocity potential u, the external problem. You see the Neumann condition on σ, and the asymptotic condition that $u = u_i$ up and downstream.

And (6.170) is the formulation in terms of u_s. This is very similar to our metal object in E field example earlier. On σ, since $\partial_n u = 0$ (no perp velocity), we know $\partial_n u_s = -\partial_n u_i$. And of course the ∞ condition is as shown, $u_s = 0$. Stak points out that this u_s formulation defines a "standard Neumann problem" since $u_s(\infty) = 0$. This is the formulation we are going to try to solve. By the way, since **v** = 0 inside σ (see below), u = constant A there, so $u_s = A - u_i$ inside the obstacle. I suspect A = 0 is OK.

Large r behavior of $u_s(x)$. (186) This is a technical detail, the conclusion is that $u_s(r) \to 1/r^2$.

Develop Integral Equation. (186-7) In the usual way, considering the region between the obstacle and the river boundary, Stak expresses $u_s(\xi)$ in terms of Green #2 surface integrals of u_s in p 186 E. The σ_r integral vanishes because flow in = flow out at river boundary, leaving just the σ integral. Meanwhile, 6.172 is Green #2 on R_i inside the obstacle, applied to u_i, where ξ is NOT inside R_i, hence RHS = 0. When we simply add this equation to 186 E we get 186 F which rewrites as 7.173, and we see the $\partial_{n\xi}$E object inside which tells us this is a "v" type equation so if we take x→s, we get "the extra term" as shown 6.174 which rewrites as 6.175. We thus end up with a Fred 2 inhomo integral equation pair:

$$u_s(x) = \int_\sigma \partial_{n\xi} E(x|\xi)\, u(\xi) \qquad 6.173$$

$$-u_i(s) = -\,u(s)/2 + \int_\sigma \partial_{n\xi} E(s|\xi)\, u(\xi) \qquad 6.175 \qquad \text{Fred 2 inhomo}$$

So the plan is to solve this second equation for $u(\xi)$ on σ [since we know $u_i(x)$], then use the first equation to find $u_s(x)$ everywhere, and then final step is $u(x) = u_i(x) + u_s(x)$.

Usually we arrive at a Fred 1 equation first, but here we get a Fred 2 first. The reason is that in the Dirichlet situation we have dealt with up to now, we had $u_i = 0$ on σ, but here $\partial_n u_i = 0$ on σ since Neumann, and this changes "which term" in the surface integral stays alive.

Is there a first-kind Fred ? Stak differentiates 6.173 then evaluates that as $x \to s$ to get this result:

$$-\partial_\nu u_i(s) = \partial_\nu \left[\int_\sigma \partial_{n\xi} E(x|\xi)\, u(\xi) \right]^{x \to s} \qquad 6.176$$

noting that it is not really a first kind Fred integral equation. It sort of *looks* like one in that it has the form (something)u = f in the sense of Ax = y, but (something) is not really a linear integral operator.

Thin Shell "Integral Equation". (187-8). We can write 6.173 above for a thin shell in this way:

$$u_s(x) = \int_\sigma \partial_{n\xi} E(x|\xi)\, \Delta u(\xi) \qquad 6.177$$

where he now defines $I(\xi) = \Delta u(\xi)$, the jump in potential u across the thin shell. Notice that in the Dirichlet problems, this was always the jump in $\partial_n u$, but here it is just the jump in u, another result of our now doing a Neumann problem! So I is no longer a surface charge type thing, just a potential thing. In 187 B,C he then takes the limit of 6.177 for $x \to s$ on the two sides of the thin shell. All he can really do is add these to get 6.179 which is also not an integral equation. If we were to somehow find $\Delta u = I$, we could use 6.179 to detangle u on the two sides of the shell.

So how might we find I for our thin shell situation? Do our same ∂_ν application but now on 6.177 and then take limit $x \to s$ and we get (the two surface sides are now folded together in one integral)

$$-\partial_\nu u_i(s) = \partial_\nu \left[\int_\sigma \partial_{n\xi} E(x|\xi)\, \Delta u(\xi) \right]^{x \to s} \qquad 6.180$$

which is the same as 6.176 but we have $\Delta u = I$ sitting inside. He again says this is not an integral equation, but claims you *might* able to solve it for $I = \Delta u$.

Comments**:** This has ALL been a 3D discussion, see text near p 186 E. He claims in his first remark here that in the 2D world if you have a circular obstruction, the equations above the "thin shell" line on p 187 are still valid, but I would have to check that. The main question would be the vanishing of great circle things which we have seen might not vanish in 2D. His second remark is that everything goes through for any u_i we like, not just parallel flow.

Thin shell example: (a 2D problem!) River blocked by **partial cylinder** as shown picture p 189. This is the thin shell situation, it has symmetry. Little r,φ coord system set up at obvious point. I agree that $u_i = x = r\cos\varphi$ so $\partial_r u_i = \cos\varphi$ and thus $\partial_r u_s|^c = -\cos\varphi$ since $\partial_r u|^c = 0$. So we have a problem formulation as p 188 B.

He then writes general form interior and exterior u_s solutions for the imagined circle r = 1 which aligns with the partial cylinder with a_n and b_n coefficients. No lnr terms. We know $\partial_n u_s$ has no jump across the circle (because $\partial_n u_i$ has no jump, and $\partial_n u = 0$ on both sides), so conclude $b_n = -a_n$ and our general forms are now p 189 E and F. Tiny algebra then yields equation H which is then true for all φ. We know of course that for $|\varphi| \le \alpha$, the LHS = 0. H is going to get used heavily below.

As in our thin shell discussion above, we define I as the jump in u = jump in u_s across the circle, and from p 189 E and F we conclude 6.183 which gives I(φ) as a sum involving the a_n. Since this is just a Fourier Series, we compute the a_n as shown in p 190 B.

Now back to H for $|\varphi| \le \alpha$ where LHS = 0 as just noted. Write this as $\cos\varphi = -\Sigma_n n a_n \cos(n\varphi)$ and insert for the a_n the projections we just found in B, and we get intermediate result 6.184. Tweak this into p 190 C using the ∂_φ^2 trick to get convergence. But we know the n sum from our Exercise 6.47 work so we end up with 6.187 with m as shown in 6.188. He calls this an integrodifferential equation that maybe we could solve for I(ψ) on the partial cylinder. We know that I = 0 on the rest of the cylinder because u is continuous there, since just a math circle boundary. So we know I(φ) around the entire circle, we can compute the a_n from p 190 B, we then have u_s everywhere from p 189 E and F, add this to u_i and we have u(x) everywhere, problem solved. So our development thread just ends with this integrodifferential equation 6.187 that maybe we could solve for I(ψ).

Now he is going to find a second and better integral equation. This will be for the LHS of equation H, $J(\varphi) = \partial_r u\,|^{r=1}$, which we know vanishes for $|\varphi| \le \alpha$ and has some value on the rest of the ring. If we can get J(φ), we do Fourier on H and learn the a_n and we march as above to the full problem solution. In fact, the Fourier projections a_n of J are shown p 191 B. If we plug these a_n integrals into 6.183 for $|\varphi| > \alpha$, where we know I = 0, we get p 191 D. Our same m-sum appears, and we then get 6.189 which looks to me like a fairly simple and standard integral equation for J(φ). Notice that the integral is over the *non*-cylinder part of the circle, different from integrodifferential equation. The only odd thing is the a_0 sitting on the RHS of this equation, which is the constant term of the circle-inside solution p 189 F. In the 2D charge-on-conductors problem we had a constant c_0 which got determined by charge normalization to Q. There must be some corresponding condition in this problem that will determine a_0 but I don't know what it is. Perhaps when you try to solve this integral equation, you will be forced to some value for a_0. Stakgold is strangely silent on a_0, probably feeling he has already run overtime on this problem.