Stakgold chap 7 meta meta notes PhL 1.2.12

For 40 years I wondered what was in this "equations of evolution" chapter, and now I know. Here I will give only a very high level overview with minimal details. This runs 11 pages. The TOC only shows level 1 and 2 headings. This is the longest Stak chapter, being 331-194 = 137 dense pages long!!

7.1 Introduction [194] raw1 notes begin here **1**

HEAT CONDUCTION THEORY 2

7.2 The Causal Green's Function for Heat Conduction [197] **2**

7.3 Methods for finding Causal Green's Functions for heat [204] **3**

7.4 Uniqueness and Continuous Dependence on the Data (222) **3**

7.5 Miscellaneous Heat Conduction Equation Topics (227) **3**

The Heat Midterm Exam [239] **3**

WAVE EQUATION THEORY RAW2 NOTES BEGIN HERE 4

7.6 Preliminaries for the Undamped Wave Equation (243) **4**

7.7 Causal Green's for the Wave Equation, and the General Wave Solution (246) **4**

7.8 Wave Problems in one spatial dimension (249) **6**

7.9 Wave Problems in more than one spatial dimension (253) **6**

7.10 Wave Equation with External Damping (257) **6**

7.11 Monochromatic Excitation and Principle of Limiting Absorption (259) **6**

The Wave Midterm Exam [262] **7**

HELMHOLTZ POTENTIAL THEORY RAW2 NOTES CONTINUE 7

Section 7.12 Helmholtz Green's Functions and Applications (265) **7**

RAW3 NOTES BEGIN HERE. 9

7.13 Half-plane excited by a line source or a plane wave (281) **9**

The Helmholtz Midterm Exam [290] **9**

7.14 Helmholtz for exterior domains (294) (verbatim) **10**

7.15 The Scattering Problem (299) **10**

7.16 The Wiener-Hopf Method (311) **11**

The Final Exam [326] **12**

7.1 Introduction [194] raw1 notes begin here

The new feature is having time t as a coordinate, in addition to having n spatial coordinates. The only two PDE's Stak considers in this entire chapter are the heat (diffusion) equation and the wave equation, but at some point he does install damping terms in these two equations which makes them different. He never mentions that the heat equation includes the Schrodinger equation with imaginary time.

Stak states the heat conduction BV problem in 7.3 and the wave BV problem in 7.5, and draws his famous cylinder in $d^n x dt$ space. In both cases we have a BC h(x,t) on σ. For the BC we always have three choices: Dirichlet, Neumann, Radiative, though Stak uses the first. In heat we have IC f(x) at t=0, and in wave we have two IC's $f_1(x)$ and $f_2(x)$ at t=0. Solution is always u(x,t) on region R. Source always q(x,t).

If we apply page 40 Green's Theorem (u and v) to this cylinder using the heat and wave L operators, we get 7.6 for heat and 7.7 for waves. I show this in full detail in "geometry/N-dim...doc". In doing this, we make use of the surface current vectors **J** for these two L operators we found in Chapter 5. These two huge equations 7.6 and 7.7 involve two arbitrary functions u and v, the operators ∂_t, ∂^2_t and ∇^2 and integrals over the boundary of the space-time cylinder. These things are just "Green's Theorem" applied to that cylinder, somewhat analogous to Green's Second Identity.

A method is given for how to handle σ when R is an infinite region (but perhaps not all of R^n).

HEAT CONDUCTION THEORY

7.2 The Causal Green's Function for Heat Conduction [197]

Whereas in Section 7.1 mentions both heat and wave, we now start our multi-section "heat portion" of this long Chapter 7.

Just as in potential theory, the Green's function can be defined, but since time is now involved, this thing is a "causal" Green's Function which is always 0 if you try to go backwards in time. It turns out that you can state the causal Green's Function Problem in two different ways, such as 7.8 and 7.8a for heat. Once you compute the Green's Function, you can write the solution to any heat BV problem of the class of well-posed problems we are usually interested in. My general formula is this: (from my BV doc)

$$u(t,\mathbf{x}) = \int_0^t dt_0 \int_{R_n} d^n x_0 \; g(x|x_0)\, q(t_0,\mathbf{x}_0)$$

$$+ \int_{R_n} d^n x_0 \; u(0,\mathbf{x}_0)\, g(x,t|x_0,0) \qquad\qquad u(0,\mathbf{x}_0) = f(x_0)$$

$$+ \int_0^t dt_0 \int_{\sigma_n} dS_{n0} \left[u(t_0,\mathbf{x}_0)\{-\partial_{nx0}\, g(x|x_0)\} + g(x|x_0)\partial_{n0} u(t_0,\mathbf{x}_0) \right] \qquad (7.14\text{ gen})$$

which is a generalization of the potential theory result

$$u(\xi) = \int_V dx\; g(\xi|x) q(x) - \int_S dS\, [u(x)\, \partial_{nx}\, g(\xi|x) - g(\xi|x)\partial_n u(x)]$$

My formula is "gen" in that you can specialize it for the three usual cases Dirichlet, Neumann, and Radiative. For example, in the Dirichlet case, g=0 on σ which kills off the very last term.

As in potential theory, you can consider situations with NO boundary conditions, and then the Green's Functions are called again the "fundamental solutions" and Stak refers to these as the C_n functions for n dimensions of space. These C_n are what I like to call the free-space heat propagators, though Stak never uses the word propagator.

Right away Stak treats the n=1 example called "the infinite rod" which has no boundary σ, but of course can have an initial condition f(x).

7.3 Methods for finding Causal Green's Functions for heat [204]

In this section, remember the title above! It is similar to the title of a section in the Laplace World. He is not only going to *find* the Causal Green's, but he is going to *use* the Causal Green's to write general problem solutions. He will do Dirichlet, Neumann and mixed Radiative examples. He does veer off a bit doing problems having nothing to do with Causal Green's, but that is fine.

My meta notes have several sections which I will just list in these meta meta notes:

A. The Method of Images and other Trick Methods.
   Half Rod Problems
   The **Wire Ring** Problem [211]
   The **Finite Rod** Problem [212]
B. The Method of Expanding in **Spatial Eigenfunctions**: h=0, initial f(x) (213)
C. Examples of the Eigenfunction Expansion Method (216)
D. The **Laplace Transform** Method (218)
Spherical hole in 3D space example (220)

In Chapter 6 we had a similar section "methods of finding Green's Functions". As usual, Stak provides nice examples for each of his methods.

7.4 Uniqueness and Continuous Dependence on the Data (222)

This section contains a set of Theorems which are just what you would expect, analogs of the same theorems in Laplace World.

The Maximum and Minimum principles say that extrema of the solution u must occur on the space-time cylinder boundary. Uniqueness Theorem says for our class, the solution is unique. The notion of Continuous Dependence is that your solution should and does change smoothly if you make a smooth change in the boundary conditions.

7.5 Miscellaneous Heat Conduction Equation Topics (227)

I will just list the various subheadings of this grab-bag section, see summaries in the meta notes.

A. The **Semigroup** Connection. (227)
B. The **Backward** Problem (228).
C. The **Diffusion** Interpretation of the Heat Conduction Equation (230)
D. Asymptotic Formula for The EV's of $-\nabla^2$ (**Weyl's Law** for n=2) (231)
E. A 3D **Composite Medium** Heat Conduction Problem (234)
F. The **Stefan** Problem (237)

The Heat Midterm Exam [239]

These are Exercises 7.4 to 7.16. (13 problems), of which I did 10 in full detail. Here are some of the problem titles (those I did). Since I spent a large percentage of my Stak time doing problems, I will give the problems good mention in this meta meta review.

Exercise 7.4 Solve the **ring** problem by doing a **Fourier Series Transform** in the x variable. (239)
Exercise 7.5 Redo the **Stefan** problem with ice at u = V < 0.
Exercise 7.6 **Freezing of water** in a cylindrical tank.
Exercise 7.7 Obtain the **Weyl** formula for an n=3 region.
Exercise 7.8 Obtain the **Weyl** formula for Neumann and Radiative BC.
Exercise 7.9 Another way to show **max and min** of harmonic u both lie on the boundary.
Exercise 7.10 **Finite rod** with dipole source in center.
Exercise 7.11 **Average Temperature** is a constant in time.
Exercise 7.12 Add a **cu linear term** into the heat equation and resolve in EF method.
Exercise 7.16. The **half-rod driven** from the left end by various methods. (11 days)
Exercise 7.17. Deriving the **Weber** Transform
Exercise 7.18. An Application of the **Weber** Transform

I ended up spending 11 days on 7.16, and I did not do the last two listed above but want to get them into this meta meta in case I someday need to learn the Weber Transform, just another SL transform involving Bessel functions.

WAVE EQUATION THEORY raw2 notes begin here

7.6 Preliminaries for the Undamped Wave Equation (243)

This then is the first section of the "wave" portion of this chapter 7. The main idea is the **Big Theorem** which I will just quote:

(1) If you compare two wave equation problems,

BC = 0	IC = 0	IC' = f(x)	solution u_f
BC = 0	IC = f(x)	IC' = 0	solution v_f

then if you find solution u_f, you can determine the unique solution v simply as $v_f = \partial_t u_f$. So in some sense specifying IC' = f(x) is more fundamental.

(2) If you find solution u above, you know solution v as stated, and then you can superpose to find that

BC = 0	IC = $f_1(x)$	IC' = $f_2(x)$	solution w = $u_{f2} + v_{f1} = u_{f2} + \partial_t u_{f1}$

Thus, you might as well only do the u_f IC problem shown above and then all else follows.

7.7 Causal Green's for the Wave Equation, and the General Wave Solution (246)

Stak here repeats the steps of the heat Section 7.2 above. Again, the causal Green's Function defining system can be set up in either of two ways, here 7.113 and 7.114. The our class of interest most general problem has a general solution nearly the same as 7.14 gen above, but here it is 7.116 gen (my BC doc)

$$u(t,\mathbf{x}) = \int_0^t dt_0 \int_{R_n} d^n x_0 \; g(x|x_0)\, q(t_0,\mathbf{x}_0)$$
$$+ \int_{R_n} d^n x_0 \; g(x,t|x_0,0) f_2(x_0) - \int_{R_n} d^n x_0 \; \partial_{t_0} g(x,t|x_0,0) f_1(x_0)$$
$$+ \int_0^t dt_0 \int_{\sigma_n} dS_{n0} \left[h(t_0,\mathbf{x}_0)\{-\partial_{nx0}\, g(x|x_0)\} + g(x|x_0)\partial_{n0} h(t_0,\mathbf{x}_0) \right] \qquad (7.116 \text{ gen})$$

where changes and new things are shown in red. In heat world, we have $f(x_0)$ as the u(x,0) initial condition, but what takes its place in wave world is f_2 which is $\partial_t u(x,0)$, and there is a whole new term now in which $u(x,0) = f_1$ appears. As before, this gen thing applies to Dirichlet, Neumann or Radiative.

And as in the heat case, we take note of the "free space propagators" for wave theory, and these are given the *same names* C_n as in the heat case, though they are different "fundamental solution" functions. All these fundy things were really obtained back in Chapter 5 for both heat and waves and other things.

Now if you have some BC's, you can of course talk about both the Green's Function problem (with g=0 on the BC's) as well as the Eigenvalue problem (with $\varphi_n = 0$ on *same* BC's). These problems are related in that g can be written as a bilinear expansion on the φ_n, and of course you can have this bilinear expansion for Laplace, Heat and Wave, and here is a comparison:

$$g(x,t\,;x_0,t_0) = \Sigma_k \varphi_k(x)\varphi_k^*(x_0)\left[\sin\{\sqrt{\lambda_k}\,(t-t_0)\}/\sqrt{\lambda_k}\right] \quad \text{// wave 7.118}$$
$$g(x,t\,;x_0,t_0) = \Sigma_k \varphi_k(x)\varphi_k^*(x_0)\left[e^{-\lambda_k(t-t_0)}\right] \quad \text{// heat 7.57}$$
$$g(x\,;x_0) = \Sigma_k \varphi_k(x)\varphi_k^*(x_0)\left[1/\lambda_k\right] \quad \text{// Laplace 6.108}$$

I suspect these look the same for the Neumann and Mixed BC's, but am not sure. Here is another little comparison I found useful, and I discuss details in the meta notes (such as the elastic support of a string)

	potential theory	heat conduction	wave equation	
u	potential	temperature	displacement	
q	charge	heat source	driving force	// since $\partial_t^2 u \sim a$
$\partial_n u$	surface charge	surface heat source	surface driving force (see comment below)	

When there is no driving force q, and when h = 0 on boundary σ, only the f_1 and f_2 terms survive in (7.116 gen) above, and a general solution is a superposition of the φ_k "normal modes"

$$u(t,\mathbf{x}) = \Sigma_k \left[\sin(\sqrt{\lambda_k}\,t)/\sqrt{\lambda_k}\right] \varphi_k(x)\, f_{2,k}$$
$$+ \Sigma_k \left[\cos(\sqrt{\lambda_k}\,t)\right] \varphi_k(x)\, f_{1,k} \qquad \text{p 248 C}$$

where eg $f_{1,k} = \int dx\, f_1(x)\, \varphi_k^*(x)$. Of course φ_k and λ_k are the EF's and EV's of the related EV problem. The coefficients $f_{i,k}$ here are just the amount that each normal mode is activated by the initial conditions.

7.8 Wave Problems in one spatial dimension (249)

I have sections on three main situations here: (in the last two cases, surface σ is one or two points)

The Infinite String (249) // no boundary σ
The Half String (249) // σ = left end
The Finite String (252) // σ = the two ends

These are all analogous to the infinite rod, half rod and finite rod problems done in the heat section.

7.9 Wave Problems in more than one spatial dimension (253)

We note that $C_3 = \delta(t - t_0 - R)/4\pi R$ and thus this fundy has a pure shell wavefront which tends to simplify life for the n=3 world. There is no "wake" as there is for C_2 and C_1. This shell wavefront leads to the notions of "retarded potentials". C_2 is $1/[2\pi\sqrt{t^2-R^2}\,]$ for $t_0 = 0$ (7.130) which blows up at t=R and has a wave behind it that eventually vanishes.

Stak discussed Hadamard's so-called Method of Descent which means obtaining a dimension n result from the knowledge of some result for larger n such as n+1. So maybe get C_2 from knowledge of C_3.

7.10 Wave Equation with External Damping (257)

Notion of "struck" (f_2) versus "plucked" (f_1) pure initial conditions.

If you add a damping term $2\gamma\partial_t u$ to the wave equation and set $u = we^{-\gamma t}$, you get the "equation of telegraphy" which is the wave equation for w with an added term $-\gamma^2 w$, which is easier to solve. Stak is then able to write solutions for w in terms of the EF's φ_k of the original undamped wave equation with coefficients $w_k(t)$ as shown page 258 D, and then the final solutions for u are as in 7.137. As expected, every term dies away. This solution 7.137 is $u(x,t) = \Sigma_k\, e^{-\gamma t}\, w_k(t)\, \varphi_k(x)$. The $w_k(t)$ can have some expo time growth but cannot overwhelm the explicit decay factor $e^{-\gamma t}$ shown.

7.11 Monochromatic Excitation and Principle of Limiting Absorption (259)

Now Stak attacks the full "damped wave equation boundary value problem". It is too hard to do I think, so he assumes $e^{-i\omega t}$ "monochromatic" time dependence for all functions. This system can have a steady state solution he calls U in addition to a transient solution he calls v, depending on the IC's.

The principle of limited absorption is this: If you solve a damped problem with $\gamma > 0$, you can define a clean limit as $\gamma \to 0$ from the positive side, and this can be stated in the way ω approaches the real axis in the ω plane (from above). The solution realm for $\gamma < 0$ is completely unstable having blowing-up solutions, so you cannot approach from that direction and expect to get a reasonable $\gamma = 0$ limit. This is all related to the notion that a solution should behave smoothly as a parameter is varied ("continuous dependence").

Also discussed in this section are "energy considerations" for wave theory. There is a notion of energy density being sum of kinetic and potential , $\mathcal{E} = (1/2)\dot{v}^2 + (1/2)\,|\nabla v|^2$ and there is a related energy flux vector **J** and one then has $\partial_t \mathcal{E} + \nabla \bullet \mathbf{J} = 0$. I did not study this stuff too much, see meta notes.

The Wave Midterm Exam [262]

These are Exercises 7.19 to 7.30 (12 problems), of which I did 5 in full detail and at least read all the rest.

Exercise 7.19. Details concerning the time averaged **current J** shown on page 262.
Exercise 7.20. Use **energy** argument to show solution uniqueness for damped wave equation.
Exercise 7.21. **Kirchhoff's Formula**.
Exercise 7.22. Kirchhoff's Formula **Application**
Exercise 7.23. A **2D example**.
Exercise 7.24 . Infinite String with **Air Resistance** using 5.169 (partial)
Exercise 7.25. **Method of Descent** Application
Exercise 7.26. Redoing the end-wiggle **half-string** problem 3 alternative ways.
Exercise 7.27. **Half-string** with **elastic support** at x = 0.
Exercise 7.28. Solve the **String Ring** problem.
Exercise 7.29. The **internally damped string** problem.
Exercise 7.30. Solving a certain 1D driven Helmholtz ODE

HELMHOLTZ POTENTIAL THEORY raw2 notes continue

Both heat and wave equations yield timeless Helmholtz equations when you do a time transform, so this entire section really applies to finding solutions to both heat and wave problems. Of course the subject is interesting in its own right. Sometimes I call this Helmholtzstatics where you think of a dumb extra λ sitting there ($\nabla^2 + \lambda$). The fundamental solutions are called E_n and these also were found in Chapter 5.

Section 7.12 Helmholtz Green's Functions and Applications (265)

In general $\lambda < 0$ implies expo solutions and $\lambda > 0$ has oscillatory and we write $\lambda = \omega^2$. Stak discusses the various free-space Helmholtz propagators E_n. Stak sometimes talks $\lambda = -k^2$ on negative real λ axis and written in terms of k we see that for n=1,2,3 the fundy decays far away.

The n=3 fundy is simply $e^{i\omega R}/4\pi R$ and we think of this as our outgoing or scattered 3D wave.
For n=2 the fundy is a more complicated $H_0^{(1)} \sim K_0$ function.
The electrostatics limit is always $\lambda = \omega = k = 0$.

At this point Stak seems to wander off a bit obtaining fancy math results like addition theorems and various expansions in polar coordinates similar to my various 1/R expansions in curvilinear ones.

Addition Theorem for Cylindrical Waves (268)

Stated in sum rule 7.166.

Riemann Surface E function defined on φ in $(-\infty,\infty)$ (270)

Treating φ itself as a complex variable, yields an integral sum rule 7.171. The Poisson Sum Formula gets used here, I have notes on it. I think this applies in fact to the next little section.

Green's Function for a 2D wedge (272)

This Green's Function is obtained by an infinite image superposition trick similar to what we did elsewhere, and the resulting g is the sum 7.174 for arbitrary wedge opening angle ψ. In this approach, although it is not totally obvious how it gets that way, things are based on the φ eigenfunctions which are sin(...φ) which vanish on both faces of the 2D wedge.

Another Form for the Free-Space Green's Function (273)

In general Stak uses the symbol E to refer to the E_2 propagator. Here, using the KL transform, he is able to write E as a KK integral of KL K functions and this gives in 7.179 an integral *version* of the sum addition theorem 7.166. This result is used at once for E in the next section.

After doing this, Stak considers the ψ = 2π limit of the Dirichlet wedge problem which we will call by many names in upcoming sections: the knife edge, the half-wire, the half-plane. Starting from scratch, and using the KL transform to exchange r for γ, Stak solves the knife edge problem but alas he makes mistakes and it took me a long time (about 19 pages of the raw2 notes) to clean all this up and my errata give the corrected results. The tour de force is to take the KL integral solution for the knife edge g and deform the contour to pick up an infinite series of pole residues and you then reach in this manner the previously derived sum formula 7.174 with ψ = 2π. My work is summarized in the meta notes.

In doing all this, I was forced to clean up my Bessel equation notes and I also wrote up the KL transform and various other transforms as well in a new transforms folder.

Hankel Transform vs Fourier Transform (275)

If a Helmholtstatics problem is azisym in φ in cylindrical coordinates, you have an r,z problem and you can separate the homo problem as in 7.184 and 7.185. This is the analog of 7.160 and 7.161 where we had an r,φ problem. Given a pair of separated equations like this, you can find the EF's of *either* equation and use those as basis functions for an expansion of the PDE solution. In the r,φ case those eigenfunctions were either the $e^{inφ}$ type or the KL K function type. In the present r,z case the eigenfunctions are either the e^{ikz} or the $J_0(kr)$ Bessel functions. Thus, you are led to use either the Fourier Transform in z, or the Hankel Transform (with order 0) in r (assuming an appropriate problem region).

Source Ring in Free Space and how related to a heat problem (276)

This is used as an example of the r-z problem and Stak solves it both the ways just mentioned, first using the Hankel Transform, and second using the Fourier Transform. As a side benefit, these two sections produce lots of integrals involving J_0 and K_0 functions.

On page 280 Stak then works backwards to show how the solution of this ring-source Helmholtz problem is relevant to a heat conduction problem where the ring is then an initial condition. Stak solves this problem two different ways as 7.204 and then 7.205. I make the point that a Helmholtz source appears as an initial condition in a heat problem, not as a heat source in a heat problem!

raw3 notes begin here.

7.13 Half-plane excited by a line source or a plane wave (281)

Line source excitation (2D point source) (281)

We continue in our study of Helmholtstatics with our knife-edge situation. Here I call it a grounded half line and we put a point source at (r_0,φ_0) as shown in the page 281 figure. Using the usual partial EF method with sin(..φ) EF's, we arrive at the Green's Function g as in 7.207. Stak now refers to this situation as a half-plane problem in the extruded 3D sense.

Stak then burns a lot of oil rewriting these g as shown in 7.217 which has two terms each of which is a similar "diffraction" style integral. This is an early set up for "scattering" theory. Note the meanings of distances R and R* from the p 281 figure: distance from observation point to source and source image.

Plane Wave Excitation (285)

Next Stak moves the point source infinitely far away ($r_0\to\infty$) and renormalizes it as in 7.219 so it can then be considered to be a unit "plane wave" coming in from the φ_0 direction. We call this a "wave", but of course we are still just doing Helmholtstatics with a far-away source. The E_2 function from this far source has near-planar contours, one might say.

Fiddling with 7.217 Stak obtains 7.220 in which r_0 no longer appears! This thing is true for any ω and is credited to Sommerfeld. But now if we go to the optical limit $\omega >> 1$ we get 7.221 where we see clearly our "incoming plane wave", our "reflected plane wave" and a "shadow region" in which u = 0. Again, this is all just a static solution u, nothing is "moving".

Causal Green's Function for half-line (287)

Here Stak computes the causal wave Green's function for the half-line, which he now calls G, with local spacetime source at (r_0,φ_0) in space, and pulsed δ(t). The way he gets this result is that he does a Laplace transform on the wave equation and this gives exactly the half-line Helmholtz problem with local source which we just solved to get 7.217 with $\lambda = -s^2$, and thus he gets p 288A. He processes this result for G into 7.223, and we then see the results in the three "regions" at several different time bands. The radicals in these expressions are the C_2 propagators coming from the source (R) and from the image point (R*).

The Helmholtz Midterm Exam [290]

These are Exercises 7.31 to 7.43. (13 problems), of which I did 7 in full detail.

Exercise 7.31 (290). 3D Helmholtz Green's Function for **point charge on +z axis**, **sphericals**
Exercise 7.32 (290). Same 3D as above, but replace the z-axis point source with a **ring source**.
Exercise 7.33 (290) 3D initial condition **heat problem, spherical hole** in ∞ medium
Exercise 7.34 (p 290) 2D Helmholtz **Green's Function** outside a **circle** on which g=0.
Exercise 7.35 (p 291) Repeat previous problem using r eigenfunctions instead of θ ones.
Exercise 7.36 (p 291) 2D Helmholtz Neumann **Green's Function in slab** with ring source.
Exercise 7.37 (p 291, read only) Find temp inside a slab with Dirichlet initial conditions.

Exercise 7.38 (p 291) **Waveguide** 3D Helmholtz Green's Function Problem.
Exercise 7.39 (p 292) Induced Helmholtz **current on half-line** with point source off left end (2D).
Exercise 7.40 (p 293, read only) Induced **current on half-line** due to incident **plane wave** (2D)
Exercise 7.41 (p 293, read only) Redo the half-line problem for **Neumann** instead of Dirichlet.
Exercise 7.42 (p 293, read only) **Sound wave** interface between two media.
Exercise 7.43 (p 294, read only) Solve the 3D Helmholtz **sphere with q(x)** distributed source.

Although we just had the Helmholtz exam, Stak has a little more to say on the subject.

7.14 Helmholtz for exterior domains (294) (verbatim)

An "exterior domain" means the region outside some localized scattering object. Stak shows that the general solution to Helmholtz in such a case outside the object is as in 7.235 where the radial functions are the $H_{n+1/2}^{(1)}(kr)$ "spherical" Bessel functions which decay at $r = \infty$. If we take the large r limit, this has the famous form $e^{ikr}/r * f(\theta,\varphi)$ so the scatterer looks like a strange point radiating this outgoing wave that has angular dependence of some sort. We are still doing Helmholtstatics, by the way. Note that e^{ikr}/r is the E_3 fundy.

The point is that outside the radiator region, far away, regardless of the medium and the physics, if you have a wave equation operating, then this is what the far field looks like, and then $f(\theta,\varphi)$ is dependent on the physics.

In 2D you get $f(\varphi)\, H_n^{(1)}(kr)/\sqrt{r}$ with "non spherical" Bessel function.

So Stak has now touched on the deep notion of a scattering amplitude and partial wave expansion and this all appears in many fields including quantum mechanics with particle waves.

Radiating Exterior Dirichlet Problem uniqueness of solution (296)

Here Stak shows that uniqueness still holds in an infinite exterior region, for 2D and 3D, and for Dirichlet, Neumann or Radiative. If the scatterer has very sharp corners, there are extra problems in doing the uniqueness proof, and our classic wedge problem has just such a sharp corner.

7.15 The Scattering Problem (299)

We are STILL doing Helmholstatics. Stak reprises the $u = u_i + u_s$ decomposition we used back in electrostatics where u_i is some sort of external field (incident) and u_s is the field generated by induced sources on the scatterer. Just as we did in electrostatics, Stak develops the Helmholtstatics version of the "integral equation method" displayed in (7.249) and (7.250) where E is a Helmholtz fundy.

Plane Wave Excitation (302)

If we take $u_i = e^{i\omega\alpha\cdot x}$ which is a Helmholtstatics "plane wave" coming in the α direction, then using the "integral equation" and going to the far field limit we again obtain $e^{ikr}/r * f(\theta,\varphi)$ but now $f = A(\boldsymbol{\beta},\boldsymbol{\alpha})$ where β is the observer's position, and Stak then has an expression for $A(\boldsymbol{\beta},\boldsymbol{\alpha})$ as a simple integral of the induced current I on the scatterer. From this he is able to derive the reciprocity principle 7.254.

Cross Section (303)

Stak defines the cross section which he calls D (since σ is so many other things) in terms of scattered power (energy flux), and then he uses his energy formulas to write an expression for D in terms of u_s. Shuffling he comes up with D expressed as the imaginary part of A(**α**,**α**) which is the **optical theorem**.

Low and High Frequency Behavior (305).

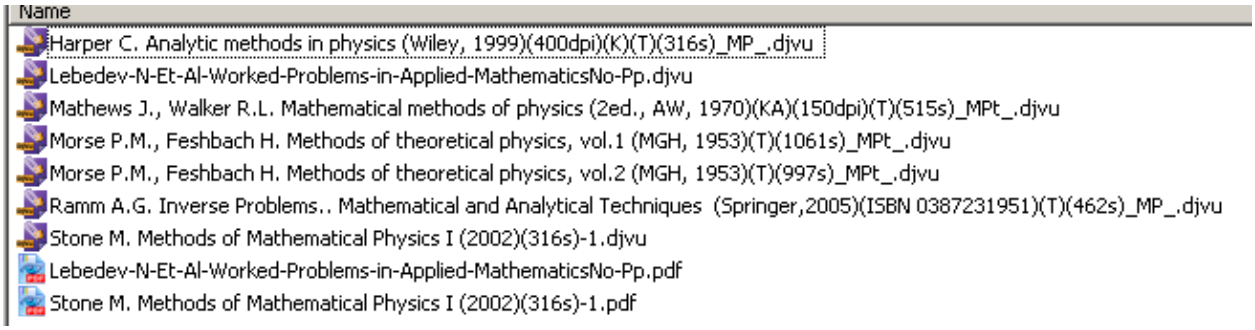
In the low frequency case, Stak writes an expansion in ω for the induced current I(x) on the scatterer and this gives a reasonable expansion ladder for doing perturbation theory around the electrostatics limit ω = 0. He relates cross section to capacitance of the scatterer, 7.266.

In the high frequency case he cannot do a 1/ω expansion, but instead does some optical work. He finds that the cross section of an ellipsoidal type scatterer is twice the projected area, a famous result.

In a last section he mentions a variational idea to improve approximation of integral equation solution I.

7.16 The Wiener-Hopf Method (311)

Noted added: I finally got directory level djvu search going. Here are the files in my Math Methods folder which contain <Weiner-Hopf> (the search is very slow, these are big docs, entire huge books)



Stone and Harper have nothing much, though Stone draws a nice matrix view which I like. Mathews has an example of solving a BV problem using WH. Ramm has nada really. Could not find the Noble 1958 book for download. So only M&F have a significant section on this stuff in the above list. But the Hochstadt integral equations book has quite a few pages on WH, see math books.

This is a fascinating 1931 method of doing Fourier diagonalization of a Fred 1 or Fred 2 integral equation with integration range (0,∞) such as appears in the half-line integral equation method. Normally one does Fourier diagonalization on (-∞,∞) and that is a trivial group theory diagonalization into ω space from x space. The WH idea is to use one-sided functions. You want to solve for some $u_+(x)$, and you have to introduce an associated $g_-(x)$. You can then be on (-∞,∞) and you can diagonalize as usual. BUT, now there are two unknown functions u and g. By doing "segregation" of the diagonalized equation (using quotient and sum splitting), you can arrive at the conclusion (given a set of assumptions) that each side of the segregated equation must vanish, and this then lets you determine both u+^(ω) and g-^(ω) and you do

inverse Fourier to get u(x) and you have solved the problem! The basic tool of this method I would say is analytic continuation and simple properties of analytic functions, and the critical sum splitting idea.

The assumptions are that all the involved functions have large x behavior as on page 314 (1)-(4).

Stak then gives two examples of the WH method. The first has a simple answer, but the second requires a fancy sum-splitting and some contour deformation. The second example is in fact finding the current on the half line when the Helmholtz point source is placed distance a off the end.

Wiki says there have been "thousands" of papers on WH methods and applications. I wonder if it was every used in S-matrix particle theory? I think yes, at least in some sense.

Notice that WH is a general integral equation diagonalization method, and we just happen to apply it in example 2 and in a problem below TO the integral equation which arises in Helmholtstatics for the half line. So we can apply WH to a Helmholtstatics "integral equation method" integral equation if the integral is over a half line.

The Final Exam [326]

These are Exercises 7.44 to 7.57. (14 problems), of which I did 4 in full detail.

Exercise 7.44 (p 326, read only). Surface approach, extra term, **Fred 2 instead of Fred 1 for I**.
Exercise 7.45 (p 327, read only) Same idea on a **shell surface**, getting I+ and I- separately.
Exercise 7.46 (p 327, partial) Redo scattering theory for **Neuman BC** on scatterer surface.
Exercise 7.47 (p 327, read only). 2D scattering from a **circle-shaped u=0 scatterer**.
Exercise 7.48 (p 327, read only). More on 2D **scattering from a circle**.
Exercise 7.49 (p 327) **Point charge off end of half-plane (a 3D problem**)
Exercise 7.50 (p 328) Plane wave hits u=0 screen with **aperture** head on, 3D problem,
Exercise 7.51 (p 328, read only) Repeat above with **Neumann** screen and aperture.
Exercise 7.52 (p 329, read only) A **diffusion** problem with absorption.
Exercise 7.53 (p 329, read only) 2D Neumann **partially blocked waveguide** problem.
Exercise 7.54 (p 330) Compute the **integral** shown p 325 B.
Exercise 7.55 (p 330, read only) Convert the Example 1 **integral equation to ODE** and solve.
Exercise 7.56 (p 330,) **Half line with plane wave from above**, use WH method.