Stakgold Chapter 7 Meta Notes PhL 5.1.11

The raw1 file contains my notes on the *heat* section of Chapter 7. My raw1 notes fill 72 pages (the Stak text is 40 pages), while the meta notes here fill 17 pages, so we have roughly 4-to-1 note compression.

The raw2 file contains my notes on the *wave* section of Chapter 7. But then raw2 got too long so I started raw3 at text page 281. So there are three raw note files, but only this one meta file.

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7.1 Introduction [194]

Stak states the heat conduction BV problem in 7.3 and the wave BV problem in 7.5, and draws his famous cylinder in $d^n x dt$ space. In both cases we have a BC h(x,t) on σ. For the BC we always have three choices: Dirichlet, Neumann, Radiative, though Stak uses the first. In heat we have IC f(x) at t=0, and in wave we have two IC's $f_1(x)$ and $f_2(x)$ at t=0. Solution is always u(x,t) on region R. Source always q(x,t).

If we apply page 40 Green's Theorem (u and v) to this cylinder using the heat and wave L operators, we get 7.6 for heat and 7.7 for waves. I show this in full detail in "geometry/N-dim...doc". In doing this, we make use of the surface current vectors **J** for these two L operators we found in Chapter 5.

A method is given for how to handle σ when R is an infinite region (but perhaps not all of R^n).

7.2 The Causal Green's Function for Heat Conduction [197]

We now enter only the heat world, waves come later. We are going to do the "general theory" first, and then look at "rod problems" as toy applications.

The PDE system defining the causal heat Green's Function g is given in 7.8 where our source is δδ and g = 0 at t=0 *and* on σ. Causality is imposed *by fiat* which means g = θ h. Given this fiat, we can rewrite 7.8 in the alternate form 7.8a where now the PDE is homo, but g = δ at t=0+. I show all this in full detail and draw my first Feynman diagram. I often refer to g as a "propagator" though Stak does not. I write $g(x|x_0)$ has sense g(←) meaning the first argument has to be in the future of the second (else g=0). I note connection of the heat equation to the Schrodinger equation t → it.

In Potential Theory we found that

$$u(\xi) = \int_V dx\, g(\xi|x)q(x) - \int_S dS\, [u(x)\, \partial_{nx}\, g(\xi|x) - g(\xi|x)\partial_n u(x)\,]$$

gives the solution u(x) in terms of g for arbitrary boundary conditions (Dirichlet, Neumann or Radiative). We obtain this result by using our page 40 Green's theorem (u and v) (which in this case is known as Green's 2nd identity), and by then applying it with u the solution of our target system and v(x) = g(ξ|x).

Just so for heat. In this case our page 40 Green's theorem is 7.6 (u and v) where again u is the solution of our target heat system and $v(t,\mathbf{x}) = g(x_0,t_0|x,t)$. When the dust settles, we get what I call (7.14 gen),

$$\begin{aligned} u(t,\mathbf{x}) \;=\; & \int_0^t dt_0 \int_{R_n} d^n x_0 \; g(x|x_0)\, q(t_0,\mathbf{x}_0) \\ & + \int_{R_n} d^n x_0 \; u(0,\mathbf{x}_0)\, g(x,t|x_0,0) \\ & + \int_0^t dt_0 \int_{\sigma_n} dS_{n0}\, [\, u(t_0,\mathbf{x}_0)\{-\partial_{nx0}\, g(x|x_0)\} + g(x|x_0)\partial_{n0} u(t_0,\mathbf{x}_0)\,] \end{aligned} \qquad (7.14\text{ gen})$$

where $u(0,\mathbf{x}_0) = f(x_0)$ and for the Dirichlet case we would set $u(t_0,\mathbf{x}_0) = h(t_0,\mathbf{x}_0)$. Stak has just this Dirichlet case so the last term does not appear since g=0 on the surface dS. [I am inconsistent in the ordering of the arguments of functions like u and h.] I draw Feynman diagrams for each of the terms in 7.14 gen in the BV doc.

When σ = no boundary, we know that g is the "fundamental solution" for heat which we found in Chapter 5

$$g(x|x_0) = C_n(x,t|x_0,t_0) = H(t-t_0)\,[4\pi(t-t_0)]^{-n/2}\exp(-|\mathbf{x}-\mathbf{x}_0|^2/4(t-t_0))$$

I note that with $t\rightarrow$ it this gives the Saxon "free particle propagator" in non-rel Schrodinger theory and as $t\rightarrow t_0+$ this does indeed become just $\delta(x-x_0)$ as required. Note that for heat things are non-oscillatory, that is to say, non wave-like.

We are now at page 12 in the raw notes and only page 199 in the text. My raw notes here give a good summary: "General Comments on What has Happened Here.". It has little sections (a) through (e) and then a Big Theorem, go read that now (this is a "must read" section).

Application to problems with an Infinite Rod (200)

Rod means n=1, see "meaning of a rod" in raw notes p 14. The infinite rod has no σ. If there are no sources, then the complete solution to the infinite rod problem is (superposition!! , linear equation!!)

$$u(x,t) = 1/\sqrt{4\pi t}\ \int_{-\infty}^{\infty} dx_0\ f(\mathbf{x}_0)\exp(-(x-x_0)^2/4t) \qquad \text{// agrees with (7.16)}$$

which I plot in 3D for $f(x) = \delta(x-2)$ which means $u(\mathbf{x},t) = C(x,t|,2,0)$.

Stak comments on "u depending continuously on the initial data", then takes $f(x)= -\delta'(x)$ as another example, for which we find that $u(\mathbf{x},t) = -\partial_{\mathbf{x}}\, C(x,t|x_0,t_0) \sim x\exp(-x^2/4t)$. This is a "dipole impulse" and I plot this too. Stak likes this alternate form of the above 7.16

$$u(x,t) = 1/\sqrt{\pi}\ \int_{-\infty}^{\infty} dz\ f(x + 2\sqrt{t}\,z)\exp(-z^2) \qquad \text{// agrees with 7.19}$$

because here it is obvious by inspection that $u(x,0) = f(x)$ since the integral is then $\sqrt{\pi}$.

Exercise 7.1: Get (7.14) for The Neumann case $\partial_n u = h$ on σ. **[202]**
Exercise 7.2: Get (7.14) for The Radiative case $\partial_n u+\theta u = h$ on σ.

I did both of the above in my "BV problems..doc".

Exercise 7.3. Interpreting the BC's as additional source terms.

I had trouble with Stak's δ notation here, but the point is clear: you can think of BC h and IC f as sources on a footing with q if you enlarge your cylinder to completely enclose the initial Stak cylinder. I think I could maybe do a better job on this than Stak did.

7.3 Methods for finding Causal Green's Functions [204]

In this section, remember the title above! It is similar to the title of a section in the Laplace World. He is not only going to *find* the Causal Green's, but he is going to *use* the Causal Green's to write general problem solutions. He will do Dirichlet, Neumann and mixed Radiative examples. He does veer off a bit doing problems having nothing to do with Causal Green's, but that is fine.

A. The Method of Images and other Trick Methods.

Half Rod Problems

The half (infinite) rod boundary σ is just the point x = 0, and the surface dS integral shrinks to just a point in 7.14 gen, so our half-rod version is just

$$u(t,x) = \int_0^t dt_0 \int_0^\infty dx_0\ g(x|x_0)\, q(t_0,x_0)$$
$$+ \int_0^\infty dx_0\ f(x_0)\, g(x,t|x_0,0)$$
$$+ \int_0^t dt_0\, [h(0,t_0)\{-\partial_{nx0}\, g(x|x_0)\} + g(x|x_0)\partial_{n0}u(tx_0,t_0)\,] \qquad \text{(7.14 gen, half rod)}$$

The g is no longer the fundamental C since we need g = 0 at x=0 due to presence of σ. But the image method quickly shows that g = C(x,t|x0,t0) - C(x,t|-x0,t0) where you put a negative image source at -x0. I give a plot of this as well. I call this g the "half rod propagator".

If h=0 and q=0, we get just u(x,t) = $\int_0^\infty dx_0\ f(x_0)\, g(x,t|x_0,0)$ where now g is our "half rod propagator". In 7.16 above we had this same u(x,t), but g was the "full rod propagator". If we set f = 1 so that the rod is uniformly heated to temperature 1 (but the left end is freezing at h=0), we find that the cold advances onto the rod and u(x,t) = erf(x/(2$\sqrt{t}$) which I plot in various ways. As a little extracurricular work, I show that this u(x,t) solves the homo heat equation. On page 206 Stak shows how you can take the large t limit of this erf solution, something he often likes to think about (ie, large t limits of solutions). Stak then does a time Laplace on our PDE to get a space ODE, solution is 7.30, and of course the inverse Laplace of this is going to be erf(x/(2$\sqrt{t}$). Instead of showing that, Stak gets into Appendix B where he worries about the large t behavior of an inverse Laplace, and shows this agrees with what he just got above. He then looks into small t for this problem.

The next problem is f = 0 but some h(t) so Dirichlet says u(t,x) = $\int_0^t dt_0\, [h(0,t_0)\{-\partial_{nx0}\, g(x|x_0)\}$ where g is our half rod propagator. He works this out as 7.35, and h is referred to once somewhere as "the boundary schedule". If h(t) = δ(t) we get u(x,t) = x exp(-x^2/4t)/[$\sqrt{4\pi}\ t^{3/2}$] which I plot. This is a frozen half rod where you apply a thermal δ pulse at the left end.

If the half rod has "**an insulated end**" at x=0, that means $\partial_n g = 0$ at that end, the Neumann case. Recall that $\partial_n g \sim \sigma$ on a boundary in electrostatics "through which **E** flux flows", and it is a surface heat source in heat theory "through which heat flux flows". In this case, as in electrostatics for a point above a

grounded plane, the solution is found with a positive image charge, and I plot this too, call it g, or call it u for the case $h(t) = \delta(t)$. Basically $g = C(x,t|x_0,t_0) + C(x,t|-x_0,t_0)$ where C is the full rod propagator.

Stak then does the case of a **radiative rod end** (p 209) using a "trick method" and the result is shown in 7.44 for the Causal Green's. He will later refer to this result when he does this same problem using the spatial EF method. [And this trick method is required in a later wave Exercise 7.27.]

We are now at page 27 in the raw notes, and page 211 of the text.

The Wire Ring Problem [211]

"Perimeter is 1 unit, wire is regarded as really a 1D problem (very thin wire, and it has insulated sides, so like a piece of our infinite rod that is just cut and bent into a circle.) The BV Causal Green's problem for this ring is shown in (7.45) [with $t_0 = x_0 = 0$] which seems very reasonable: this causes in effect the solution and all derivatives to be continuous at the linkage point which is ± 1/2 . If you look at the picture on page 212, you see that we are in fact going to form our wire ring from the piece of straight wire that runs along (-1/2,1/2) on the rod axis."

In this linearized picture, we can meet the boundary conditions with one set of positive image charges located at the integers: this meets the conditions of 7.45 on (-1/2,1/2) and solution is unique! The clumsy series 7.46 is replaced with the more convergent and reasonable series 7.48 using the Poisson Sum Formula which I discuss on page 26 of my Chapter 1 raw notes: sum of function at integers ≈ the sum of the Fourier Transform at the integers. So in 7.48 we have $g(x,t; 0,0) = \theta(x,t)$ as shown, and we note in passing that it has the correct "bilinear heat form" but not obvious since $x_0 = 0$ so $\cos(2n\pi x_0) = 1$ does not appear.

The Finite Rod Problem [212]

Here we need solution = 0 at both ends x = 0 and x = $\boldsymbol{\ell}$ (σ is now two points) and this is achieved with two infinite sets of image charges. Each series is summed and transformed using the same Poisson Sum trick and we get 7.50 as our Causal Green's solution. I note that this is *very* similar to the Laplace problem for a strip with g = 0 on each edge. [In fact, each solution is the general bilinear form in its respective world, as we shall see below.]

B. The Method of Expanding in Spatial Eigenfunctions: h=0, initial f(x) (213)

We first consider a class of problems I call **"cold boundary problems"** (h=0 on boundary, Dirichlet). We can look for separated solutions of the form p 213B with separation constant λ and we get 7.52 and 7.53. The first is just *the Laplace World EF problem* with u = 0 on σ as usual. The second is a trivial ODE whose solutions are $\exp(-\lambda t)$. We solve the Laplace problem for $\varphi_i(x)$ and λ_i.

If we have no sources q = 0 and some initial $f(\mathbf{x})$ for $u(\mathbf{x},0)$, the solution is found to be 7.56 using the obvious Smythian form method and finding the coefficients c_i. If we specialize to $f(\mathbf{x}) = \delta(\mathbf{x}-\mathbf{x_o})$, the solution is then the Causal Green's 7.57 which is the bilinear heat form mentioned just above where we had a cold boundary at x = 0 and x = $\boldsymbol{\ell}$ on our finite rod.

"We can then of course use this g in our general 7.6 to solve any heat problem in our class of problems where the boundary σ does not change over time" and where the boundary is "cold" h = 0 Dirichlet. That is, we have from 7.6 a solution for some arbitrary source distribution q.

Note that the spatial EF's used here are simply Laplace Equation EF's (from potential theory).

Stak repeats this analysis for the Neumann **"insulated boundary problems"** where eigenvalues are now called μ_i which now include $\mu_0 = 0$ allowing for a constant component in the solution. Stak shows the expected result that, since no heat can escape the region R, as $t \to \infty$ the solution approaches the constant value which is the spatial average of f(x), and this is what goes with $\mu_0 = 0$! [think those bilinear $e^{-\mu i\ t}$] .

Stak comments that if R = infinite, the λ and μ spectra become continuous and he will treat this situation only in special cases, no general theory.

For his closing *tour de force* in this section, Stak launches a direct attack on the general problem 7.3 (without using a Green's Function g) to get an alternate general solution of 7.3. This seems *like* the partial eigenfunction method Stak used in his Laplace section in that he uses the same spatial eigenfunctions $\varphi_i(x)$ used in the cold boundary problem, allows coefficients here called $u_i(t)$ instead of $c_i(t)$ which are functions of the "extra coordinate", here t, and he then ends up with a simple ODE for the $u_i(t)$ which he can then solve and throw back into his partial EF expansion. In the just-mentioned Laplace section, we were looking for a Green's Function, and so we ended up with an ODE for $g_i(\xi)$ where ξ was the extra variable, but the ODE was second order and driven by a δ. In other words, in the Laplace case we reduced things to a 1D Green's Problem which we knew how to solve, whereas here we reduce to a 1D time ODE which we know how to solve. This is the general idea of doing a spatial EF expansion. The final alternate solution is 7.60, see raw notes.

C. Examples of the Eigenfunction Expansion Method (216)

In **Example 1** we replicate our Causal Green's solution above for the finite rod, but here we use the spatial EF method and the general heat bilinear formula. Of course this has a discrete λ_i spectrum.

Example 2 concerns using the EF expansion method on the *full half rod* with a *radiative* BC at x=0, and this is our first example of a problem with a continuous λ spectrum. My raw notes show I was *very* confused by Stak's logic flow in this discussion starting on page 217.

Associated with this half rod Laplace EF problem is a Laplace *Green's Function* problem stated in p 217 B where you see the radiative BC. In the raw notes p 33 I have "Radiation explained" which you should go read now to understand the interpretation of the BC in 7.61 and p 217 B with $\theta = 1$.

My confusion was WHY Stak suddenly rolled out this Laplace Green's function problem which I will answer momentarily. Stak first uses the Green's Function called $r(x|\xi; \lambda)$ to obtain a δ completeness relation as in p 218 D and E (in the usual manner) and of course this leads to a (nameless) symmetric transform shown in 7.62 and 7.63. Recall that every SL problem has an associated transform! We expect that the functions appearing in this transform are in fact our desired eigenfunctions, but here "pseudo" since they cannot be normalized on the infinite range. He has switched from λ to $\nu = \sqrt{\lambda}$. Thus it is on page 218 that he writes $\varphi_i(x)$ as $\varphi_\nu(x)$, the continuum EF, as in 7.64. This is just a linear combination of sinx and cosx that solves 7.61 as I could no doubt verify, so he could have solved 7.61 "by inspection" instead of doing all this Green's stuff. I think part of the issue is obtaining the correct *scale* of the EF's. I think I might have done that by showing $\delta(\nu-\nu')$ orthogonality, which he never mentions. In any event, once we have the EF's as in 7.64, we can write our general bilinear Causal Green's heat solution as in 7.65 where we replace our former λ_i sum with a λ integration, here a ν integration, again, $\lambda = \nu^2$. He claims this fancy integral answer agrees with the 7.44 [via the trick method] which I noted above.

So to answer the question posed above, the reason Stak introduces the associated Laplace Green's problem is that from this function we can find the spatial EF completeness relation and the SL transform

and the continuum EF's and we can then write the Causal Green's as bilinear 7.65 which is our end goal. As an alternative, he could have solved this problem starting with a finite rod and then taken the limit $\ell \to \infty$. Recall that he has required the radiative BC to keep his variety flowing!

What I have just outlined above is Stak's "second method" mentioned top page 218. In the raw notes I outline his implied "first method" which is a continuum version of having coefficients like $c_i(t)$ which depend on time (like TD perturbation theory etc etc). I do all the details of this first method in the raw notes and end up with a time ODE 1D Green's problem for $g_\lambda(t)$ which is the TD coefficient of the full g.

D. The Laplace Transform Method (218)

If we take our general heat problem and Laplace transform t into s, we get a purely spatial problem where s is just a parameter,

$(-\nabla^2 + s)\, \tilde{u}(x,s) = \tilde{q}(x,s) + f(x) = q_{eff}(x,s)$ // includes the "initial condition" f(x) 7.68

$\tilde{u}(x,s) = \tilde{h}(x,s)$ for x on σ // this is the "boundary condition"

and we note that f(x) is part of the effective q due to "the way Laplace works" which I now know so well. Our standard-issue way to solve the above Helmholtz full problem is to first solve the associated Green's Function problem (think $-\lambda = s$)

$(-\nabla^2 - \lambda)\, G(x|\xi; \lambda) = \delta(x-\xi)$ 7.69

$G(x|\xi; \lambda) = 0$ for x on σ and he adds that G should be square integrable on R

But this is a standard Stak spatial Laplace Green's function problem whose solution of course depends on the boundary σ. So if we solve this thing for G, then our solution to the system above (Dirichlet) is

$$\tilde{u}(x,s) = \int_V d\xi\, G(x|\xi; -s)\, [\tilde{q}(\xi,s) + f(\xi)] - \int_S dS_\xi\, [\tilde{h}(\xi,s)\, \partial_{n\xi} G(x|\xi; -s)] \quad \text{// agrees with 7.70}$$

and then we have to do an inverse time Laplace to get u(x,t) and we are then done!

In the special case that q = 0 and f = δ, our system solution is $\tilde{u}(x,s) = G(x|\xi; -s)$ which we know we can write as $\Sigma_i \varphi_i(x)^* \varphi_i(\xi)/(\lambda_i - [-s])$ where φ_i and λ_i are the EF's and EV's for σ, which you see has poles in the s plane which yield residues when you do the Laplace inversion formula. When you add up all these residues, you end up sure enough with the general bilinear heat formula for g p 219 C. This is just a sanity check done by Stak. [Much more Helmholtz lies far ahead, starting page 265.]

Spherical hole in 3D space example (220) The region R is the (infinite) exterior of a spherical hole (radius 1, centered at our origin) in a 3D heat-conductive medium. We have the IC f(x) that u=0 in the infinite medium, and the BC that u=1 on the hole boundary (and this is held for all t), so we are going to heat up the medium. Despite the Green's Function discussion above, we are going to solve this problem *directly* -- we can do that since we really only have one spatial variable r. We transform t to s and get p 220B where note the transformed BC as 1/s. Stak solves the ODE in r "by inspection", throwing out the solution that blows up at r = ∞, hence p 221 B. Doing the inverse Laplace (this involves a cut discontinuity) we get the erfc /r solution p 221D. So starting at t = 0 (medium all at u=0), heat starts to flow out from the hole boundary, and we eventually reach the steady-state solution u = 1/r where u = 1 on

the hole boundary, and u = 0 at infinity. The sink is so large and we never heat it all up, so this differs from the corresponding frozen half-rod problem (for which I just now added a solution on page 27 of the raw notes) where the whole rod eventually heats up to u=1. At t=∞ the 3D problem is really an electrostatics problem (but temperature-wise) and we have V = 1/r outside our shell of uniform charge σ which here is a uniform spherical heat source. This example demonstrates: (1) an infinite region; (2) a heat problem in isotropic 3D; (3) the Laplace time transform as a solution method ; (4) an example where the heat source cannot heat up the entire medium no matter how long you wait.

2D version of the above example (221 bot). Things go along in similar fashion, and the s-space solution turns out to be a $K_0(\sqrt{s}\,\rho)$ function as in p 222B. The inverse Laplace integral has a simple pole at s=0 and a cut and the result is 7.76 which is a rather nasty looking Bessel integral that Stak does not attempt. I think this reappears later on as some sort of custom transform deal (it is the Weber Transform, see Exercises 7.17,18 p 242,3). Stak's point about this 2D solution is that it is like my 1D solution quoted above: eventually the entire medium heats up to 1. So the 3D case is the first dimension in which this does not happen.

Notice that these last two examples do not use Green's functions at all, they are done directly. And so this brings to an end this very long section 7.3 entitled methods of finding Causal Green's Functions.

7.4 Uniqueness and Continuous Dependence on the Data (222)

This section contains a set of Theorems which are just what you would expect, analogs of the same theorems in Laplace World.

[223] We certainly expect that if u = 0 on the bottom and sides of the cylinder, that the unique solution is u = 0 everywhere inside. This is the content of **Theorem 1** on page 223, which Stak proves two different ways. The first proof is on page 225. The second is deferred a while.

[224] The Maximum Principle (Theorem 2) says that <u>if u is bounded</u> in both initial and boundary conditions ($|u| \le M$) , then u is similarly bounded everywhere inside the cylinder. In other words, at no point in the interior can |u| exceed its max on the boundaries. I suspect that we may conclude that, barring the case that u = constant inside, the max of u must occur somewhere on the boundary. This is very similar to our conclusion regarding solutions of the Laplace equation (Stak does not mention this), and is also reminiscent of a fact about analytic functions in the z plane. Stak proves this Principle on page 224.

[225] The Minimum Principle is trivially also true therefore, so barring constant, the minimum value of |u| must also occur on the boundary. All as in Laplace Potential Theory. So the initial conditions, assuming nothing is infinite in them, bracket the solution everywhere inside, in that $A \le u \le B$.

So the "second proof" of Theorem 1 is this: If u = 0 for IC and BC, then the max and min principle tell you that inside we must have $0 \le u \le 0$ and we conclude then that u = 0.

Uniqueness Theorem 3: If you had two (bounded = continuous) solutions of 7.81 with the same BC's, then the difference would be a solution with 0 BC's. But then Theorem 1 says that difference solution

must be 0. An alternate proof is to appeal to the min and max principles which forces $0 \le \Delta u \le 0$ so $\Delta u = 0$ everywhere.

[226] Theorem 4 is the thing about continuous dependence of the solution on the boundary data. If you make small variations of ε in the boundary data, then you get a small variation in the solution, and this is guaranteed by the min and max principles, nothing to prove.

At this point, right in the middle of things, Stak says that it is pretty hard to prove the **existence** of a solution, although we have proved uniqueness, and he is not even going to attempt a proof of existence. The reason is that it is hard to demonstrate a solution by construction in the general case, as we have done in past existence proofs. But of course in any specific problem, if we can build a solution, then it must exist. [Question: isn't the general bilinear formula for g a constructed solution? And can't you then use it in the general formula like 7.14 gen?]

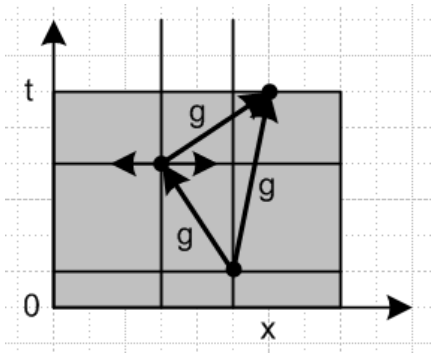
Theorem 5 is an extension of Theorem 1 to unbounded regions: u = 0 on all boundaries means u = 0 inside. It follows I think that we can also extend uniqueness Theorem 3 to unbounded regions. Yes, that is the corollary on top of page 227.

7.5 Miscellaneous Heat Conduction Equation Topics (227)

Yes, this is a "grab bag" of interesting "topics" which involve "heat conduction".

A. The Semigroup Connection. (227)

Using the 1D full rod as an example in 7.83, Stak points out that you can think of the propagator motion equation as a linear integral operator G_t which "moves you ahead in time by amount t". It seems pretty reasonable that $G_{t1}G_{t2} = G_{t1+t2}$ so these operators have closure under multiplication. $G_0 = \delta(x-y)$ so the identity exists. Since the inverse does not exist (see raw notes text for why), the operators G_t don't form a group, they only form a semigroup -- that is the point of this section. The conclusions apply not just to the 1D rod propagator, but do *any* Causal heat propagator as in 7.87. This is why I use the word "propagator" because operator G with kernel g "propagates" you ahead in time. I draw 7.87 as a little Feynman diagram:



B. The Backward Problem (228).

This section is of special interest to me since it involves diagonalization of an integral equation. The raw notes have a detailed discussion, here I will summarize. If you look at the integral equation for forward propagation, you have $u_t = G_t u_0$, say. The question is: given u_t, can we invert this to get u_0 ? We said the answer is "no" in the semigroup section above. This seems to conflict with the fact that we can diagonalize the integral equation $u_t = G_t u_0$ [which we write as $a_x = \Sigma_x G_{x,x'} b_{x'}$ in the raw notes] not with group theory, but simply by going to the "i basis" where i means the i of $\varphi_i(x)$. The diagonalized form of the integral equation is then just $a_i(t) = \Sigma_j G_{ij} b_j = G_{ii} b_i = e^{-\lambda_i (t-t')} b_i(t')$ and in the example of a finite rod (where i = n and where we move ahead $\Delta t=1$) this becomes $a_n = \exp[(n\pi/\ell)^2] b_n$ where b_n is in the future and a_n in the past. So you would think "gee, since I know the future b_n projections of the solution, I also know the a_n past projections, so I know u in the past, so I *can* go backwards in time. " Math always has a way of "working out" and what happens here is this: Yes, you can compute the a_n as just shown, but then when you construct u from these a_n you will in general find that the series diverges so there is no solution! You can see the diverging tendency in the expo factor.

It turns out that in certain cases the series does not diverge and you *can* go backward in time. If the solution in the future is C^∞ then yes, you can go backwards and the series converges. This relates to the general theory of projection and recovery which in the Fourier case brings in the term Schwartz Functions, the class of functions which allow you to converge in both directions. But even in this case, Stak shows that the solution (ie, u in the past) "does not depend continuously on the data" meaning the projections of u in the future, and for this reason such a solution, though it exists, does not represent a stable physical solution to a problem. In other words, the "backward problem" is "ill-posed". That is the single main point of this section. Physically of course we can imagine that you could arrive at the same future u from many different initial u, so intuitively we can see that you cannot go backwards in a unique manner. The backward path to each different initial u exists, but is not "stable". I could imagine doing further reading on this subject.

C. The Diffusion Interpretation of the Heat Conduction Equation (230)

"Recall that electrostatics, static heat flow, no-curl fluid flow, and static diffusion are all Laplace potential theory worlds. The claim here is that non-static diffusion is like non-static heat flow where u is particle density of the solute in the solvent. Zero temperature u = 0 corresponds to "no particle density", so I guess this is a place where particles are sucked out of your system, just the way u = 0 sucks out heat. So if u = 0 on a boundary, that is an absorbing boundary, it absorbs solute particles. And if $\partial_n u = 0$, no particles cross (since no diffusive force to do so) so an insulating boundary becomes a particle reflecting boundary. I have NEVER done any of this.

Stak shows how the diffusion equation gets modified if there is a "steady drift" of velocity $v_x = 2\gamma$ in the x direction say (so $\ell = x$) , and this is 7.92. This is an "example" of the **Fokker-Planck equation,** says Stak, yet another massive hole in the PL math world, I see it is all tied in with "stochastic processes". But in our simple example here, Stak shows how you can remove the drift and get back to the heat equation by changing from function u to v. Stak then does a toy example showing how you might start in 1D with a δ source of solute and watch the drift current spread it out. We just solve our normal heat equation for v using our rod propagator C, then convert that back to u. I think I can "see" in the solution 7.95 that our initial puff of solute drifts to the right and spreads out at the same time. The first pair of expos gives a

combined 1 when $\gamma(x-x_0)-\gamma^2 t = 0$ which means $(x-x_0) = \gamma t$, so $x(t) = x_0 + \gamma t$ so the peak here drifts to the right at speed γ. The other factors describe the spreading out always centered at x_0."

D. Asymptotic Formula for The EV's of $-\nabla^2$ (Weyl's Law for n=2) (231)

My raw notes hide the main result in a mass of details. The main conclusion of this section is something called **Weyl's Law** which is stated in 7.101, namely, $N(\lambda) \sim \lambda A_R/4\pi$. Here, $N(\lambda)$ is the number of eigenvalues less than λ of the Laplace equation with $\varphi=0$ on a boundary σ, but the formula is meant to be valid only when λ is large. Stak only treats this problem for n=2, so that A_R is the area of a 2D "membrane" surrounded by boundary σ. It does seem rather shocking that $N(\lambda)$ is a function only of A_R and not a function of the specific shape of σ ! The second fact of this formula is that $N(\lambda)$ is proportional to λ for large λ, which says that for very large λ the eigenvalues are evenly spaced. Notice now the title of this section. Asymptotic means "large λ" and "The EV's" means the eigenvalue$<\lambda$ count $N(\lambda)$.

Weyl's Law is a statement about potential theory and has nothing to do with the heat conduction equation. However, Stak makes *use of* the heat conduction equation to obtain this law! Stak derives Weyl's Law on pages 232 and 233, you just follow each step.

A key step in his derivation is that for small t, you can approximate a Causal g by g_0, which is the no boundaries "free space" propagator. In my raw notes section "Question: Why is $g \approx C$ for small t ?" I explain why this is justified, and compare it to the electrostatics idea that near a point charge you can ignore the effects of induced charges on boundaries. In the heat case, "heat" propagates outward from internal and boundary sources at a certain (time dependent) velocity, and at small time if you are out in the middle somewhere, you don't yet feel the influence of the boundary sources. My section here I think is pretty good and brings in a lot of info that is useful to keep in mind.

Stak then goes on to derive a first order correction to Weyl's Law which is shown in 7.102. For large λ you can see that the correction term (the second term there) is much smaller than the first term due to the factor $\lambda^{1/2}$ versus λ in the main term. In the second term L_R is the perimeter length of the boundary!

Suppose the membrane were just a square of edge ℓ. In this case we know $\lambda_{nm} = (\pi/\ell)^2(n_x^2 + n_y^2)$ which we could write as $\lambda_n = (\pi/\ell)^2 n^2$. How many eigenvalues lie inside a disk of radius λ_n? The answer is the number of dots in a circular 2D integer grid of radius $n = (\ell/\pi)\sqrt{\lambda_n}$ dots, which is the same as the area of such a disk, which is $\pi [(\ell/\pi)\sqrt{\lambda_n}]^2 = \pi (\ell/\pi)^2\lambda_n = N(\lambda) = (1/\pi) A_R \lambda_n$, but we must only count the dots in the first quadrant to avoid overcounting the EV's, so get $N(\lambda) = A_R \lambda/(4\pi)$ in agreement with Weyl. I presume the correction term relates to the fuzziness of the dot count near the perimeter of the disk, but I have not looked into that detail.

E. A 3D Composite Medium Heat Conduction Problem (234)

The problem here is to have an interface at r=1 between an inner sphere of medium 1 and an outer infinite region of medium 2. We start the inner ball at u = 1 and the outer region at u = 0 and we watch the heat flow out in some way from the inner ball. On page 235 Stak writes in A and B the heat equations for the two regions, while C and D are the interface boundary conditions (see raw notes for details). We use our time Laplace method to change equations to E and F which we solve by inspection to get G and H. Matching at the interface gives coefficients A and B as shown in I and J.

So the first point to make is that we have a complete exact solution to this problem in (r,s) space. The second point is that A and especially B are messy functions of s, so the inverse Laplace is going to be

difficult to compute. There are branch points at s=0 from the $s^{1/2}$ factors, and perhaps others as well. Even if we restrict our interest to the central point r = 0, our exact s-space solution is 236 A which has that messy B(s) function sitting there.

So Stak does not attempt to get an exact solution to the problem. Rather, he works up a small-s expansion for the RHS of p 236 A and when the dust settles, the leading term for small s is what you see in p 236 H and this results in $t^{-3/2}$ behavior in t as in 7.103. That is the main point Stak makes. The large t dependence will be controlled by the rightmost singularities in the s plane due to e^{-st} sitting in the inverse transform.

Aside: If you look at p 236 H you see that our leading term is $\sim s^{1/2}$. This *by itself* is not a "legal" F(s) as I discuss in my Laplace transform notes, but it is legal in the sense of finding large t behavior from this component of the full (and legal) F(s) which we are here approximating. This is discussed in the last part of Appendix B, and the result is consistent with Schaum p 164 32.28 continued to the value n = -1/2.

At the top of page 237 Stak then makes a fascinating argument. If the two media are the same, then for large times you cannot really tell if heat came from a ball or a point "big bang" of the same total heat. In the latter case, we know the problem solution u is just the 3D free space propagator (see 5.140 p 60 with n=3). For large t, this propagator goes as $t^{-3/2}$ confirming the earlier result above. All the constants work out as well.

F. The Stefan Problem (237)

I found a whole Google book just on this class of problems, where a phase change boundary moves (known as free boundary problems). In 1889 Mr. Stefan (a Slovene) was wondering about the way ice forms on the top of a lake, and the way in which ground freezes. With a uniform cross section, you can think of this as a 1D heat conduction problem, but Stak talks about it in this 3D sense. You have $x \geq 0$ filled with ice at u=0 and at t = 0 you apply a metal plate at u = U against this ice (plate in the x = 0 plane). The ice starts to melt and a water layer forms and gets wider with time. We don't worry about density change here, we assume water exactly fills the space of the melted ice. We assume a planar phase interface which moves to the right. The problem then is to find an expression $x = \xi(t)$ which describes the interface location in time, and also to find an expression u(x,t) for temperature in the (ever-thickening) water layer. Temperature is u = 0 to the right of this layer, and we don't care about $x < 0$ and can assume it is U there if we like. Stak proceeds to set up and then solve this problem. The key fact is this: in a thin layer Δx at the interface, the heat required to melt the layer has to be supplied by the water layer on the left. A feature of this problem is that the boundary moves with time, and one of the BC's reflects this fact.

The solution interface position is given by 7.107 where α is a constant you have to get by solving a transcendental equation D (arising from the BCs). Then the temperature in the water region is given by E. Very elegant.

Stefan is also known for empirically determining the T^4 power law for black body radiation, and his student was Boltzmann who provided the theory to derive the formula, the Stefan-Boltzmann Law!

"And so ends the heat conduction portion of Chapter 7, apart from the exercises which follow. Think of this as "the heat exam" at the end of this mini course. I will do the exam, then move into the wave equation. The heat part of this chapter was p 194-243, or about 50 pages. The wave part goes to p 331 and so is 90 pages, yeouch!"

The Heat Final Exam [239]

Exercise 7.4 Solve the ring problem by doing a Fourier Series Transform in the x variable. (239)

Earlier Stak used the image method to get the Causal Green's for the wire ring as 7.48. Here we use the complex Fourier series method (not Laplace, and space, not time) to solve this problem for an arbitrary f(x) IC with the following steps: (1) restate the ring problem with $\theta = 2\pi x$ in $(-\pi,\pi)$. (2) use the complex Fourier series from transforms.doc to expand $u(\theta,t)$ onto $e^{in\theta}$ basis functions. (3) the PDE becomes an ODE in time t which can be solved by inspection for the $u_n(t)$ with A_n as coefficients which are then expressed as integrals of f(x). The solution at this point is

$$u(\theta,t) = \Sigma_n A_n \exp[-(2\pi n)^2 t] e^{-in\theta} \qquad \text{where} \quad A_n = (1/2\pi) \int_{-\pi}^{\pi} d\theta\, f(\theta)\, e^{+in\theta}$$

$$= \Sigma_{n=0}^{\infty} A_n\, \varepsilon_n \exp[-(2\pi n)^2 t] \cos(n\theta)$$

If we then set $f(x) = \delta(x)$, we replicate the image method result 7.48. I give full detail in the raw notes.

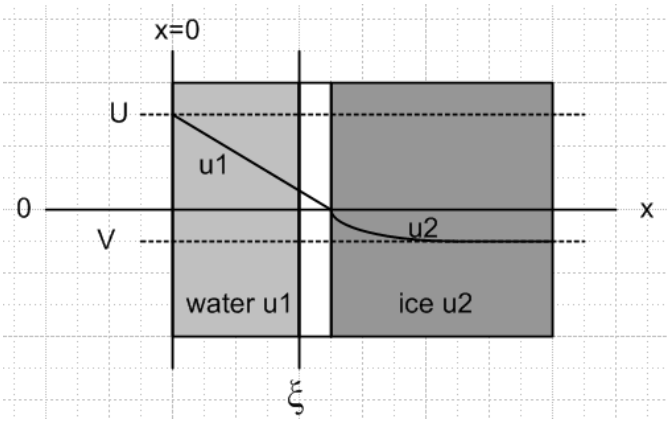
Exercise 7.5 Redo the Stefan problem with ice at u = V < 0.

This is the same as the text's problem where we apply a heat plate at x=0 and temperature U > 0 to a block of ice, but this time the ice block starts at temperature V < 0, whereas in the text problem it was at V = 0. This complicates the problem substantially, but I obtain the solution in the raw notes showing all steps. There is a messy transcendental equation one must solve for a constant α_1 (messier than in the text problem), and then the position of the ice surface as it backs away from the heat wall is $\xi(t) = 2\alpha_1\sqrt{a_1 t}$, where a_1 is the thermal diffusivity of water $a_1 = k_1/c_1$. Note that $\alpha_1 = \alpha_1(k_1,k_2,a_1,a_2,\nu,\rho,U,V)$ where ν is the latent heat of the water/ice phase change. The final solution to this problem is as follows:

$$u_1(x,t) = U[\,1 - \text{erf}(x/2\sqrt{a_1 t}\,)/\,\text{erf}(\alpha_1)\,] \qquad \text{// agrees with p 238 E}$$

$$u_2(x,t) = V[-\text{erf}(\alpha_1\sqrt{a_1/a_2}\,) + \text{erf}(x/2\sqrt{a_2 t}\,)\,]\;/\;\{\,1 - \text{erf}(\alpha_1\sqrt{a_1/a_2}\,)\,\}$$

The form of the water solution $u_1(x,t)$ is the same as the V=0 text problem, but of course α_1 is different. Here is a very crude plot of the solution at some t > 0 : [note that u=0 at the phase boundary] [the white is water but is extremely thin]



Exercise 7.6 Freezing of water in a cylindrical tank.

Another good Stefan problem. We have a vertical water tank filled (at t=0) with water at temperature V>0, and the weather causes the top tank surface to be at some temperature U < 0. The tank sides are insulated. So ice starts forming at the top, and eventually the ice/water interface reaches the tank bottom and the entire tank is then frozen at U < 0. The question is: how long does this take? One could think of the tank as a cylindrical lake of depth h freezing in the winter. Stak does not state the answer to this problem, but I do get an answer in the raw notes. My answer is $t = h^2/(4\,\alpha_1^2\,a_1)$. I obtain my result by sort of transforming the previous exercise into this one. If I were to write this up again, I think I could do a cleaner job than I did. Notice that the time is quadratic in the tank height h. Stak asks a second question in this exercise which I chose to ignore.

Exercise 7.7 Obtain the Weyl formula for an n=3 region.

Here I mimic Stak's n=2 work in the text, and I am able to show that

$N(\lambda) \sim \lambda A_R/4\pi$	n=2	as in the text
$N(\lambda) = (V_R/6\pi^2)\,\lambda^{3/2}$	n=3	new result for n=3, agrees with Stak's stated result

where V_R is the volume of our 3D region. Although Stak asks for the correction term as well, I decided not to do that part of the problem. We then consider a special case where the 3D region is a cubic box. In this case we know that $\lambda_{nmk} = (\pi/\ell)^2(n_x^2 + n_y^2 + n_z^2)$ which we could write as $\lambda_n = (\pi/\ell)^2\, n^2$. How many eigenvalues lie inside a sphere of radius λ_n? The answer is the number of dots in a spherical 3D integer grid of radius $n = (\ell/\pi)\sqrt{\lambda_n}$ dots, which is the same as the volume of such a sphere, which is $(4/3)\pi\,[(\ell/\pi)\,\sqrt{\lambda_n}\,]^3 = (4/3)\pi\,(\ell/\pi)^3\lambda_n^{3/2} = N(\lambda) = (4/3\pi^2)\,\ell^3\,\lambda_n^{3/2}$, but we must only count the dots in the first octant to avoid overcounting the EV's, so get $N(\lambda) = (1/6\pi^2)V_R\,\lambda_n^{3/2}$ in agreement with Weyl.

Exercise 7.8 Obtain the Weyl formula for Neumann and Radiative BC.

I tentatively conclude in the raw notes that the formulas are the same for all three BC cases!

Exercise 7.9 Another way to show max and min of harmonic u both lie on the boundary.

This has to do with cupping up and down and hilltops inside a region, see raw notes.

Exercise 7.10 Finite rod with dipole source in center.

We already know the finite rod propagator g, and we know the general formula 7.14 in terms of given functions q, f and h. In this problem the finite rod has u=0 at both ends [so h = 0] , and we assume no internal sources [so q = 0], so we just have $f(x) = -\delta'(x-\ell/2)$. Inserting this into 7.14 gives this result

$$u(x,t) = \Sigma_{n=1}^{\infty}\,(-1)^n\,(4n\pi/\ell^2)\,\sin(2n\pi x/\ell)\,\exp(-\,4n^2\pi^2t/\ell^2)$$

Stak then asks us to obtain this same result using a set of image dipole sources and I discuss this a bit but don't follow through with the complete solution.

Exercise 7.11 Average Temperature is a constant in time.

My raw notes here are quite convoluted, I will try to summarize. Here is the general scenario: we have a finite region R bounded by a σ on which we impose Neumann BC's which means no heat flows in or out. We have some initial f(x) temperature distribution at t = 0 and we watch as time advances. We want to show that the average temperature in our volume V is constant.

Obviously the total heat Q inside R cannot change. The heat in some tiny volume dV would be dQ = u c dV where c is the heat capacity of the medium, so total heat is $Q(t) = \int dx\ u(x,t)\ c(x)$. But all our heat analysis, including propagators, assumes c(x) = constant, so $Q(t) = c \int dx\ u(x,t)$. Saying that Q does not change in time is then the same as saying that $\int dx\ u(x,t)$ = constant which is the same as saying <u> = $\int dx\ u(x,t)/ V$ = constant, which says the average temperature does not change. So we *know* that this is true just "from the physics" of the situation. But Stak wants us to show this "directly" somehow.

My first step is to take the special case that $f(x) = \delta(x-x_0)$ and watch what happens. In this case, there exists some propagator g for the Neumann BC region R, and we have u = g. I can write the bilinear expansion for g in terms of EF's $\psi_i(x)$. Using this expansion, I show directly that <u> = 1/V = constant which is the desired result. I show that only ψ_0 contributes to <u>. I then argue vaguely that by superposition, this conclusion that <u> = constant holds for any f(x).

Making this vague last argument more specific, I write the general formula 7.14 for u(x,t) with Neumann BC. Since heat does not flow through the boundary, we set $\partial_n u = h = 0$ on the boundary, so there is no surface integral in 7.14, and this leaves only the f(x) integral . I then time average to get <u> = $\int dx\ f(x)$ <g>. But I know that <g> = 1/V so this says <u> = (1/V) $\int dx\ f(x)$ and this says that the average temperature is constant, and that is what we were supposed to show.

Exercise 7.12 Add a cu linear term into the heat equation and resolve in EF method.

As the problem says (maybe read the problem p 240), such a linear term models radiation from the sides of a rod in a 1D heat problem, or models disintegration (loss) of particles in a diffusion problem. The question then is this: how does the presence of this new linear term affect our eigenfunction method of solving the problem?

To answer this question, I do the separation u=XT and incorporate the new c effect into the time ODE. The result is that the spatial ODE is *the same as it was* with c = 0, and the time EF is $e^{-(\lambda_i+c)t}$. So the upshot is that everything is the same except in our formulas but we replace λ_i by $\lambda_i + c$. If you think of the factor $e^{-(\lambda_i+c)t}$ as the decay of solute density or decay of temperature, you see that "c" (c>0) just enhances the rate at which the decay occurs in each "partial wave".

Stak asks for a general solution if u=0 on the boundary σ for some IC f(x), and one way to write that solution is then $u(x,t) = \Sigma_i\ \langle f,\varphi_i\rangle\ \varphi_i(x)\ e^{-(\lambda_i+c)t}$ where φ_i are the usual spatial eigenfunctions.

Exercise 7.13 Maximum principle for fancy diffusion equation.

Here we have a "fancy" diffusion equation which allows for "creation or destruction" of particles (the cu term) , allows for a drift velocity α (Fokker-Planck term), and the diffusion coefficient k(x) is a function of position. We are supposed to rethink the max and min theorems derived earlier. If c > 0 you are destroying particles (reducing temperature), so you would expect the max stuff to still work, since c acts to reduce u. But not the min. And vice versa if c < 0. I did not do this problem!

Exercise 7.14. Show Theorem 1 page 223 for Neumann and for Radiative BC's.

Theorem 1 of Section 7.4 says that if Dirichlet u=0 everywhere on a closed boundary (IC + BC), then continuous heat conduction u = 0 inside. This is similar to the Laplace equation fact. Here we are supposed to show this is also true if we assume Neumann or Radiative boundary conditions (θ>0). I did not do this problem.

Exercise 7.15 Show Theorem 1 page 223 for the fancy diffusion equation 7.108 if c ≥ 0.

This sounds like more of the same. Put these last two problems on the rainy day list. I don't see any big issues here. Good to know the conclusions.

Exercise 7.16. The half-rod driven from the left end by various methods.

This problem bogged me down for 11 days, see "Stakgold Chap 7 Exercise 7.16 .doc" for the full bloody details.

Part (a) states a very special problem where we have $e^{-i\omega t}$ time on both the q and h BC for the half rod problem. If we look for a separated XT solution with $T = e^{-i\omega t}$, we do NOT find ourselves doing the spatial EV problem as I once thought. Rather, we discover a certain steady state solution for the rod that has been driven at x=0 by $Ae^{i\omega t}$ for all time. That solution is p 242A, and the real part is 242 B (also a solution as I show) which is the actual physical solution: a damped wave traveling to the right, and I made a Maple animation to show this. For this solution, the IC f(x) just *comes out* a certain sinusoidal way (a photo of the steady state solution at t=0). You cannot also specify f(x) because this problem is not in the same "class" as our generic 7.14 type problem. Here we forced a certain solution class by assuming separation with $T = e^{-i\omega t}$. There are no "transients" as part of the solution because this is a steady-state solution, though Stak does not use that term. [Ie, this is not an "initial value problem".]

Part (b) treats the same problem in the regular class we are used to where f(x) = 0 (though he neglects to say this). Using a **time Laplace** transform, I am able to show result 242C where both terms solve the homo heat equation, and the second term is what causes us to achieve u(x,0) = 0. This is a typical situation, you add a homo solution to tune the boundary conditions, here the IC boundary condition. I did find the leading term of the transient integral for large t as requested.

Part (c) I call it asks for the same solution via Fourier Cosine spatial transform. I first erroneously did this with a Laplace transform and ran into various tricky problems (most of my days were here), and I concluded that you cannot solve the problem this way! I then much later tried the Fourier Cosine method and determined eventually that it has the same "missing information" problem that the Laplace method has. I then finally realized that you CAN do this problem using the Fourier Sine method, and this is all written up in "Stakgold Chap 7 Exercise 7.16 .doc" as section (e) in which I encountered a rare Maple error.

Exercise 7.17. Deriving the Weber Transform

I am not going to do this because I spent so much time on 7.16. Also, I can see that 7.17 is a fairly standard (though singular at one end) Sturm-Liouville 1D problem. We take the Bessel ODE and look for a Green's Function on the interval (a,∞), which is the new feature here (finite lower endpoint). We need g = 0 at the left end of the interval and g = finite s-norm (limit point) at the right end -- this is what p 242 E says. That solution is found to be p 242 F which involves Bessel functions including a special linear combination of J_0 and $H_0^{(1)}$ which Stak calls Z_0 (this is a "Smythian form" combination which vanishes at x = a). The L_λ type L is assumed, which allows us to obtain the spectrum in the usual manner and completeness comes out as in p 243 D. The resulting transform is called **The Weber Transform** and I imagine one would use it for an *exterior* problem in cylindrical geometry where u = 0 on the cylinder of radius a. I looked in Google books and found there are three errors on page 243, two of which are the same minus sign, and one of which is the omission of the projection in F (which led me to peruse Google).

Exercise 7.18. An Application of the Weber Transform

We are supposed to first derive 7.76 (p 222) by doing the required inverse Laplace Transform, and *then* we are to redo this page 222 cylinder problem using this new Weber Transform. This problem involved an infinite medium with an infinite cylindrical hole taken out where u = 1 (h = 1 reservoir BC) on this inner boundary and u = 0 in the medium (IC f = 0). There we did a *time* Laplace transform and got a fancy answer 7.76 using the ILT, whereas here we want to do this thing with a *spatial* Weber transform. So this is precisely the kind of "external" cylindrical problem that I imagined one would use this transform on.

I did not do either part of this problem. Notice u = 1 on cylinder, but u=0 in the Weber thing, so you might have to add a constant 1 or some such.

This is where the raw1 heat conduction notes end and the raw2 wave notes begin.

7.6 Preliminaries for the Undamped Wave Equation (243)

We now have a BC but *two* IC's instead of one. Stak shows that the solution of the wave class problem 7.109 is *unique*, and I do all the details in my raw notes. He then shows this theorem:

p 245: Big Theorem with two parts:

(1) If you compare two wave equation problems,

BC = 0	IC = 0	IC' = f(x)	solution u_f
BC = 0	IC = f(x)	IC' = 0	solution v_f

then if you find solution u_f, you can determine the unique solution v simply as $v_f = \partial_t u_f$. So in some sense specifying IC' = f(x) is more fundamental.

(2) If you find solution u above, you know solution v as stated, and then you can superpose to find that

BC = 0 IC = $f_1(x)$ IC' = $f_2(x)$ solution w = $u_{f2} + v_{f1} = u_{f2} + \partial_t u_{f1}$

Thus, you might as well only do the u_f IC problem shown above and then all else follows.

We are now going to "redo" most of our heat section results here in our new Wave World.

7.7 Causal Green's for the Wave Equation, and the General Wave Solution (246)

The two versions of the Causal Green's System, and the General solution formula 7.116

Recall in Heat World that we could express the Causal Green's problem in two ways, which were 7.8 and 7.8a on page 198, where in the second way we have a homo PDE and a requirement that $g \to \delta$ at $t = 0$, as well as $g=0$ on σ. The analogous wave Causal Green systems are **7.113 and 7.114**. In the first inhomo version, we simply have $g = 0$ on σ. In the second homo version we still have $g = 0$ on σ, but we now have $g = 0$ as one IC, and it is the other IC $g' \to \delta$ (ie, this is the fundamental IC setup as noted above). I show all details in the raw notes. Since we are saying $g = 0$ on σ, we are doing this all in the Dirichlet case, but I think the other two cases work out the same.

Stak then claims that, whereas our heat g's were smooth in t, our wave g's won't be. And whereas our heat g's were functions, our wave g's will often be distributions. We saw evidence of this back in Chapter 5 where the causal propagator was smooth for heat, but involved θ and δ functions and their derivatives for waves.

Equation **7.115** is just a statement that we have $g(\leftarrow)$ whichever time argument labels you use. He has not commented on the possible symmetry or lack of it for this causal g in the x variables (but see below). We know there is no symmetry in the t ones due to causality.

At this point, I went off and added a new section to my BV document. I had already derived the wave equation Green's theorem (7.7) in my "N dim" doc, but there was still much to be done, the current J_μ is different for the wave equation, and we end up therefore with an extra term. I have fully derived equation 7.116 in the fully general Dirichlet/Neumann/Radiative case (I call it **7.116 gen**). The extra term now is that ∂_{t0} term and it has f_1, so you would argue that f_2 term is also different in that f_2 is the $\partial_t u$ IC. Note that this 7.116 is the wave version of the general heat solution equation 7.14 on page 199, but the subscripting has been swapped (see the BV doc just mentioned).

Space Eigenfunction Expansion and the Wave Bilinear Formula (247)

Instead of doing a "separation of variables" approach as he did in Heat World, Stak manipulates the homo Causal Green's system by doing a "partial eigenfunction expansion" of the Causal Green's g onto the usual Laplace spatial eigenfunctions called $\varphi_k(x)$ [Dirichlet, u=0 on σ] with coefficients $g_k(t\ ;\ x_0,\ t_0)$. He ends up with a little time ODE for these g_k which he solves by inspection, and he then ends up with a bilinear form for g. Here is the triple comparison

$g(x,t\,;\,x_0,t_0) = \Sigma_k\,\varphi_k(x)\varphi_k^*(x_0)\,[\,\sin\{\sqrt{\lambda_k}\,(t-t_0)\}\,/\,\sqrt{\lambda_k}\,]$ // wave 7.118

$g(x,t\,;\,x_0,t_0) = \Sigma_k\,\varphi_k(x)\varphi_k^*(x_0)\,[e^{-\lambda_k(t-t0)}]$ // heat 7.57

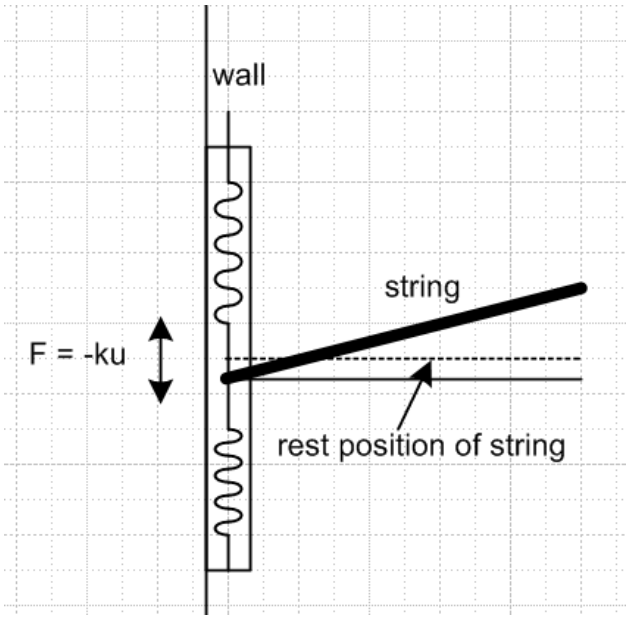
$g(x\,;\,x_0) = \Sigma_k\,\varphi_k(x)\varphi_k^*(x_0)\,[\,1/\lambda_k]$ // Laplace 6.108

Comparison. I then pause to take note of the comparison between electrostatics, heat and waves

	potential theory	heat conduction	wave equation	
u	potential	temperature	displacement	
q	charge	heat source	driving force	// since $\partial_t^2 u \sim a$
$\partial_n u$	surface charge	surface heat source	surface driving force (see comment below)	

Comment: In heat theory, $\partial_n u$ at a boundary means there is a Δu radially at the boundary, and we know from a simple heat conduction model that this drives heat across the boundary at that point. That is why we associate the Neumann $\partial_n u$ with a "surface heat source" (or sink). In potential theory we know $\partial_n V$ is the radial electric field and we have Gauss's law which associates this with surface σ. The question then is this: what is the meaning of $\partial_n u$ in wave equation theory? If you look for example at the end of a string you will find that if the string has a slope $\sin\theta$ at the boundary (say x = 0) then there is a force acting *vertically* on the boundary which is $T\sin\theta$ where θ is the angle of the string away from horizontal and T is the tension in the string. But $\sin\theta = du/dx$ at the boundary, $= \partial_x u = \partial_n u$. So for a string we can associate the Neumann quantity $\partial_n u$ with a force on the string exerted by the boundary in the direction of the displacement u. For a 2D surface, we would associate $\partial_n u$ with a perpendicular (to the membrane) force *per unit length* of the boundary perimeter, which we could call then a pressure as force/length. I both 1D and 2D cases, your displacement u is perpendicular to all the spatial dimensions of the membrane. For a 3D medium, the wave model would require that u be a displacement perp to the x,y and z axes, so you would need 4D space to plot this displacement u versus x,y,z and you would then find at the boundary that $\partial_n u$ was in fact a force per unit area or a true pressure, and that force would be directed in that fourth dimension, so a bit hard to ponder.

Now suppose for a 1D string we insist that $\partial_x u = ku$ at a boundary. We just said that $\partial_x u$ is the force exerted on the boundary by the string, so force = ku. But this is Hooke's spring law so physically this would mean that we have the string end somehow mounted on a spring constrained to be perpendicular to the string itself. Here is how you might implement such a connection:



The thin vertical cylinder provides a frictionless constraint for the two springs and is glued to the wall. Later in Exercise 7.27 page 264 Stak refers to this arrangement as an "elastic support" of the string end.

Normal Modes. In a problem with no driving force (q = 0) and Dirichlet u=0 on σ, everything is run by the IC functions f_1 and f_2. By inserting the above wave bilinear expansion into 7.116 Stak shows that (and so did I)

$$u(t,\mathbf{x}) = \Sigma_k [\sin \{ \sqrt{\lambda_k}\, (t) \} / \sqrt{\lambda_k}]\ \varphi_k(x)\ f_{2,k} + \Sigma_k [\cos \{ \sqrt{\lambda_k}\, (t) \}]\ \varphi_k(x)\ f_{1,k}$$ p 248 C

where eg $f_{1,k} = \int dx\ f_1(x)\ \varphi_k^*(x)$. These are just the familiar normal modes of free oscillation for such a system, each mode driven by a component of the initial displacement or velocity function, $\omega_k = \sqrt{\lambda_k}$.

Time Laplace on 7.114 to get alternate derivation of the bilinear formula (249)

This alternate method still uses the spatial eigenfunctions $\varphi_k(x)$, but instead of solving the time ODE for the $g_k(t)$, Stak transforms Laplace time t into variable s, solves the Helmholtz thing in s space for $G_k(s)$, and then does the inverse transform back to t space, wherein we pick up an infinite number of simple pole residues and this then becomes the sum 7.118 as shown 16" above. I did most of the details in the raw notes.

7.8 Problems in one spatial dimension (249)

The Infinite String (249)

From Chap 5 we know the "struck at x=0" propagator to be $C_1(|x|,t) = (1/2)H(t-|x|)$, so we know that the response to a "pluck at x=0" is $\partial_t C_1(|x|,t) = (1/2)[\ \delta(x+t) + \delta(x-t)\ ]$. I plot both these responses in 3D using Maple and correlate with Staks 2D drawing. [struck = init velox δ; plucked = init displace δ.]

I then have a long digression to clear up various confusions about combining Heaviside functions with absolute values. Conclusions are:

Theorem 1: If t > 0, then H(t-|x|) = H(x+t) - H(x-t)

Theorem 2: H(y) + H(-y) = 1

Theorem 3: $\partial_t H(t-|x|) = \delta(x+t) + \delta(x-t)$ but only one or the other δ can have a hit (dep on signx)
$\partial_x H(t-|x|) = \delta(x+t) - \delta(x-t)$

We already have 7.116 as our "general solution" so if we set $g = C_1$ and realize there is no boundary so no last term, we find after much fiddling this fully general result

$$2\,u(t_0,x_0) = \int_0^{t_0} dt \int_{x_0-(t_0-t)}^{x_0+(t_0-t)} dx\ q(t,x) + \int_{x_0-t_0}^{x_0+t_0} dx\ f_2(x) + [\ f_1(x_0-t_0) + f_1(x_0+t_0)\ ] \qquad 7.122$$

where Stak seems to have a sign error on the f_2 term. If you do a "pure pluck" with some shape f_1 and no f_2 and you have no q, you can see that the response is that shape holding perfectly and one f_1 shaped pulse going to the right and another copy to the left, with overall 1/2. This is the essence of "the wave equation" -- it supports waves with no distortion. (undamped).

The Half String (249)

The propagator C_{half} is obtained using the same image method we used for heat, where we add a negative "strike" on the left at $x = -x_0$, and this maintains $C_{half} = 0$ at $x = 0$ as required. Thus,

$$\begin{aligned} 2C_{half}(x_0,t_0|\ x,t) &= 2C_1(x_0,t_0|\ x,t) - 2C_1(x_0,t_0|\ -x,t) && \text{// as per 7.123} \\ &= H(t_0 - t - |x-x_0|) - H(t_0 - t - |x+x_0|) && \text{// as per p 250 A} \end{aligned}$$

We can of course install this C_{half} into our "general solution" 7.116 in a mechanical way and we could then write down the general solution. Rather than do this, Stak suggests we assume $f_1 = f_2 = q = 0$, and consider just the "boundary driven" term in 7.116 which involves h(t) applied at the x=0 string end. In this case, we learn from 7.116 that [recall that g=0 at string end, $g = C_{half}$ is a Dirichlet propagator]

$$u(t_0,x_0) = -\int_0^{t_0} dt \int_{\sigma_n} dS_n\ [\ h(t,x)\partial_{nx}\, g(x_0|x) - g(x_0|x)\partial_n h(t,x)\ ] = -\int_0^{t_0} dt\ h(t)\partial_{nx}\, g(x_0|x)$$

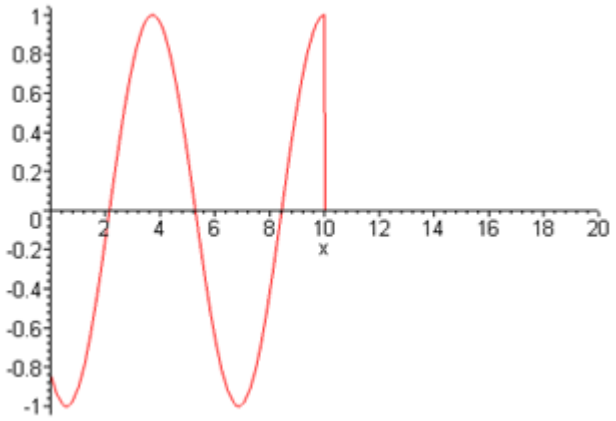
With some effort we then compute then that

$$\partial_x C_{half}(x_0,t_0|\ x,t)|^{x=0} = \delta(t_0-t\ -x_0) - \delta(t_0-t+x_0) \qquad \text{// agrees with p 252 C}$$

and when this is installed into the above equation, we find as our solution

$$u(t,x) = h(t-x) \quad \text{if } x < t \qquad \text{we know that } x > 0 \text{ already}$$
$$= 0 \quad \text{if } x > t$$

Recall that we have $f_1 = 0$ so the initial string is flat (and is also unstruck). This accounts for the second line above, since for x>t the wave disturbance has not yet reached point x so it is still at x = 0. The first line just says the applied stimulus h(t) is "carried off as a waveform" that moves to the right on the string at velocity 1 in our case. I did a simple Maple plot for if h(t) = cos(t) to get



where the red line is on the axis to the right of x = 10.

Now in this case, we have h(0) = 1 which is inconsistent with u(0,0) = 0 by 7.124 and u(0,0) = h(0) = 1. I think this is OK, because we have u(x,0) = 0 almost everywhere. In my example, u is discontinuous at x = 10. The issue here is that the wave disturbance travels at a certain finite speed and has not yet reached the string beyond this point. [Recall that the ω = 0 A-heated half-rod has the erfc solution so small influence "travels" instantaneously to all points on the rod. In contrast, the ω=0 A-displaced at the end string will have the influence travel in the form of a square wave at finite velocity 1.]

Stak claims we got this solution u(x,t) in Chapter 5 using the "method of characteristics".

Stak states that you can alternatively solve this problem using a spatial Fourier Sine transform and in the raw notes I explain why I agree. Then I show that you can get the same solution by taking an infinite string and wiggling it in the middle with an applied q, and I show that picture.

The Finite String (252)

Method of Images versus the Bilinear Formula for g

What is the propagator g in this case? We of course know the $\varphi_i(x)$ for a string tied down at both ends, and then we can insert that into our general wave bilinear formula to get 7.126, so there it is. On the other hand, we can construct this same g using the method of images, and I show how that works and the result is 7.125 which certainly looks a lot different than 7.126. I then wrote a Maple animation which shows what 7.125 looks like in time, and the result was fascinating and unexpected. At t = 0 you have your "delta strike" occur at all the image points, and little square waves appear and start to spread, starting as thin matchsticks. But after the pulses overlap, you end up with a wavetrain of positive pulses moving to the right and negative ones going to the left, and things "thin up" when there is overlap, and we always

maintain the required result that g = 0 at all the finite string segment boundaries, so this superposition of simple square waves must be the unique solution and it must equal 7.126. See graphs in raw notes.

I then do a plot of 7.126 and get this same set of graphs, though there is ringing because I take a finite number of terms.

General solution u(x,t) for the finite string

We limit our interest here to a pure "pluck" initial condition (so q = 0 and f_2= 0) and explore the general form 7.116 to see what happens. Inserting our complicated g, we end up with this result (B and C p 253)

$$u(t,x) = \Sigma_{k=1}^{\infty} \cos(k\pi t/\ell) \sin(k\pi x/\ell) f_k \qquad f_k = (2/\ell) \int_0^{\ell} dx_0 \, f(x_0) \sin(k\pi x_0/\ell)$$

But, if we expand the cos*sin product we get E and finally we get F where we see our $f_1(x)$ waves going off in opposite directions. Stak uses an asterisk on these functions to indicate that we have extended the initial condition $f_1(x)$ [which he calls f(x)] off the limited interval $(0,\ell)$ so that it exists on the entire infinite string, as if such a string existed. We do this by first drawing the odd reflection of $f_1(x)$ in $(-\ell,0)$ and by then replicating what we then have for $(-\ell,\ell)$ to the entire infinite string (see pencil drawing p 253). I sometimes refer to this as "oddizing" the function f_1 from its initial specification in $(0,\ell)$. You can see from the left equation above that u(t,x) must be odd in x if you interpret the formula outside $(0,\ell)$. I comment more on this in my transforms.doc in the discussion of applying a Fourier Sine transform to a function defined only on $(0,\pi)$. Such a function is not really "odd" until you "oddize it". This extended solution is of course exactly in agreement with the method of images construction above. In my example I had just a square wavetrain moving left and another moving right, whereas here we have the more general $f_1(x)$ shape moving left and right as p253 F indicates.

7.9 Problems in more than one spatial dimension (253)

Problems in 3D with no boundary: propagator is C_3

As with all odd-n dimensions, C_3 (the no-σ 3D wave Causal Green's Function) has only a δ wavefront, and specifically it is $C_3 = \delta(t-R)/(4\pi R)$ where $R = |\mathbf{x}-\mathbf{x}_0|$. The lack of wake has major implications as outlined below in that "influence" only occurs when t = R, so to speak. A comparison of the wave equation to Jackson's φ and A equations shows that we can regard our solution φ as a time-dependent potential that is driven by q = ρ with, as we shall see, an appropriate time delay.

It takes some work, but Stak inserts this $g = C_3$ into the general solution 1.116 and the result is 7.129 on page 255. Here we assume there is no boundary σ, which is why we can use $g = C_3$, and which is why there is no h. You see that things depend, as always, on f_1, f_2 and q, where f* are certain spherical surface averages as noted in the text. Looking at the q term, we see that u (our φ) is given by ρ(at earlier time)dx /4πR which is just the electrostatic result but with the appropriate "wave influence time delay"! This kind of term is usually called "the retarded potential". If q (=ρ) has no time dependence, and if we let $t_0 \rightarrow \infty$ to enlarge the integration region to all space, we just get "electrostatics", and that explains why C_3 has the same 1/4πR factor which appears in electrostatics.

The f_2^* is worth some comment. If you observe u from location x_0 at time t_0, the only portion of f_2 that can influence this u is that portion of f_2 that is located on a spherical shell of radius t_0 centered at your observation location x_0. . It is a shell, not a sphere, due to the no-wake δ influence notion mentioned in the first paragraph above. If you think of this influence shell as being little pieces $f_2(x)dx = \delta(x-x_1)dx$, then if you then add up the effect of the entire shell, you get in effect the average of $f_2(x)$ on that shell which is what $f_2^*(x_0)$ is. The f_1^* term is similar in general nature, but different in details. It still has that same δ influence to a shell notion.

I did my own little exercise to show *directly* that C_3 really works, which means I showed that

$$(\partial_t^2-\nabla^2)\,[\theta(t)\delta(t-R)/4\pi R] = \delta(t)\delta(R) \qquad (1)$$

This was much harder than I thought it would be (Stak did not do it) and I had to clarify a lot of Delta Function Rules, see doc of that name in the Stak Distribution Stuff folder.

Regarding wakes, in contrast to the 3D, the 1D Green's function is $C_1 = \theta(t)\, H(t-R)/2$ and the "wake" in this case is permanent and is the same size as the wavefront. In 2D we get halfway between these two solutions. There is a wake, but it is maximal just behind the wavefront which is infinite but integrable. We have $C_2 = (1/2\pi)\, H(t-R)/\sqrt{t^2-R^2}$, and this looks like the charge density on a disk of radius t and coordinate $\rho = R$! I'll bet there is some connection here. Remember that all these Green's functions can be interpreted as "a wave" (a solution to the homo wave equation) that results from a $\delta(x-x_0)$ "velocity impulse" at t = 0, not a "displacement impulse".

Hadamard's Method of Descent (255)

Stak shows how you can start with knowledge of C_3 and apply it to a 2D geometry (line of source) to obtain the C_2 function. This idea of "if you know more, you know less" is precisely Hadamard's idea -- you "descend" from n=3 to n=2. This idea is normally applied to the wave equation, but I suppose you could apply it to any equation you wanted. Stak then does another example, going from C_3 down to C_1 by having not a line charge, but a plane charge.

In the raw notes, I comment on other "math methods" that have a similar sounding name but which are unrelated. The Method of Steepest Descent is a way to find local minima (including saddle points) of a function, and that can be used in the Method of Stationary Phase to approximate integrals with fast winding exponential factors. I show some nice pictures regarding these methods.

7.10 Wave Equation with External Damping (257)

There are several major ideas here.

First, Stak considers the damped wave equation with no driving function q but with an $f_2 = f$ strike IC as shown in 7.131, and also with u = 0 on the boundary σ. If we were to take f(x) = δ, then 7.131 defines the "struck" Causal Green's Function for the Damped Wave Equation. I explain in the raw notes why the form of the damping term is reasonable (linear in velocity with correct sign to be damping).

Second, Stak shows that the solution u to 7.131 tells you the solution v to 7.132 which is the "plucked" BC system. The two solutions are related as in 7.133. Stak proves this and I did the details. The

extra term was not present of course when we proved this same Big Theorem for the undamped wave equation.

Third, Stak converts the system 7.131 into the "equation of telegraphy" as shown in 7.135, by extracting the factor $e^{-\gamma t}$ as in 7.134. I comment on this equation in the raw notes somewhat.

Fourth, Stak solves this telegraphy thing using the usual method of expanding onto the spatial Laplace eigenfunctions, which leads to a time ODE for the coefficients here $w_k(t)$. He solves this ODE system and gets the result shown in page 258 D, but he has two big typos which I have fixed. With these coefficients, the final solution for u is as shown in 7.137.

I then show that if you were to go to the Green's Function limit and take f(x) = δ, then this 7.137 becomes "the general bilinear form for the damped wave equation Causal Green's Function", and I compare this to the other three bilinear forms I know of (Laplace, heat, undamped wave).

Notice that we have NOT solved the system 7.131 if there is a q term present. We would have to trace that term into the telegraphy equation and learn what it does. So we have the solution to f_2 only, and the Big Theorem gives us then a solution to f_1 only, and we could then linearly combine these to get a solution to f_1 and f_2 only. It is not clear what happens to the big theorem if q is present. Also, we have assumed all along that h=0 on the boundary σ, so really we have only done a small piece of the general solution to the damped wave equation. I could imagine developing the full blown theory for the damped equation, where we get a modified version of 7.116 and find the Causal Green's function and then we would have a general solution.

Stak points out that those coefficients $w_k(t)$ which go with eigenfunctions $\varphi_k(t)$ have slightly different behavior depending on the relative size of λ_k and γ, such that in a given mode k you could have the usual underdamped, overdamped, or perfectly damped cases. Here is the Google corrected result:

Therefore

$$w_k = \begin{cases} f_k \dfrac{\sinh\,(\gamma^2 - \lambda_k)^{1/2} t}{(\gamma^2 - \lambda_k)^{1/2}} & \text{for } k \text{ such that } \lambda_k < \gamma^2; \\ f_k t & \text{for } k \text{ such that } \lambda_k = \gamma^2; \\ f_k \dfrac{\sin\,(\lambda_k - \gamma^2)^{1/2} t}{(\lambda_k - \gamma^2)^{1/2}} & \text{for } k \text{ such that } \lambda_k > \gamma^2. \end{cases}$$

7.11 Monochromatic Excitation and Principle of Limiting Absorption (259)

In 7.138 Stak states what I would call the full damped wave equation problem, with f_1 and f_2 and q and h all involved. But rather than attempt this full thing, he at once assumes $e^{-i\omega t}$ forms for q and h, so we are only looking at a special case (hence the title "monochromatic excitation"). We assume a solution of the form 7.139 and then try to find U(x) and v(x,t) that makes it all work. [Note that we could arrange for q and h to have some constant phase difference by assuming an $e^{i\varphi 0}$ factor in q or h.]

One possible solution is U that solves 7.140 and v that solves 7.141. Notice that the U system involves only q and h, whereas the v system involves f_1 and f_2 and U(x).

First consider the v system. The general claim is that if we knew U(x), we could completely solve the v system using methods of the previous section. But we are not going to solve for v because we know that v will be some damped transient form that goes away in time, and we are going to be only interested in

the long term steady state solution! Thus, we are going to ignore v(x,t) in 7.139, now that we understand it to be a function that decays in time and does not persist in the steady state.

So now we can concentrate on function U(x). I show in Plan B in the raw notes, using "the usual" Green's Theorem development, that the solution U(x) as a function of q and h is given by 7.143. This is the unique solution to the U problem. I show in the raw notes that the second term (a surface integral) satisfies the homo damped wave equation, and this involves some tricky δ function stuff.

To summarize, we have found the unique steady state solution to system 7.138 in the special case that both q and h have $e^{-i\omega t}$ time dependence. This steady-state solution is then just $e^{-i\omega t}$ times U(x) as given in 7.143. Note that λ is the specific complex number shown in 7.140, a function of ω and γ.

The rest of this section discusses the limit $\gamma \to 0$. At first this seems a trivial matter, but then we see there is a bit more to it. This limit is in fact an unstable region especially if region R is unbounded, since we know that for $\gamma > 0$ our system is going to "blow up" and go out of control, and the same might happen at $\gamma = 0$ if you approach it wrong. There is going to be a cut somewhere, and we need to approach the cut from the correct side. In general, the right thing to do is imagine there is some small $\gamma > 0$, solve the problem that way, at least approximately for small $\gamma > 0$, (the problem will then have a fully defined and clean slightly damped solution), then take the limit of that solution as $\gamma \to 0+$. One way to think of the limit is that you are working with $\omega + i\varepsilon$ with $\varepsilon > 0$. That is the essence of 7.144, and this method of taking the $\gamma \to 0$ limit is called "the principle of limiting absorption": you assume some absorption is present, then you take the limit of its going away.

Energy Considerations.

We need to first review page 244 A which is energy conservation (with $\gamma = 0$) stated in differential form. If we combine the first and third terms together, and use symbol **J** as shown there, we can write 244B in this manner

$$\partial_t \mathcal{E} + \nabla \bullet \mathbf{J} = 0 \qquad \mathcal{E} = (1/2)\dot{v}^2 + (1/2)\,|\nabla v|^2 \quad = \text{kinetic + potential}$$

For a tiny volume dx, this says that the decrease in the total stored energy in dx must equal the rate at which energy flows out as an energy flux through the boundary of dx. This can of course be stated a little more reasonably by integrating over a region R, and then we have 244B which says

$$\partial_t \int dx\, \mathcal{E}(x) = \int dS\, \mathbf{n} \bullet \mathbf{J}$$

Now if we have a damping factor γ, there will be a frictional energy loss and we get 7.145 where we have the new velocity dependent term on the RHS. Stak claims that this is very easy to derive and says how to do it, and I suppose I did it while reading the text, I have no raw note details on this, but it does make sense. You see that my $\mathcal{E}$ above agrees with p 262 A. The integrated version of 7.145 is 7.147 and this applies to the damped wave equation in a source-free (q=0) region.

Now I have a large digression on obtaining an expression for $\mathbf{J} = -(\partial_t u)\nabla u$ where we want to use for u our steady-state damped wave equation solution U(x) $e^{-i\omega t}$ found in the previous section. But we want to do the real world now, so we have to use $u = \mathrm{Re}[U(x)\, e^{-i\omega t}]$. If we ignore fast oscillating terms and keep just the slow time averaged stuff, I show that J is given by (he calls this J_a for averaged)

$$\mathbf{J_a} = -(\omega/2)\,\mathrm{Im}\,(U(x)\,\nabla U^*(x)) = (\omega/4i)\,\{\,U^*(x)\,\nabla U(x) - U(x)\,\nabla U^*(x)\,\} \qquad 7.148$$

and then I show that

$$\int dS\ \mathbf{n}\bullet\mathbf{J_a} = \text{as shown in 7.149} = \text{proportional to } \gamma$$

If we have no damping, this last integral is therefore 0, and there is no time averaged flux going across the boundary, and that means that on average the total energy (E_a) in our region is constant. I think in the microscale there might be small t changes in E due to activity close to the boundary, but when averaged these details wash out.

I recall that this complex U business was used by Jackson and others in playing with Poynting vectors where we assumed complex E and B fields so we could use $e^{-i\omega t}$ time dependence. You do this anytime you have monochrome time dependence and something that is bilinear in the field of interest.

Comment: I can imagine that $(1/2)\dot{\mathbf{v}}^2 d^n x$ is the "kinetic energy" where **v** is the displacement at position x in n dimensions, think mass density m = 1 perhaps. But it is unclear to me why $(1/2)\,|\nabla v|^2$ should be the "potential energy". Somehow this represents a stretching of the medium. The current **J** is also unclear. This is "continuum mechanics" stuff where I am very weak. Stak just mentions things "in passing" without much derivation at the micro scale.

Exercises: The Wave Midterm Exam

Exercise 7.19. Derive 7.148 .

I already did this in the earlier raw notes, and I then show how you get the same result if you take Im{...} instead of Re{...}.

Exercise 7.20. Show that the uniqueness idea is maintained when we add the γ term.

On page 244 Stak derived uniqueness of the wave equation solution by considering the system for the difference function v. The question is to see how this little proof goes if we add our γ term. I wrote this up and I think I got the thing done right.

Note: Here is where my meta review ended on 5.26.11 and I am now again to this point 7.3.11 (having in the last few days reviewed this entire meta review to here), so there has been a Stakgold work hiatus of 37 days! Ouch! I "did" the rest of these exercises in the raw2 notes but did not get them into the meta review, so that is what I will do next.

Exercise 7.21. Kirchhoff's Formula

First of all, go back to 7.129 on page 255. There we write the 3D solution $u(x_0,t_0)$ of the full wave equation in all of R_3 (all 3D space) in terms of an integral of the regarded potential $q/4\pi R$ [this was the integral called I] plus two terms involving surface integrals over certain math spheres of the IC's f_1 and f_2. The integral of $q/4\pi R$ is restricted by $R < t_0$. Now look at the Kirchhoff Formula 7.151 on page 263. We have the same first integral, but in *this* problem we have a finite region R with boundary σ (don't confuse this R with $R = |x-x_0|$), so the first integrals have the same integrands, but different integration regions. In addition to this volume integral first time, 7.151 has three surface terms which are also very different from those in 7.129. I comment in the raw notes that this Kirchhoff formula is interesting but requires a lot of tricky information on the boundary σ.

In the raw notes I do a full derivation of 7.151 starting from the cylinder fact 7.7. There are a lot of steps to this derivation and even in pretty dense notation it takes 6 pages of raw notes!

Exercise 7.22. Use Kirchhoff's Formula do Derive 7.129 on page 255.

In the previous exercise I compared 7.129 and 7.151, and now we are suddenly asked to derive the first from the second. I do this in full in the raw notes and it comes out perfectly. The secret always is picking a local spherical coordinate system around observation point x_0 in which $\mathbf{r} = \mathbf{x}-\mathbf{x}_0$ and $|\mathbf{r}| = r$. Vector $\mathbf{n}$ is always the unit normal at some point on σ and we get to say $\partial_{\mathbf{n}} r = \mathbf{n} \bullet \nabla(r) = \mathbf{n} \bullet \hat{\mathbf{r}}$ and so on. So at least we have found something "useful" to do with this Kirchhoff's Formula baby.

Exercise 7.23. A 2D example of the general solution form 7.116

The exercise here is to take the 2D infinite space propagator $C_2(x,t) = (1/2\pi)\, H(t-r)/\sqrt{t^2-r^2}$, convert it to standard form $C_2(\mathbf{x}_0, t_0; \mathbf{x}, t) = (1/2\pi)\, H(t_0-t-r)/\sqrt{(t-t_0)^2-r^2}$, and then jam that as g into the general solution form 7.116. I do all this and obtain this solution result, but get no verification from Stak:

$$2\pi\, u(t_0,\mathbf{x}_0) = \int_{\mathbf{r}<t_0} d^2x \int_0^{t_0-r} dt\; q(t,\mathbf{x})/\sqrt{(t-t_0)^2-r^2}$$

$$+ \int_{\mathbf{r}<t_0} d^2x\; f_2(x)/\sqrt{t_0^2-r^2} + \partial_{t_0}\left\{ \int_{\mathbf{r}<t_0} d^2x\; f_1(x)/\sqrt{t_0^2-r^2} \right\}$$

"I might add some comments. For n = 2, there is a "wake" in C_2 and that is why the two initial condition terms are integrals over disks and not just circles. Similarly, for given r, the effect of q(x,t) is limited such that $t_0 - r > 0$ which says $t_0 > r$."

Exercise 7.24. Infinite String with Air Resistance using 5.169

We know the propagator for this problem to be $C_1(r,t;\gamma) = (1/2)e^{-\gamma t} I_0(\gamma\sqrt{t^2-r^2})\, H(t-r)$. We are asked to provide the general solution to the initial value problem which I interpret as the general solution in the presence of q, h, f_1 and f_2. We cannot just stick this C_1 in as g in our 7.116, because 7.116 only applies when $\gamma = 0$. My conclusion is that this problem is asking us to redo the entire theory for $\gamma > 0$ including that Green's Theorem starting point. I declined to do the problem, feeling it is asking too much right now. But it certainly is a reasonable problem. I think the telegraphy equation might be involved in one approach.

Exercise 7.25. Method of Descent Application

"In 5.168 we have C_3 which is the damped wave equation free propagator with $\gamma = 1$, and it involves the function $I_0'(\sqrt{t^2-r^2}\ )$. Our task here is to first superpose a line of sources along the z axis, say, and from this superposition obtain a formula for C_2 (Hadamard's Method of Descent). Then we consider a whole sheet of sources and we are supposed to find C_1. We know C_1 from 5.167, and we could get C_2 using methods of Chapter 5 (iteration idea shown top page 58 in 5.131). We know from p 68 C how to reinstall $\gamma \neq 1$ into C_n. So I think I could mechanically do this problem, but I am going to skip it. Unlike the previous problem, I know HOW to do this problem and it is just a matter of doing it. In the previous problem I am facing uncharted waters. "

Exercise 7.26. Redoing the end-wiggle half-string problem 3 alternative ways.

"System 7.124 is for the half string with $f_1 = f_2 = q = 0$ and we just have an h driving the left end x=0. The text solved this problem on page 251 by using the surviving σ term in general formula 7.116 for the wave equation and the solution is the wiggle solution for which I did a Maple animation. In this exercise, we are supposed to solve this same problem by three alternate methods. Those methods are time Laplace, space Fourier Sine, and a difference function method. The first two ideas are similar no doubt to the general idea of Exercise 7.16. So this is really 3 problems in one, each would take me at least an hour. So this goes into the rainy day bin. Right now I am just trying to understand what the problems are! "

Exercise 7.27. Half-string with elastic support at x = 0: Find the Causal Green's Function

The first complication here is to figure out what the elastic support means. I did that, and it turns out it is the same as having a radiative boundary condition at x = 0 in the heat world. Stak did that problem for us, images did not work, and it was a bit complicated. We had to replace g by $v = \partial_x g - \theta g$. When you do that here in wave world, you end up with a certain system for function v and you can solve this system using the standard 7.116 method where you set $f_1 = q = h = 0$ and $f_2 = \partial_x \delta(x-x_0) - \theta\, \delta(x-x_0)$ and where you set $g = C_{half}$, the half string propagator which we know is a certain C_1- C_1. So that is the method, and I declined to carry out all the mechanical steps, don't think there is anything radically new here. Also, Stak does not give the answer, so I would have no verification.

Exercise 7.28. Solve the String Ring problem.

We did this several ways in heat world (heat conducting ring of 1D wire), we are supposed now to do that in wave world in two different ways: (1) images; (2) eigenfunctions.

" Recall that on page 211 we did "the wire ring" in heat world with BC's like those stated here. We took the ring and unwrapped it to get an infinite wire with two sets of image sources. The ring then corresponded to the piece of wire $(0,\ell)$. We can do the same thing here, but each source has a response that is the infinite string wave propagator C_1 instead of the infinite string heat propagator C_1. This will produce a result that we will then want to transform into another sum by the Poisson sum formula. Recall that we also did this "finite rod" problem using the eigenfunction method on page 216, so I presume we could also use that method here with the "finite string" in Wave World. "

So again I declined to carry out the problem, but think I could do it.

Exercise 7.29. The internally damped string problem.

"Another heavy duty problem. Up to now we have assumed damping was a $\gamma\, \partial_t u$ velocity type term as might apply with external air friction. But here Stak attempts to model the notion of "internal damping" of an infinite string by considering as a string a series of equal masses connected by equal springs with frictional dashpots of some sort. Sadly Stak does not have a picture, so we don't quite know what he means. Perhaps these dampers are velocity dampers one of which parallels each spring. The distinction here would then be that these dampers are affected by displacement along the string, not displacement perp to the string. Whatever he has in mind, if we take a continuum limit somehow, he claims we end up with a PDE which is the wave equation with the addition of a new and ugly looking $\partial_t \partial_x^2 u$ term. So this is an entirely different L operator, neither heat nor wave nor damped wave as discussed in the text. Our problem is first to derive this PDE, and then to find the normal modes of the PDE, presumably in a fashion similar to what the text did with the externally damped wave equation (with only a striking function f(x) as in 7.131). This could be a tough problem, perhaps you have to come up with some intermediate step like the telegraphy equation. Or maybe you do a time Laplace transform. Probably various ways to handle it. I understand the statement of this problem except for the picture, but I don't want to undertake doing a solution. "

Exercise 7.30. Solving a certain 1D driven Helmholtz ODE

"The k^2 constant is λ, more or less, and we have a specific and simple driving function. On page 265 B Stak shows that this ODE has a homo solution which is the first two terms, which will be quite easy to show, but then we have to find the particular solution shown in C which we then add to the homo solution. I think this gets tied in with the Weyl stuff of Chapter 4, and he claims there is only one finite-s-norm solution relative to $|x| \to \infty$, so maybe limit-point. This is the case if λ is off the positive real axis. But if λ is on this real axis, then solutions only exist if $a = n\pi/\sqrt{\lambda}$. I guess we could interpret these as EV's for $L = -\partial_x^2 - f_a(x)$ where we then study $Lu = \lambda u$. After all this work, we see that he is concocting an example of the **limiting absorption principle**, but that was in terms of the damped wave equation. Maybe this $f_a(x)$ acts in a damping fashion here. Perhaps there is an issue of which direction you select to approach the real positive λ axis. So doing this problem would take me on a long review voyage of the Weyl world, and then I would have to get the solution, and then somehow fit this peg into the limiting absorption hole. This last part is very unobvious to me right now. The real axis solutions are only stable at the eigenvalue points, so I suppose the issue is how we approach these stable solutions in the λ plane. But I don't think there is any λ cut on the right side, so any approach ought to be OK. But maybe you only have stable solutions above these points and not below them?

OK, so I actually did 5 of Stak's 12 problems here. He probably spent 30 years getting these problems developed, I cannot do all that work in my passing rowboat. But I do think it is important to understand what all the problems are asking, and I have done that above. "

7.12 "Helmholtz Potential Theory" (265)

The title is mine. We are going to repeat a lot of our Laplace potential theory with an extra λ term sitting in the ODE's operator L, and I call this "Helmholtz potential theory" where $L_\lambda = -\nabla^2 - \lambda$. The motivation

of course is that in both heat and wave equations (mostly wave), if we do a time-Laplace to replace t by s, we end up with a Helmholtz potential theory problem where s is a fixed parameter. Stak give a list of where this kind of thing has happened in our past work.

Free-space Fundamental Solutions (266)

In this section Stak reviews our "fundy" Helmholtz solutions from earlier work, but only in n = 1,2,3 dimensions, called E_1, E_2, E_3. Yes, we did the general E_n case back in Ch 5. He makes a big deal about where λ is located in the λ plane, but for me this is not such a big deal. When λ is on the neg λ real axis, he writes $\lambda = -k^2$. And when on the positive λ axis, he writes $\lambda = \omega^2$. He writes E_3, E_2 and E_1 in terms of λ and k^2 and ω on pp 266-267. We are going to do a lot of fiddling with the E_2 fundy, expressing it as various sums and integrals! Only E_2 is a "special function".

Addition Theorem for Cylindrical Waves (268)

First, I will note the results of this section, then I will comment on the methods used to get those results. The results are 7.163 or 7.165, and then 7.166 where E=E_2 is 7.159. First, we already know from 7.155 that

$$E_2(r,\varphi|r_0,0;\lambda) = (i/4)H_0^{(1)}(\sqrt{\lambda}\, R) = (1/2\pi)K_0(kR) \qquad \text{the last when } \lambda = -k^2$$

So here are the results:

$$E_2(r,\varphi|r_0,0;\lambda) = (i/4)\sum_n e^{in\varphi} J_n(\sqrt{\lambda}\, r_<) H_n^{(1)}(\sqrt{\lambda}\, r_>) \qquad 7.163$$

where n sums over all integers. Then 7.165 gives the folded ε_n form. Then get $I_n(kr_<) K_n(kr_>)$ with $(1/2\pi)$ in the case that $\lambda = -k^2$ and this is 7.166. Now for method. What we are really doing is regarding E_2 as some $f(r,\varphi)$ and expanding this on the complete set $e^{in\varphi}$ and then finding the "coefficient" $a_n(r)$. This is the old technique Stak called "the partial eigenfunction expansion method for finding Green's functions in potential theory". Your fundy Green's equation becomes a 1D Green's equation in r which you solve, and that is just what he does here in Helmholtz potential theory. So we don't need any fancy interpretation of the expansions shown here, they are just "what you get" when you expand on the $e^{in\varphi}$ basis functions. In general, whatever basis functions you start with, you will be getting some kind of "expansion" for your Green's functions in terms of homo equation solutions.

Riemann Surface E function defined on φ in $(-\infty,\infty)$ (270)

In this section Stak provides an "alternate derivation" of 7.163 noted above.

The presentation is quite convoluted. We know that the fundy solution E_2 is driven by a $\delta(\varphi)$ term in its Green's ODE where we think of φ in $(0,2\pi)$ say. But we know that we can "extend this to" (I would say, "interpret E_2 as") some E defined on all real φ since our E_2 is obviously periodic in φ. Then E must solve 7.167 as its Green's. If we write $E = \Sigma_n E_s(2\pi n)$ so to speak (as in p 271F), then E_s solves 7.168 with just $\delta(\varphi)$. The idea is that E is singular at all these φ points, but E_s only at $\varphi = 0$. The "Riemann" word is used because we are thinking of E and E_s as being defined on the entire φ complex plane and each 2π

wide strip maps into a different Riemann sheet E = E(φ), or $E_s = E_s(\varphi)$. Once we have some f(φ) defined on the entire real axis for φ, then instead of doing Fourier series as $\Sigma_n e^{in\varphi}$ we can do Fourier integral as $\int dn\, e^{in\varphi}$ and Stak uses this transform to convert the ODE from 7.168 to p 271 C (he uses α for n). We are using the same "partial eigenfunction method" as before, but here we have continuous n. We get our 1D Green's in r (just what it was before p 271 D) and we end up with E_s as a dn=dα integral p 271 E. But what we really want is the fundy E_2 = E (by interpretation) so we have the extra sum as in p 271 G. Notice that the sum index appears here in a very simple way (I would use integer m maybe). When you then apply the famous Poisson Sum Formula to this, we get our alternate derivation of 7.163 (the PSF interprets the integral as a Fourier projection). Finally, note that 7.169 can be written as 7.171 when $\lambda = -k^2$.

Another Poisson Formula Example (from next wedge section)

Here I am doing a little math problem whose result will be required in the next subsection. The problem is to show that p 272 C (a certain image sum of E_s functions) is equivalent to 7.173. I do this in the raw2 notes, and when we use E_s = 7.169, the equivalence is just an application of the Poisson sum formula.

Green's Function for a 2D infinite wedge using images (272)

This is a Helmholtz potential theory analog of the electrostatics problem of putting a charge between two parallel metal plates of separation D in the z direction. In that 3D electrostatics problem, if we add two infinite sets of image charges outside those plates), we can arrange for the potential to be 0 on both plates. In the 2D version of this we arrange for the potential to be 0 on both edges of a strip of width D (and that is the problem we once solved using this image method). In our current problem, the z direction is the infinite φ direction and in that direction we have two "parallel plates" (having φ = constant) separated by distance ψ. We have our one true "point charge" between these plates (here, a Helmholtz point source which makes its 2D fundy solution E_s) and we again add in the two infinite sets of image sources, and that is what p 272C says. This sum causes the Helmholtz "displacement" (field, potential..) to vanish on both those "plates" (as I show in the raw notes). Of course this is a model for the 2D problem of an infinite wedge where the potential vanishes at φ = 0 and φ = ψ. From our little math problem above, we find that we can rewrite our solution not as a sum of images, but as a the sum 7.173 which is what we would have obtained had we done the "partial eigenfunction sin(nφ) expansion method". Notice that we somewhere did this infinite wedge problem in regular Laplace potential theory (I think).

Another Form for E_2 (273-5) : summary of raw doc very long effort

My raw2 notes on this 2 page section of Stakgold fill about 19 pages! I had a lot of stuff to get straight, and of course every waypoint equation had to be derived, some not so easy to do.

(1) The sum rule. First, I will note the result of this section, then I will comment on the methods used to get that result. The result is 7.180 where we express E_2 as a dγ integral whose integrand contains $K_{i\gamma}(kr)$ $K_{i\gamma}(kr_0)$. Since E_2 is itself $K_0/2\pi$, this 7.180 is an interesting integral sum rule involving only K functions (and a ch).

As for the method, as usual we start with our defining fundy Green's PDE for E_2. This time we transform this PDE using the Kantorovich-Lebedev (KL) projection formula and the transformed Green's ODE is then p 273 A with certain BC's. With the KL we are replacing r by conjugate variable γ so the resulting ODE is in φ. Note that this is just the opposite of what we have been doing up this point -- replacing φ by n or $n=\alpha$ and having an ODE in just r. The solution of this φ ODE by inspection is p 273C. Then the KL expansion formula (which has $K_{i\gamma}(kr)$) gives us our sum rule result. The point here is that the KL thing removes r from the problem and gives us a fairly simple 1D Green's problem in φ. We end up with "yet another weird expansion for E_2". [The KL expansion allows us to expand an arbitrary reasonable function f(r) on $(0,\infty)$ onto a set of "harmonics" $K_{i\gamma}(kr)$ where $\lambda = -k^2$. These K functions form a complete set on our r interval $(0,\infty)$. This set is associated with the SL problem which is the homo Helmholtz equation, see doc elsewhere. Not clear how you recover some kind of Laplace result as $k\to 0$ where the K function limit I think is $\sim (kr)^{-i\gamma}$.]

(2) the knife-edge problem. (274-5) This is not an application of the above sum rule, but just something that uses a similar development. The knife edge problem is really the infinite Helmholtz wedge problem with $\psi = 2\pi$. The point source is put on the left (picture p 274) at $\varphi_0 = \pi$. We then use the "Dirichlet method of finding a Green's Function" by writing $g = E_2 + v$. In terms of v, our 2D Green's PDE is 7.182 which has no delta source since that is taken care of by the E_2 term. We now have $v = -E_2$ on the two boundaries. We apply the KL projection to replace r by γ, and the resulting ODE in φ is so trivial that we can just write the solution as in p 275 B. We solve for A and B to get E (huge Stak error here), and then we use the KL expansion to find v and then g. Unfortunately, Stak never states the resulting v or g so I had to forge my own way here through his various errors. Here are some of my results

$$\tilde{v}(\gamma,\varphi|r_0,\pi) = -K_{i\gamma}(kr_0)\, ch[\gamma(\varphi-\pi)] / [\gamma\, sh(2\pi\gamma)] \qquad \text{// corrected p 175 E} \qquad (2)$$

$$v(r,\varphi) = -(1/\pi^2) \int_0^\infty d\gamma\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, ch[\gamma(\varphi-\pi)] / ch(\gamma\pi)$$

$$g(r,\varphi|r_0,\pi) = (1/\pi^2) \int_0^\infty d\gamma\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, \{\ ch[\gamma(\pi - |\varphi-\pi|)] - ch[\gamma(\varphi-\pi)] / ch(\gamma\pi)\ \}$$

(3) show same as wedge solution (275). In the raw notes I show how you rewrite the $d\gamma$ contour as $(-\infty,\infty)$ and then deform it to pick up poles of $ch(\gamma\pi)$ and the above then gives 7.183 which agrees with our wedge formula founded earlier in the case $\psi = 2\pi$.

My raw notes contain *lots* of extra support material in little subsections which I will summarize here in the following lettered sections. This is then followed by the *'d sections in which I solved the knife-edge problem and did the conversion of that solution from integral to sum.

(a) Bessel ODE Review. Here I ponder solutions to certain two-constant Bessel's equations:

$$-(rR')' - arR - br^{-1}R = 0 \qquad b = -c = -n^2 \qquad J_n(\sqrt{a}\, r) = J_{\sqrt{c}}(\sqrt{a}\, r) = J_{\pm i\sqrt{b}}(\sqrt{a}\, r)$$
$$-(rR')' + drR - br^{-1}R = 0 \qquad b = -c = -n^2 \qquad K_n(\sqrt{d}\, r) = K_{\sqrt{c}}(\sqrt{d}\, r) = K_{\pm i\sqrt{b}}(\sqrt{d}\, r) \quad (*)$$

(b) Bessel ODE in Vol I . I go back now and do a huge Volume I K-L transform exercise on page 317 which I neglected to do while reading that volume. I apply part (a) above to page 317 and conclude that

$$-(xw')' + \mu x w - \lambda x^{-1} w = 0 \qquad \text{// homo}$$
$$-(xg')' + \mu x g - \lambda x^{-1} g = \delta(x-\xi) \qquad \text{// Green's}$$

and we shall use (*) above to make these replacements

(*)	p317
R	w
r	x
d	μ
b	λ

Our solutions are then going to look like $K_{\pm i\sqrt{\lambda}}(\sqrt{\mu}\, r)$. Stak is always changing the names of the Bessel constants, and I am just confirming here that on page 317 he has the right solutions.

(c) KL Transform Notes (moved). At this point I made lots of notes here concerning the KL transform, but I have moved them all to a separate doc of that name in the Transforms folder. Go see that doc for details of the KL and how it is derived.

(d) Express the K-L transform and Derive p 273 A. Here I just translate the Volume I KL transform into the version we see in Volume II on page 273 as (7.175,6), and then I apply this to the PDE and end up with a 1D equation in variable φ which is shown in p 273 A.

(e) Solve the ODE for $\tilde{E}$ etc. I show that p 273 C is the solution of ODE p 273 A

(f) More on the $\tilde{E}$ formula p 273. I show that this is the correct $\tilde{E}$ if we have the source at $\varphi_0 = 0$. This is significant because in our knife application, we need to move the source to $\varphi_0 = \pi$.

(g) Solve the "very thin wedge" (Knife edge problem) (274)

* Verify that you can just add twiddles to get the γ space BC's:
* Solve for A and B
* Compute the right version of p 275 E for $\tilde{v}$
* Compute v from $\tilde{v}$
* Verify that (7.183) is the correctly stated wedge limit of 7.174
* Transform our integral for g into the sum shown in 7.183
* Series for the E piece of g.
* Series for the v piece of g and then add to get g = v + E
* Before continuing, we have to clarify the poles of 1/ sh(2γπ) .
* The final steps

(i) Hankel Transform (275)

One use of 2D Helmholtz potential theory is in 3D problems with azimuthal symmetry in cylindrical coordinates. In this case, if you do the usual "separation of variables" you find decoupled equations in r and z as shown in 7.184 and 7.185. As we keep seeing in this chapter, you can choose to "start" with either equation and consider the separation constant as an "eigenvalue" and consider an appropriate "transform" to work with. The transform which is "appropriate" is the one which is associated with the eigenfunctions of your selected equation. In this case, those functions for the z equation are e^{ikz} type functions and the appropriate transform is the Fourier Transform since the z range is $(-\infty,\infty)$. For the r equation, the eigenfunctions are the $J_0(kr)$ type functions since this is a Bessel equation of that form, and the appropriate transform in this case is the $\nu=0$ Hankel transform, hence the title of this subsection. See doc about the Hankel transform derivation and comparison with the KL transform. In the raw notes I show more details of this idea.

(j) Source Ring in Free Space (276)

Here we set up a "Helmholtz Poisson equation" problem, a phrase I hope makes sense. The "charge density" appearing on the RHS is distributional ring of radius a and total charge 1, such that if you were to take $a\to 0$, this ring would become a point charge allowing you to check results in that limit. Of course since this is not potential theory per se, we refer to the ring as a "source" not a "charge density". The homo PDE for this problem is of course our H equation in cylindrical coordinates, and we are then faced with the choice of r or z exactly as discussed above in (i), and we now consider each in turn.

(k) Hankel Transform of Order Zero on the Radial Coordinate (276)

Working here with the r coordinate, we state the PDE including our source driving term. The solution we seek to this PDE is u(r,z). We apply the Hankel Transform projection to the PDE, do some manipulations (such as parts integration) and end up with an ODE 7.190 for $U_H(\gamma,z)$ where of course γ is the Hankel variable conjugate to r. This ODE is a Green's Function ODE and we find its solution as in C. Then we use the inverse Hankel and get our result for u(r,z) as shown in 7.191. This is the main result of this section, it is the Helmholtz potential due to the driving ring source.

Then Stak considers the $a\to 0$ limit as discussed above, in which case we know that $u = E_2$, and this leads to a series of interesting integrals that one could add to a list like that of GR7.

Special attention is paid to the Helmholtz parameter λ. The integrand of our solution 7.191 is of course a function of λ, and there is seen to be a cut in the λ plane starting at $\lambda = \gamma^2$ and going off to $\lambda = +\infty$. As long as we avoid this cut, our solution is clearly defined. For example, when $\lambda = -k^2$, we are on the left and far from the cut. If we want to understand the solution on the cut, we follow our "principle of limited absorption" and take the limit from above the cut. See raw notes for details.

In the original 2D PDE 7.188 (but homo), the parameter λ can be interpreted as the eigenvalue of a 2D differential operator. But when we convert this to an 1D ODE for $U_H(\gamma,z)$ we get 7.190, and in that equation, taken as homo, we find that $\lambda' \equiv (\gamma^2-\lambda)$ is the eigenvalue. From our Volume 1 spectral theory of 1D differential operators, we know that since we have an infinite range for z, the spectrum for λ' will be (is likely to be) the entire continuous positive real axis. If we observe that cut in λ space, we get the cut just noted above. So an interesting idea here is that the original λ eigenvalue gets "shifted" when you work with a 1D system resulting from doing your transform.

(l) Fourier Transform on z (278)

Here we repeat the previous section, but this time we apply a z Fourier transform and end up with a 1D ODE Green's Function system for u^(r,α) p 278 C, where α is Fourier conjugate to z. [Aside: α is often used for this instead of my traditional k because k can then be used as the Bessel argument scaling factor.] We solve for the 1D Green's function as in E, and get our solution u(r,z) as in 7.196. This of course must be equal to our solution done in the last section. Stak then takes the a→0 limit again and we get more integrals to add to our GR7 collection.

In a brief 1" section on page 279 Stak writes u(r,z) as a double transform integral, and I think he has an error here. [I was wrong, his form is correct.] It is pretty clear what this is, but not clear why it would be useful, though it does "elevate" the solution in such a way that you "see" in the integrand the two kinds of eigenfunctions noted above. In other words, the r and z dependence is very simple in this double-elevated integral representation of the solution u(r,z). This might allow you to differentiate the solution $\partial_{\mathbf{r}}$ or $\partial_{\mathbf{z}}$, or perhaps do some other integral of u(r,z) against some other functions, etc. [This equation never reappears in Ch 7.] Sneddon often does things like this.

The rest of page 279 is a list of more integrals that we deduce from 7.197's Fourier inversion. These are all integral representations of K_0.

Application: Now on page 280 we leave the presentation of "Helmholtz potential theory" and do our first "application" of this theory to an evolution problem! That problem is the 3D heat equation with an initial condition f(**x**) for temperature which is that same ring we used in our Poisson equation above. At t = 0 our initial temperature distribution in all of 3D space is u = 0 everywhere except on this ring, so thermal energy is concentrated there at t = 0. We then consider t>0 and ask for u(**r**,t) for all time. At first I was confused by this problem because I was thinking of the ring as a heat source, but it is not a heat source in the Poisson sense, it is just an initial condition, such as a 1D rod starting off with u = 0 everywhere except perhaps u = 1 on some finite segment of the rod.

As a first method of solving this problem, Stak thinks of the ring initial condition as a superposition of "point" initial conditions. We know the 3D free space propagator which tells us the temperature everywhere with such a point initial condition, so we just integrate that propagator with point initial conditions around the ring as in A, and we do the integral (cylindrical coordinates) and get our solution v(r,z,t) as in 7.204. This is just a little strange and I know I have commented on it elsewhere. We are superposing solutions to different problems to get the solution of our desired problem. This only works because in each problem the initial condition is 0 for all the other problems!

Now the second method is really the main act here. We start with the heat equation PDE and do a time Laplace on it to get p 280 D which is a Helmholtz equation, and is in fact exactly the problem we solved two different ways above (with λ = -s) . We pick one of those two solutions 7.191, and then we jam that back into the inverse time Laplace and the result is 7.205. Presumably we could go look up this integral in GR and we would then find the solution 7.204 ! (Do I detect an exercise coming?)

An excellent problem. Notice what is happening here. In our heat equation problem, the ring is an initial condition. But when we do the time Laplace Transform of $\partial_{\mathbf{t}}$u, we get sU(s)-u(0) and that u(0) is our ring initial condition, so the ring that starts off as an initial condition ends up as a Poisson type ring driving function in our transformed heat equation which is then a Helmholtz equation with a ring source.

Concluding Comment: This concludes our lengthy 17 page section 7.12 where we took a break from evolution equations to study the Helmholtz equation and "Helmholtz potential theory". And our last effort

above was a use of the Helmholtz work to solve a simple heat flow problem (maybe not so simple!). We did a time Laplace to convert the heat equation into a Helmholtz one.

We are now at 79 pages in this raw2 log, so I will start raw3 now (but will continue here in this meta doc)! We still have 50 text pages to go in Chapter 7.

7.13 Half-plane excited by a line source or a plane wave (281)

Stak sets up the "half plane" scenario with a nearby "line source", so we can think of this problem either in 3D as a line source near (and parallel to) a half plane (but otherwise in an arbitrary location), or in 2D as a point source near a half-line, which is the $\psi = 2\pi$ limit of our infinite wedge scenario, and which is also our "knife-edge" scenario (except source placement is now general). Stak first solves this problem using the partial eigenfunctions method $g = \Sigma_{n=1}^{\infty} g_n \sin(n\varphi/2)$, we solve a 1D radial Green's for g_n and get for g the result 7.207 we got earlier for this problem. The source is at $\mathbf{r}_0$ while the observation point is at $\mathbf{r}$.

Remember we are solving for Helmholtz spatial Green's (source is point in 2D) with parameter λ. There is no "wave yet" and no time either, but that will be coming below.

In the raw notes I discuss how the 3D and 2D problems are related in terms of the size of the source.

Stak now launches into a rewriting of the known half-plane result in a new form which is 7.217, the sum of two integrals. To arrive at this new form, he decomposes the solution g into g_1+g_2 based on reflection symmetry in the half plane, and we interpret g_1 as the "field" of the line source and its - image in free space, while g_2 is the same line source and its + image but also with the g=0 half plane. I argue that you can think of these really as the superposition of 3 problems, and the first two are very simple and have just E_2 solutions! Back in the wave equation context, the first problem will be the incident wave, the second the reflected one, and the complicated third problem's solution g_2 will be "diffraction". Stak burns a lot of oil getting the g_2 solution using elaborate methods. But in the end he adds the two g_1 and g_2 solutions to get 7.217. The first integral in 7.217 is associated with the source location, and the second with the image location.

At this point, I examine the upper endpoints of the two integrals and, by taking a sort of improper limit $r_0 \to \infty$, I see that there are "three regions" and certain integrals are 0 in certain regions. I jumped the gun on this because the limit is not well-defined yet, but the regions are still correct as are the limits of the "cosine" in the upper endpoints. Based on region, the upper endpoints are either $+\infty$, or they are $-\infty$ in which case the integral vanishes. [premature because integrands also depend on r_0]

Plane Wave Excitation in presence of the half plane. (285)

We now move the line source far away so that its E_2 wavefronts look like plane waves at the half plane edge. We slip back into the wave equation momentarily to talk about what plane waves look like, then we are back in Helmholtz space. We find that we have to "rescale" the E_2 line source in size so that when it is distant, the wave at the half-plane edge is a unity plane wave. We then process our form 7.217 and now we can take the formal limit $r_0 \to \infty$ and we get the two-integrals of form 7.220 which is the exact wave solution to this plane-wave problem! The integrals here are things like erf and Fresnel functions. The next step is to take $\lambda \to \infty$ so we go into the short wavelength limit, and out pops 7.221 with the extremely clear interpretation of incident wave, reflected wave, and shadow region - picture on page 287.

I expended some effort deriving 7.220 in detail. So Stak has snuck up on optics "scattering theory" !

A Time Dependent Problem Example: Make the line source be pulsed at t=0.

We start here with the time domain wave equation p 287 A and process it into B with the usual Laplace time transform turning t into s. The Green's δ(t) becomes just 1, we get our Helmholtz-space Green's equation as usual with $\lambda = -s^2$. We recognize this equation as precisely the equation we solved to get 7.217, so he quotes this result with $\sqrt{\lambda} = +is$, bottom page 287. We want then to take this back to the time domain and we do that in p 288A where the time deltas arise from the e^{-as} factors in 7.222 which become δ(t-a). It turns out that all the δ hits create a term of the same form $+[t^2-R^2]^{-1/2}$ (or R*) which I later realized is just the E_2 wave propagator (with its infinite front edge at t = R and its following wake). The next step is to analyze the cosine factors again in the Regions I,II,III to see whether the δ hits actually occur or not in each case. After many lines of math, we end up with the massive result on page 289. At this point I have to go to an earlier point in my notes called "the meaning of the Source and Image" and there I have a detailed interpretation of the massive result. It is really all quite clear.

[On 12.10.11 I reviewed the above meta notes starting at Section 7.12, "Helmholtz potential theory". The day before, I updated a certain doc which shows a special form for a 2D L that separates and which form is heavily used in this Stak book section.

Exercises: The Helmholtz Midterm Exam

The problems I did here appear in separate docs in the Ch 7 support folder. Ones I did not do say read only at the start.

Exercise 7.31 (290). 3D Helmholtz Green's Function for point charge on +z axis, sphericals

We consider here the 3D Helmholtz Green's PDE with a point source on the + z axis, so there is φ azisym. Our task is to solve the PDE for u(r,θ), so we have a 2-variable problem. We select θ for eigenfunctions which are $P_n(\cos\theta)$, and in the usual way this leads to a 1D Green's in variable r which we solve in terms of certain J and H Bessel functions, and the resulting u(r,θ) is the RHS of 7.224. But since there were no local BC's in this problem, that RHS must also be E_3 which is then the LHS of 7.224. In the usual way we rewrite this equation for $\lambda = -k^2$ and then the Bessel I and K functions appear in 7.225.

Exercise 7.32 (290). Same 3D as above, but replace the z-axis point source with a ring source.

It turns out that, when you write the δδ for this ring source, it is exactly the same as the δδ source for the previous problem except now we have $\delta(\theta-\theta_0)$ instead of δ(θ). We are still azisym so still an r,θ problem. The exact same solution method then applies and you get the exact same result except where we used to have a 1, we now have $P_n(\cos\theta_0)$.

Exercise 7.33 (290) 3D initial condition heat problem, spherical hole in ∞ medium

Problem Statement: Consider an infinite 3D uniform heat conducting medium with a spherical hole of radius r = a at the origin. The hole contains vacuum which we assume does not conduct heat. The "boundary" here then is the sphere at r = a and the great sphere at r = ∞ and we assume that u=0 on the

boundary. We take the entire medium to be at u = 0 for time t< 0. At time t=0 we turn on a spherical shell initial condition at radius $r_0 > a$ lying inside the medium. This is not a continuous heat source, it is just an initial condition, like saying the left inch of a frozen rod is at u=1 at t = 0+ε. The heat energy in this initial condition region is going to spread out (with inwards and outwards), and that is our problem. This problem has both θ and φ symmetry, so is a 1D problem in variable r. Recall how a heat source initial condition turns into a Helmholtz equation driving source, and that happens here.

To solve this problem, we first do a time Laplace to replace t by s, and we then get a 1D Helmholtz Green's problem in r for U(r,s): $-(r^2U')' - \lambda r^2 U = \delta(r-r_0)/4\pi$ where $\lambda = -s$. We realize that this is just the special case n=0 of a Green's equation we encountered in Ex 7.31 (whose function was called $a_n(r)$) so at this point we know the general atomic form of the solution must be $Z_{1/2}(\sqrt{\lambda}\, r)/\sqrt{r}$. Also, we have certain BC's on U(r,s) at r=a and r=∞ (0 in both places). We then solve the Green's problem in r to get

$$U(r,s) = (i/8\sqrt{rr_0}\,)\,[\,H_{1/2}^{(1)}(\sqrt{\lambda}\, a)J_{1/2}(\sqrt{\lambda}\, r_<) - J_{1/2}(\sqrt{\lambda}\, a)\, H_{1/2}^{(1)}(\sqrt{\lambda}\, r_<)]\,[H_{1/2}^{(1)}(\sqrt{\lambda}\, r_>)/\, H_{1/2}^{(1)}(\sqrt{\lambda}\, a)]$$

But then since $\lambda = -s$ this gets rewritten tat

$$U(r,s) = (1/4\pi\sqrt{rr_0}\,)[\;K_{1/2}^{(1)}(\sqrt{s}\, a)I_{1/2}(\sqrt{s}\, r_<) - I_{1/2}(\sqrt{s}\, a)\, K_{1/2}^{(1)}(\sqrt{s}\, r_<)][K_{1/2}^{(1)}(\,\sqrt{s}\, r_>)/\, K_{1/2}^{(1)}(\,\sqrt{s}\, a)]$$

But these simple K functions are in fact just elementary functions, allowing the above to be written as

$$U(r,s) = -(1/[8\pi r\, r_0])\{\quad \exp[\sqrt{s}\,(2a-r-r_0)\,] - \exp[-\sqrt{s}\,|r-r_0|]\quad\}\;/\,\sqrt{s}$$

I verify at this point that this meets both BC's. Each term above has a trivial Schaum inverse Laplace, and we then get this very simple result

$$u(r,t) = (1/8\pi r r_0)\{\,\exp(-|r-r_0|^2/4t)/\sqrt{\pi t}\;\; - \exp(-\,(2a-r-r_0)^2/4t)/\sqrt{\pi t}\;\;\}$$

I then take limit $t\to 0$ to verify that we get the initial condition shell δ. I plot the solution in Maple.

Exercise 7.34 (p 290) 2D Helmholtz Green's Function outside a circle on which g=0.

The problem is to find the 2D Helmholtz exterior Green's function outside a circle on which g=0, and we solve the problem by two different methods.

The first method is my "Green's Function by Dirichlet" method where we then get a homo PDE with a certain BC for v where $g = E_2 + v$. We solve the homo by first selecting $e^{in\theta}$ and having unknown $v_n(r)$. The Green's ODE then for $v_n(r)$ is something we already figured out near page 269B with homo solution $Z_n(\sqrt{\lambda}\, r)$, but in this simple exterior case we must have $Z_n(\sqrt{\lambda}\, r) = H^{(1)}{}_n(\sqrt{\lambda}\, r)$ so now $v_n(r) = V_n\, H^{(1)}{}_n(\sqrt{\lambda}\, r)$ and we have only to find the constants V_n. The trick is that we can look at r=a where we know $v_n(a) = -E_2$ and this (with an addition theorem for E_2) then tells us the V_n and we get our solution which is

$$g(r,\theta|r_0,\theta_0) = (i/4)\;\Sigma_n\, e^{in(\theta-\theta 0)} * \qquad r_< = \min(r,r_0)$$
$$\{\;\; J_n(\sqrt{\lambda}\, r_<)H^{(1)}{}_n(\sqrt{\lambda}\, a) - J_n(\sqrt{\lambda}\, a)\, H^{(1)}{}_n(\sqrt{\lambda}\, r_<)\;\;\}\,[H^{(1)}{}_n(\sqrt{\lambda}\, r_>)\,/\,H^{(1)}{}_n(\sqrt{\lambda}\, a)]$$

The second method is more conventional. We start with the δδ Green's, select the same $e^{in\theta}$ eigenfunctions, and get then our usual 1D ODE Green's in r which we then just solve in the usual manner and the same result is obtained as shown above.

Then just for fun I took the "electrostatics limit" $\lambda \to 0$ of the above result and got agreement with 6.100 which was the solution we got eons ago back in potential theory for this same problem.

Exercise 7.35 (p 291) Repeat previous problem using r eigenfunctions instead of θ ones.

My doc on this problem runs 34 pages because I made a certain K error and did various other things. Since it is so long, it has its own summary section.

The problem is to repeat the previous problem, but instead of selecting $e^{in\theta}$ as our eigenfunctions, I at first selected $K_{i\gamma}(kr)$ of the KL transform, and this led to a certain 1D Green's ODE in variable θ which I then solved for $u_i(\theta)$. But then I realized that I was solving the wrong problem, one with a = 0, not a = a. The KL transform for interval (a,∞) is *different* and has a discrete spectrum, not the continuous γ spectrum. So I had to go back to r-space and figure out the new KL series transform. It turns out that the functions are then $K_{\mu i}(kr)$ where μ_i are points where $K_{\mu i}(ka)$ has a zero. This is similar to the Fourier-Bessel transform situation. It is also similar to Smythe's electrostatics cone solution. Here we need μ_i to be a zero to meet the BC that u = 0 on the circle! The θ 1D Green's ODE is the same as I found at the start and I solved it again toward the end. The final solution comes out being

$$u(r,\theta) = -\Sigma_i \; K_{\mu i}(kr)\, K_{\mu i}(kr_0)\, I_{\mu i}(ka)\, \text{Res}[1/K_\mu(ka)]|^{\mu=\mu i} \; [\sin(\mu_i \pi)]^{-1} \cos[\mu_i(\pi - |\theta|)]$$

The only catch is that the zeros of this K function are not tabulated as they are for $J_{\mu i}(ka)$. I looked a bit on the web and found some work on that subject.

Exercise 7.36 (p 291) 2D Helmholtz Neumann Green's Function in slab with ring source.
(partial solution only in separate doc)

Ring Helmholtz (λ_H) source at $z=z_0$, radius r_0, inside infinite slab z = (0,h). I write the usual PDE with its δδ source. I manually separate with sep constant $\lambda=\kappa^2$ and I see that the atoms are $\exp(\pm\sqrt{\lambda-\lambda_H}\, z)$ and $J_0(\kappa\rho)$. This is cylindrical coords with φ azisym. The BC's are Neumann ones, $\partial_n u=0$ on z=0,h so no heat goes through the slab faces so to speak. I only review the solution methods below, I don't carry the solutions through to completion.

I first do Method A using the $J_0(k\rho)$ (spectrum real k) eigenfunctions and as usual this gives me a 1D Green's problem in z which I solve in this generic form where $\kappa^2 = -\,[k^2 - \lambda_H]$

$$U(k,z) = [J_0(kr_0)/2\pi] \cos(\kappa z_<) \,[\, A\sin(\kappa z_>) + B\cos(\kappa z_>)\,] \qquad z_> = \max(z,z_0) \text{ etc}$$

but then I don't complete the solution and leave A and B as constants unknown. The two conditions will be the jump condition and the BC at z = h.

Next I do Method B using $\cos(k_n z)$ (spectrum $k_n = n\pi/h$, integer n) eigenfunctions to meet the two BC's. I change from z to θ to make use of the Fourier cosine series transform. After fiddling, I come up with the expected 1D Green's problem in variable ρ which has solution of form $C\,(1/\pi h)\cos(n\theta_0)\, J_0(\kappa\rho_<)\, H^{(1)}{}_0(\kappa\rho_>)$, but I don't evaluate C.

I could finish off this problem in a few hours, but I feel I have done enough of this type of problem already. The method is very clear to me, and NOW it is very clear to my why that method always works. Originally I was very confused on this "why" question, but that question is now answered in detail in a separate doc "separation and transforms 2D chap 7".

NOTE: It was at this point near the start of August 2011 that I stopped my forward motion in Stak Chap 7, and I went off and first wrote my Stackel separation doc and then my tensor doc. Amazingly these took 3 months to do, and only today Dec 10, 2011 am I resuming Stak Chap 7. Much time has passed and I don't really feel like doing any "extra" mid term problems! But I will at least read them!

Exercise 7.37 (p 291, read only) Find temp inside a slab with Dirichlet initial conditions.

This is a full heat problem using the same slab as in the previous problem (but no ring source anymore I assume). Our boundary condition is u=0 at z=0 and u = f(ρ) at z=h. I assume that the initial condition is that u = 0 everywhere inside the slab, but the solution is probably independent of what initial condition you select. With the u=0 initial condition, heat will flow in from the upper slab surface and we will end up at t=∞ with some temperature u(ρ,z) inside the slab, and that is what we are supposed to compute.

Exercise 7.38 (p 291) Waveguide 3D Helmholtz Green's Function Problem.

This is a genuine 3D Helmholtz problem with all three variables active. Our 3D region is obtained by extruding some finite 2D region R in the z direction. If that region R were a disk, the 3D region would be a cylinder, but Stak I think calls it a cylinder regardless. He is thinking of this as a **waveguide** in a true wave problem where u = 0 on the guide surface which I suppose might apply to a certain wave mode like TE or whatever. But he is not worried about the actual wave problem, just a related Helmholtz problem where u = 0 on the cylinder wall.

We are asked so find a separated solution of the form Z(z) φ(x,y).

The first approach is to look at the transverse PDE that we get from separation which is just a 2D Cartesian Laplace equation and we imagine finding the eigenvalues and the 2D eigenfunctions of this transverse problem. We then use these transverse 2-variable eigenfunctions in an attempt to solve the 3D Helmholtz Green's problem with a point source inside the waveguide. You find that you are left with a 1D Green's function in the variable z which you would solve as something of the form $e^{-a|z|}$ with the jump at z=0. You then insert your resulting 1D z-Green into your expansion on those 2D transverse eigenfunctions, and the result should have the form 7.227. From this form we find that if $\lambda < \nu_1$ (the lowest eigenvalue), waves damp out and cannot propagate (I recall this as being a waveguide cutoff thing).

The second approach is to instead use $e^{-i\alpha z}$ eigenfunctions and we know this will then produce a 2D Green's function problem in x and y which he writes out, the unknown being $\hat{g}(x,y)$. As I think about it, I don't think Stak ever wrote down the generic bilinear φφ* form for a Helmholtz Green's function. For heat we know it is $g(x,t\,;\,x_0,\,t_0) = \Sigma_k[e^{-\lambda_k(t-t0)}]\,\varphi_k(x)\varphi_k^*(x_0)$ so if we were to Laplace this thing, the expo becomes a pole in the s plane which is perhaps α^2 in this problem, and that then would explain the final result he gives on page 292. [see solution for this bilinear formula.]

Exercise 7.39 (p 292) Induced Helmholtz current on half-line with point source off left end (2D).

We are back to our half-plane Helmholtz problem with a line source off the edge of the half plane ($\varphi_0 = \pi$). We are first asked to simplify the general formula 7.217 for this special position of the line source. From this g we could certainly compute the jump he shows at the bottom of page 292. In electrostatics, this would be the charge density (sum on both sides) for a surface, and we would expect a quadratic blow up at the edge. In some way I don't understand right now, in the corresponding wave problem (which becomes the Helmholtz problem), this charge density becomes a time dependent current density on the metal surface. It certainly seems reasonable. See solution doc.

Exercise 7.40 (p 293, read only) Induced current on half-line due to incident plane wave (2D)

More plane-wave Helmholtz. We start with the $r_0 = \infty$ but λ = finite scattering formula 7.220 which recall is exact. We still regard this as a Green's function for our very far away 2D rescaled point (3D line) source which is at some angle φ_0. And the formula bottom page 292 still gives the total "current" density on the half plane, so if we apply that formula to this 7.220 Green's we are supposed to get 7.229. Notice that r_0 does not appear in this result because we are in the r_0 rescaled plane wave limit. In contrast, the previous problem was a finite-r_0 problem. You cannot directly take the $r_o \to \infty$ limit of its result, something we have seen elsewhere, so we cannot compare 7.228 to the $\varphi_0 = \pi$ limit of 7.229. The last part of 7.40 is to set $\varphi_0 = \pi/2$ for the source hitting flat onto the half-plane, and I quickly did that to verify 7.230

Exercise 7.41 (p 293, read only) Redo the half-line problem for Neumann instead of Dirichlet.

In our entire half-plane analysis, we always used u=0 on the half-plane, as if it were grounded. Here we are asked to repeat the entire analysis for $\partial_n u = 0$ on that half plane. So we did Dirichlet, and we are asked now to do Neumann (and of course this is still Helmholtz). So we have to go way back and redo the 2D problem perhaps for finite angles with this BC, then set angle 2π, etc etc.

Exercise 7.42 (p 293, read only) Sound wave interface between two media.

This is a sound wave problem. We have a plane interface between two different media. He means to say medium 1 is on the $z > 0$ side and this is where the point source is put. If we replace time t with frequency ω using FT for monochromatic, I guess we get the two wave equations he shows first, where q are some arbitrary sources. Instead of both media having velocity 1, they are different so there are two k's floating around now. The big mystery for me is why he suddenly switches from "the velocity potential u" to something he calls "the complex potential U". I presume $u = U e^{i\omega t}$ as on page 285, OK. Now our problem involves not one but two equations and two regions. If we go to cylindrical coordinates, the problem is φ azisym, so our only two variables would then be ρ and z. I would rewrite that first equation and get it into our usual $\delta\delta$ form, then I would maybe use eigenfunctions in ρ (the $J_0(k\rho)$) and come up then with a 1D Green's for the other coordinate z which has a $\delta(z-z_0)$ hit. This is why he mentions Hankel transform. So somehow you have to deal with the two regions and the BC's at the interface. A typical bunbuster Stak problem.

Exercise 7.43 (p 294, read only) Solve the 3D Helmholtz sphere with q(x) distributed source.

This is a 3D Helmholtz for a sphere with u=0 on the boundary and driven by some arbitrary distributed source q(x) inside. I guess we are first supposed to know the internal Helmholtz Green's function. Did we do this somewhere, or do we have to figure it out here? I guess we did this for the source point on the z axis in Ex 7.31, so we could maybe modify that result to get a more general Green's. Then we integrate that thing against q(x) to get the solution to this problem as an integral. Then evaluate that Green's function at sphere center, and you are supposed to get the expression shown! OK, some very rainy day maybe I will come back and try this "read only" problem.

7.14 Helmholtz for exterior domains (294) (verbatim)

Up to now we have really only done the half-plane problem where the "scatterer" is not localized. Now we want to consider an arbitrarily shaped 3D "scattering object" which is localized, which means you can enclose it in a sphere of some radius r_0. I am very happy to see Stak confirm my claim made long ago that the implication is that outside that sphere, you can represent the k-space wave equation solution u (ie, the Helmholtz solution) as an expansion on spherical harmonics. [Note: if you separate Helmholtz instead of Laplace in sphericals, the only difference is that the radial functions are $J_{\pm(n+1/2)}(kr)/\sqrt{r}$ instead of r^n and r^{-n-1}, say Moon and Spencer, confirmed below.] Projecting the Helmholtz equation with these harmonics (atoms), you get 7.233 as the radial equation where $u_{mn}(r)$ are the coefficient functions, which he redefines into v_{mn} solving p 295 A. But this is one of our Bessel forms and the solution vanishing at ∞ is going to be $H^{(1)}_{n+1/2}(\sqrt{\lambda}\, r)$ and so we obtain our exterior solution 7.235 for $r>r_0$. This is all very well done and completely clear. If we now take the famous large r limit of this expansion and keep only the leading term in r, we find that u has the form $u(r,\theta,\varphi) = f(\theta,\varphi)e^{ikr}/r$ where $k = \sqrt{\lambda}$ in my private notation. [Note that this is the 3D Helmholtz fundy solution times a coefficient.]

Comment: In this section, Stak never talks about an "incident wave" that gets scattered. He is just talking about an outgoing solution to the Helmholtz equation which far away looks like a decaying E_3 fundy solution modulated by some $f(\theta,\varphi)$ that we don't know, and coming from the region of the source near r=0. The purpose of the incoming wave later is to create those induced charges which then radiate the outgoing wave. But here we just assume that something at the center is creating this outgoing wave some how. It could be a radio transmitter and antenna and then $f(\theta,\varphi)$ would be the antenna pattern. It could be a sound generator of some sort, a loudspeaker. Or an excited atom decaying. The point is that outside the radiator region, far away, regardless of the medium and the physics, if you have a wave equation operating, then this is what the far field looks like, and then $f(\theta,\varphi)$ is dependent on the physics details of the source.

As long as λ has a positive imaginary part, we are getting expo decay here! Stak points out that in Laplace with $\lambda = 0$ we only got *power* decay as shown in p 295 D. Without further ado, I recognize the above form as the outgoing wave (I guess $H^{(2)}$ might have yielded the seldom-used incoming wave). And of course $f(\theta,\varphi)$ is the famous "scattering amplitude". This is then the "partial wave expansion" of the solution of the wave equation in this context, though perhaps that phrase is used more often when there is no φ dependence and you have only $f(\theta)$.

Side note: If we were doing QM and the SE rather than the wave equation, we might interpret u as a quantum amplitude and then $|u|^2 d\Omega$ would be the probability of scattering into $d\Omega$, and then if we started

with a single particle in some sense, unitarity would say $\int |u|^2 dV = 1$ which says $\int d\Omega\, |f(\theta,\varphi)|^2 = 1$ and we see that the 1/r in u squared cancels the $r^2 dr\, d\Omega$ in the volume element. [no absorption]

What is the 2D analog of the above discussion. Outside circle of radius r_0 we can expand our solution on the atoms $e^{in\varphi}$ and it turns out that the radial equation solutions are then $H^{(1)}_n(\sqrt{\lambda}\, r)$ and then the large r form is $u(r,\varphi) = f(\varphi)e^{ikr}/\sqrt{r}$. Remember that in 2D you always get the "straight" Bessel functions, while in 3D you get the "spherical" ones which have n+1/2 half integral orders. So in 2D the partial waves are much simpler, and we have that $\sqrt{r}$ replacing r in the denominator which I know is just a fact of "unitarity". Again, it we were in QM, we could find $\int d\varphi\, |f(\varphi)|^2 = 1$ and the $1/\sqrt{r}$ in u (squared) cancels the r dr dφ in the volume element. These are just passing comments and I intend no rigor.

Radiating Exterior Dirichlet Problem uniqueness of solution (296)

Now we look more closely at our "scattering" problem. We assume that our arbitrarily shaped localized object has u = f(x) on its boundary σ. We imagine some very distant perhaps spherical boundary σ_r. We assume an "outgoing" solution at r=∞. So this is the exterior Helmholtz Dirichlet problem in the scattering context! On page 297 Stak proves in the usual manner that <u>if a solution exists, it must be unique.</u> He defers the existence proof (also as usual; recall that you usually have to construct a solution as part of such a proof). He then has 2.5 pages of "remarks", which are my "comments". I will summarize those remarks here. They basically all relate to the uniqueness proof.

(1) He assumed λ = complex in his uniqueness proof, but it goes through as well with $\lambda = -k^2$ real.

(2) If $\lambda = \omega^2$ conclusion is the same using "**limiting absorption**". But as an alternative, Stak claims that you can get uniqueness by requiring a certain strange condition on u shown in 7.242. He shows that this condition causes there to be a positive outgoing energy flow through a distant spherical boundary he calls σ_r. He claims that if this condition is met, you can show uniqueness with $\lambda = \omega^2$ without appealing to limiting absorption. This seems quite an obscure little sidelight, the **Rellich-Sommerfeld** Radiation Condition it is called.

(3) In 2D the uniqueness proof is as in 3D.

(4) It is possible that the scatterer might have such a **sharp corner** that it causes a singularity in the solution u or its gradient, see page 299 picture. Such a singularity impedes the usual application of Green's Theorem used in the uniqueness proof. The trick is to replace the actual boundary σ by σ' which does not have the sharp corner. Then things will be OK as long as a small spherical integral σ" (p 299A) around the corner problem vanishes. If it does, then the uniqueness proof goes through OK. There are more details here which you can go read p 299. He makes the strange comment that the singularity at a corner or vertex must be weak enough that "there is no source concentrated there". In electrostatics you always have some σ surface charge, especially at a vertex, so I don't know what he means. Defer. I think Stak means there are no non-induced sources at/on the scattering object.

(5) Uniqueness goes through if you assume **Neumann** or **Radiative BC** instead of Dirichlet.

(6) For the usual infinite **wedge problem** which has a sharp vertex, the uniqueness proof is harder.

7.15 The Scattering Problem (299)

It behooves us to do a little review before "entering" this section. Back on page 181 while doing electrostatics, we considered $u = u_0 + u_s$ where u_0 was the potential due to some glued-down charges in space (this includes the notion of a uniform E field due to two walls of infinitely-far-away charges), and we had some localized metal object which developed an induced surface charge σ which in turn created the potential u_s, and the total Laplace solution was then u and u=0 on the grounded object surface. Using the usual "integral equation method" we wrote on p 182 a Fred 1 integral equation for induced σ (inside the integral) driven by u_0 evaluated on the conductor surface, this was equation 6.159. The integrand included the 3D fundy solution there called E. Stak then showed how an alternate Fred 2 integral equation could also be developed as in 6.160.

I recall attempting the charged disk problem using the Fred 1 method and eventually succeeding (by looking up the integral equation!) as described here.

D:\Work\My Interests\Physics\E&M\Electrostatics\ring and disk\the charged disk problem.doc

Now this whole concept recurs in our wave - Helmholtz world. We now have $u = u_i + u_s$ with the new interpretation that u_i represents some kind of wave incident source (converted to Helmholtstatics space with wave number k), and now u_s represents a "scattered" wave. This is in exactly analogy to the Laplace situation, and u_s is generated by induced "currents" on the object surface, and again u = 0 on the surface. In the DC limit that $k \to 0$ this Helmholtz case will replicate the Laplace situation. Due to this new interpretation of the Helmholtstatics case, the subject is now "scattering theory" !!! The analogous integral equation is 7.250 where I and E appear in the integrand, and u_i(surface) is the LHS. So we are supposed to solve this for surface current I, and then use 7.249 to find u_s (and thus u) everywhere. The E here is of course the fundy Helmholtz solution in some dimension n, not the Laplace fundy,

In the study of waves, we end up after time FT with the Helmholtz "static" equation with a fixed parameter. This parameter Stak writes as $\lambda = \omega^2 = -k^2$ (velocity = 1 always). What we learn in this section is that you can repeat *the entire development* of the "integral equation method" in the Helmholtz context and you obtain results that have exactly the same form! We have an object $I = \partial_n u$ which at a surface is the analog of the charge density $\sigma = \partial_n V$ (ignoring sign), and it is an "induced source" in what I call Helmholtz-statics. We know it is an induced source because on the RHS of p 301B each point of this source generates E just as would an isolated point of "real source". This is the same way we know in electrostatics that I is induced charge. I don't yet know how to relate this to current, though that seems reasonable. Now p 301 B is the integral equation you are supposed to solve for I, and then you insert that I into 7.249 to get u(x) everywhere. It is all the same, but with a few twists: (1) the integral equation which determines I has $u_s(s) = -u_i(s)$ on σ for its LHS because u=0 on σ. (2) then the other equation gives u_s.

I have not gone through the details of this section. Note that the incident wave u_i can be from point sources nearby, or it could be a plane wave, things are totally general. It is assumed that there is no interaction at all outside the region R* of the scattering object. It does use Dirichlet u=0 on the scatterer's boundary and I am never quite sure what that means, it depends on what kind of wave scenario you have.

[p 302] Stak then assumes u_i is a plane wave and in effect comes from a point source infinitely far away so $x \to \infty$ in direction **α**. In this limit of 7.249 mentioned above we get 7.252. This formula says that if you view the scattering center from far away (incoming in direction **α**, observer in direction **β**), what you see is a simple E source that is putting out fundy $e^{ikR}/4\pi R$ *but* with an additional factor A(**α**,**β**) that, rather than being a constant, is a function of the two directions. This coefficient A is a surface integral of the induced source I_α times a phasor in the **β** direction, and we call it the far-field scattering amplitude. Nothing penetrates the boundary σ in this Dirichlet picture, scattering is all from the "metal" surface. Stak uses his various expressions then to proved that A(**β**,**α**) = A(-**α**,-**β**) which is the **Reciprocity Principle**.

Notice that in this section Stak does not attempt to solve the Fred 1 integral equation for I. All he does is write the *form* of the scattered Helmholtz wave u_s. In quantum mechanics, if you compare Schiff p 318 (37.11) to Stak (7.252), you see the same "form" applies in quantum scattering, and in the usual distant approximation Schiff gets p 324 (38.1) which then looks exactly like Schiff (7.252). Thus, Stak is basically doing what QM calls "the Born approximation" EXCEPT in QM Schiff has a volume scatterer whereas Stak has a surface scatterer. Also, Schiff has to assume that the scattered field is small compared to the incident one (that is the real "Born approx") whereas Stak does not have to assume this. Both assume the far field. Some day I might pursue the connection here in more detail.

Comment: But this form A(**α**,**β**) $e^{ikR}/4\pi R$ exactly matches the form $f(\theta,\varphi)e^{ikr}/r$ we obtained in the previous section just from spherical coordinate "analysis" of the Helmholtz equation, and assuming an outgoing wave. *Here* we are instead getting this same result from an analysis of our "integral equation method" equation 7.249 which is now in the context of some u_i incident action. Here something is causing the induced charge, whereas previously we just had a transmitter. But in either case, really, we have transmitter action which makes the outgoing wave and that is why in either case the form is the same.

Scattering Cross Section (verbatim)

Stak now uses a general power flux formula developed earlier to compute P_i and P_s, the incident and scattered energy flux. **Cross section** (total) is then $D = P_s/P_i$. Playing with the power formula, he obtains p 304 A which says that D is the energy of u_s crossing out through the entire far boundary. He then comes up at page bottom with the familiar **Optical Theorem** (but I did not do the steps) relating total cross section to imaginary part of forward amplitude.

I have not yet done the details, but there is a big point here: all these results come just from Helmholtz PDE analysis. Any physics for which the Helmholtz applies (for example, sound with amplitude not too large) will show these results. In particular, you get the Helmholtz equation in non-rel QM from the SE (where $\lambda = \pm E$ I suppose), so these conclusions apply to non-relativistic QM scattering. [Note that it is the SE there, not the wave equation, but that both result in Helmholtz.] Probably you could prove all these same things using a KG or Dirac equation to replace the SE. But I am getting off topic. It is nice to see Stak do the basics here with a completely general Helmholtz equation without regard to where it came from. At some point I will do some integration.

It is natural now to examine the small and large ω limits of our results.

The Small ω Behavior (verbatim)

He suggests expanding the induced source density as power of ω with coefficients $A_n(x)$, and expanding the u_i plane wave as well in the very obvious manner. Putting these into the integral equation $u_i = \int dS\ E\ I$, we can balance powers of ω on both sides. For ω = 0 we are going to get electrostatics results and A_0 must be charge density σ as in 7.261. Each power gives you an integral equation for A_n in terms of A_{n-1}, so you have to build your way up through these coefficients. They alternate real and imaginary. Using the same expansion in D from 7.259, we get p 305 B and eventually p 306 A and we find that for small ω, you can write $D = C^2/4\pi$ where C is the capacitance of your object! That is new to me, I don't recall ever seeing such a low frequency claim made before.

The Large ω Behavior

Stak cannot produce a simple series in 1/ω to analyze this situation, so instead he considers scattering off a convex object like the ellipsoid in the page 308 figure,

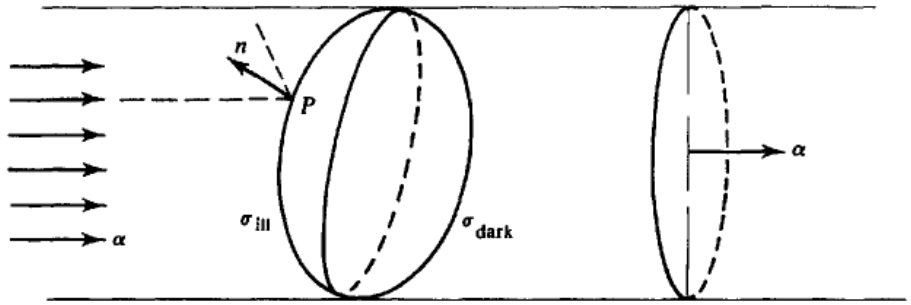


FIGURE 7.16

For very large ω, meaning very small λ compared to the smooth dimensions of the ellipsoid, he argues that you can treat any piece of the Dirichlet ellipsoid surface as a plane mirror. He does a plane mirror analysis and derives a simple expression for the current I as a function of the incident direction **α** and the surface normal **n**, as shown p 308 G. When this current is installed into 7.259 for cross section D, we find that D = twice the projected area of the ellipsoid! (this projected area is shown on the right above). He does not comment on the factor of 2, but I do somewhat from Schiff and it relates to the problem setup. So this is basically an optics large ω limit.

Stationary Principle for the Cross Section (verbatim)

This is a "technical detail" section as far as I am concerned. The idea is this. Suppose you can't exactly solve your integral equation for $I_\alpha(x)$, where α is the direction of your incoming plane wave. Suppose you come up with some kind of approximation for I_α. Then 7.268 is an approximation for A(α,α) and thus for cross section D. Stak shows that you can get a better formula for A(α,α) as in 7.277 which is insensitive to errors you have made in your approximation for I_α. You just install $u = I_\alpha$ and $v = I_{-\alpha}$ into 7.277 and this is supposed to give you an improved estimate for A(α,α) and D. I have not bothered to track through this section, it sounds reasonable, and it falls into the realm of scattering approximation theory. My PDF (djvu?) search says that he is never going to use this result.

Since most of the "final exam" problems apply to the above section, I am going to move them here. The last few problems related to the Wiener-Hopf method and I show them below the next and final section of the book.

Final Exam Part I: (the last 4 problems are after the WH section below)

Exercise 7.44 (p 326, read only). Surface approach, extra term, Fred 2 instead of Fred 1 for I.

If you apply a normal-approach-to-point-s ∂_ν operation onto 7.249, and then take the limit x→s on σ, that "extra term" pops up just as it did back in Chapter 6. I think we are approaching from the exterior of our scattering type object, and I think that I(s) = $\partial_\nu u$ at the boundary since there is doubtless no source on the inner surface. Then $\partial_\nu u_s = \partial_\nu(u-u_i) = \partial_\nu u - \partial_\nu u_i$ = I(s) - $\partial_\nu u_i$ and then we have in p 327 A a full I(s) on the left and a half I(s) on the right and this gives exactly 7.309. This same issue came up on page 172 in the Laplace analysis with perhaps some sign difference, see also the start of my raw 2 notes and perhaps Ch 6 meta notes. So the upshot is that you can have a **Fred 2** integral equation for I(s) instead of the Fred 1 equation 7.250! We know that such equations are always easier to handle, though Stak does not comment here on that fact.

Exercise 7.45 (p 327, read only) Same idea on a shell surface, getting I+ and I- separately.

We repeat the last problem for an **open shell** where we can then have source on both sides, so things are then a little different and we get a new version of 7.309 as p 327 B. He claims that if you use this and the Fred 1 7.250, you can deduce the separate I_+ and I_- contributions. I don't care much about open shells today, so I won't do this problem.

Exercise 7.46 (p 327, partial) Redo scattering theory for Neuman BC on scatterer surface.

Here we are asked to redo "everything" in our **scattering** analysis using the **Neumann** $\partial_n u = 0$ on σ instead of the Dirichlet u= 0 on σ. We would start I guess on page 300. There would be a change in 7.247 that I think is pretty clear, then we have to take it from there. This of course would force me to verify all the stuff in that section which I skipped over!

Exercise 7.47 (p 327, read only). 2D scattering from a circle-shaped u=0 scatterer.

In the text, we considered scattering in 3D off some localized obstacle with boundary σ and developed the integral equation method etc. Here Stak wants us to redo this in 2D. Then in particular, he wants us to examine **2D plane wave scattering off a circle** (σ) with u=0 on this circle. This would apply to the real world with plane wave scattering from an infinite cylinder if the plane wave k vector is normal to the cylinder axis vector. I think this would be a 2D canonical scattering problem and of course the origin would be put at circle center and we might as well have our incoming wave coming in along some axis like the x axis. Polar coordinates would be appropriate.

Exercise 7.48 (p 327, read only). More on 2D scattering from a circle.

We are now asked to find a series for the scattering **cross section** in the above 2D problem. Perhaps that means the ω series, or perhaps it will just be a trig series. A good problem. I could do all these problems, but it would be more weeks! Just reading and understanding the problems I feel is still useful.

Exercise 7.49 (p 327, read only) Point charge off end of half-plane (a 3D problem)

Earlier we studied a line charge off the end of a half plane, which was really the 2D problem of a point charge off the end of a half line. Here, however, we want to do the harder 3D problem of a ***point*** **charge off the end of a half plane** sitting at x = -a. I agree with the Helmholtz Green's problem statement p 327 C. We assume Dirichlet so v=0 on both sides of the half plane. I agree that if you apply the z FT projector to both sides of the PDE, the result will look like 7.295 with a new k. But then we are supposed to use some result in the Weiner-Hopf section to obtain the current on the half-plane as shown p 328 A which involves a K'_0 function. We are then to take the electrostatics limit. I am not sure Stak or I have ever done this electrostatics canonical problem. It shows the expected $1/\sqrt{x}$ blowup at the edge for the charge density. I still don't quite get the connection between H source as current and L source as charge density. This would be a good problem to do.

Exercise 7.50 (p 328, read only) Plane wave hits u=0 screen with aperture head on, 3D problem,

This is a 3D "screen with **aperture** A" problem, plane wave coming in from the left z < 0, the screen at z = 0 and u=0 on the screen. Stak replaces the solution u with a different function v which is "small" in a certain sense (it vanishes as A→0). We are then to restate our integral equation stuff in terms of this v and its current J (rather than u and its current I), and the result should be p 328 F. Stak claims that if you know v in the hole, you know it everywhere. Another one to do! This sounds like a very canonical optics diffraction problem !!

Exercise 7.51 (p 328, read only) Repeat above with Neumann screen and aperture.

Repeat the previous **aperture** problem with $\partial_n u = 0$ on the screen. Somehow we are then supposed to connect this to diffraction of a disk that matches the aperture. These are classic problems indeed!

Exercise 7.52 (p 329, read only) A diffusion problem with absorption.

This was discussed in Ex 7.12 on page 240. Here k^2 is the time transform variable (replacing ∂_t etc). The term cu of 7.12 must be this thing $-k_0^2(x)$, but the sign seems wrong for absorption according to the p 240 text. But I think he is trying to set up a situation where absorption occurs only inside the region R. We have some solute and it is getting consumed inside R at a rate which is position dependent. We also have a point source of solute outside R. It would have been nice if Stak had related this to a real world situation of nuclear physics or chemistry or whatever he has in mind. Since this is a 2-media problem, it is not clear how you solve the exterior Green's function problem. He claims you can relate u at any exterior point to the values of u inside R with an integral equation deal. OK, another day at least to do this I would guess. All are good.

Exercise 7.53 (p 329, read only) 2D Neumann partially blocked waveguide problem.

This is a **2D waveguide that is half blocked** by a line segment. As usual we can think of this as a 3D problem between two infinite parallel plates with a half-blocking plate. We inject a plane wave from the left side $z < 0$ and watch what happens. The wave is going to diffract through this aperture. We are asked some questions concerning an integral equation for this situation. On all surfaces we assume $\partial_n u = 0$ Neumann.

7.16 The Wiener-Hopf Method (311)

The opening section is not about the WH method, it is about how to handle a standard inhomo Fred 2 convolution equation on $(-\infty,\infty)$, example being 7.279. You of course diagonalize this with the FT to get 7.280. This is a particular solution of the integral equation.

To this you can in general add homo solutions of 7.281. Suppose $k-\mu \sim (\omega-\omega_0)$ near $\omega=\omega_0$. Near the zero 7.281 reads $(\omega-\omega_0)\, u\hat{}(\omega) = 0$. We need a solution $u\hat{}(\omega)$ which solves this equation at the zero, and which also solves $(k-\mu)\, u\hat{}(\omega) = 0$ away from the zero. Function $u\hat{}(\omega) = \delta(\omega-\omega_0)$ meets both these requirements, and then $u(x) = \exp(-i\omega_o x)$ agreed, so this is a homo solution.

Stak then generalizes to a zero of order p instead of just a simple 0.

But how exactly does this solve the homo equation back in x-space? Stak provides a simple example starting bottom page 312 to show how it all works.

The **Weiner-Hopf** begins on page 313 bottom. The exact same integral equation except on $(0,\infty)$, so now simple convolution won't work. Kernel k, solution function u, eigenvalue μ, driver function f. I think in fact W-H is exactly what I once did with such an equation 30 years ago!! You extend u and f by adding θ functions and call them u+ and f+, right-sided functions. You invent a new negative side function g and extend it to positive, so it is then g-. Kernel k remains 2-sided, full range. You arrive at 7.285 which now has the full $(-\infty,\infty)$ convolution form, and you diagonalize this *as usual* to get 7.286. This is the first step.

Question: why can't you just solve 7.286 for $u+\hat{}(\omega)$ and be done with it? The reason is that there is another unknown $g-\hat{}(\omega)$! This is very Sneddon-like, by the way.

The second step is to ponder the nature of the FT of something like f+. From page 23

$$f+\hat{}(\omega) = \int_{-\infty}^{\infty} dx\, e^{+i\omega x} f+(x) = \int_{0}^{\infty} dx\, e^{+i\omega x} f(x)$$

You can see that if you move ω up in the ω-plane so $\omega = +i\,\mathrm{Im}(\omega)$, the expo is $e^{-\mathrm{Im}(\omega)x}$ and eventually will force convergence provided f(x) is limited by some $f(x) < e^{d'x}$ in its blowup. If the integral converges, then $f+\hat{}(\omega)$ = some finite number for that ω and for all ω of this type, and this means $f+\hat{}(\omega)$ is analytic for ω above a certain horizontal line in the ω plane! So $f+\hat{}(\omega)$ is analytic in the + half ω-plane and you see that indicated on page 315. You do this same analysis for u+ and for g+ and you end up having to make a set of assumptions as shown in (1)-(4) p 314 for the blow-up of all your functions. Perhaps you cannot meet these assumptions, but assume we can. Since you don't know what u and g are, it might take a little more work to determine their blow up bounds.

We now come to the third step. The idea is to somehow express $k\hat{}(\omega)$ in terms of + and - functions, then to partition 7.286 so that all + functions are on the left and all – functions are on the right. This might require some sum-splitting and quotient splitting, technical matters discussed below.

At this point you have $L+(\omega) = R-(\omega)$. Let's assume that the two functions' analytic regions have a strip of overlap in the ω plane. Then the RHS provides the analytic continuation of the LHS to the entire

lower half plane below the strip, and similarly in the other direction. So there is now one analytic function and our equation can be written $E(\omega) = E(\omega)$.

This brings us to the last step. Suppose we know that $L+(\omega) \rightarrow 0$ going up in the ω plane, and suppose we also know that $R-(\omega) \rightarrow 0$ going down in the ω plane. Then these two limits apply to $E(\omega)$, and by Louiville's Theorem we know that $E(\omega) = 0$. The solution is then $L+(\omega) = 0$ OR $R-(\omega) = 0$. At least one of these equations hopefully will include $u+(\omega)$ and not $g-(\omega)$, and you then find your solution $u+(\omega)$ and then you do the inverse FT to get $u+(x)$ and you have solved the problem !! This IS the W-H method (usually called the WH "technique" I think).

History: I just found a great 2007 PDF by Laurie and Abrahams on this history of the W-H technique. Wiener is the actual Norbert Weiner of MIT cybernetics fame, Hopf had a similar fame, Jew and German. Their original paper is this

Wiener N, Hopf E (1931) Über eine klasse singulärer integralgleichungen. Sem–Ber Preuss Akad Wiss 31:696–706

It seems they were interested at that time in stellar atmospheres. The method can be applied to digital integral equations and this ties it all in with noise theory and all that stuff. Reviewers say that since 1931, there have been thousands (!) of papers elaborating this W-H technique. **Sneddon** was in fact involved, and the W-H Technique bible was a book by **Nobel** published in 1958, republished in 1988.

SECOND (UNALTERED) EDITION

The present work is published at New York, N.Y.
in 1988. It is a Second (textually unaltered) Edition
of a work first published at London in 1958. The
size of type is larger than that of the First Edition.
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Much of this 1988 book appears in google books, but not all.

Sum Splitting. This is the first of two technical issues and I wrote my own little paper on this subject called "analytic functions defined by integral.doc". The general subject is the analyticity of a contour integral in a variable appearing in the integrand. In certain situations, you can write some general $a(\omega)$ as a sum $a+(\omega) + a-(\omega)$, and this is then needed sometimes in getting segregation of the WH projected equation as discussed above. Stak does a little example page 317 with $a(\omega) = 1/(1+\omega^2)$ and he comes up with the correct a+ and a- functions. My paper shows there is more to it than meets the eye.

Quotient Splitting. This is the second technical issue, and an example is writing $k(\omega) = k+(\omega)/k-(\omega)$ and in fact you can treat this as the sum splitting case doing logs on both sides. In practice, both types of splittings are often done "by inspection" using partial fractions.

Two Examples of the W-H Technique

Example 1: (Solve the integral equation 7.292, pp 318-321). In this example (I did ALL the details in the raw notes) , as **Step 1** Stak first finds the Im(ω) lines c,d,d' etc for the various functions involved, and this takes some work especially for u. In **Step 2** he computes f+^(ω) and k+^(ω) and our diagonalized equation is then $[(9+\omega^2)/4(1+\omega^2)]\ u_+\hat{}(\omega) = i/\omega + g_-\hat{}(\omega)$. In **Step 3** we do the quotient split for the bracketed factor and we arrive at

$$(\omega+3i)/4(\omega+i) * u_+\hat{}(\omega) = (\omega-i)/(\omega-3i) * i/\omega + (\omega-i)/(\omega-3i) * g_-\hat{}(\omega)$$

In **Step 4** he then does a sum split to get $(\omega-i)/(\omega-3i) * i/\omega = i/3\omega + 2i/[3(\omega-3i)]$, which leads to this fully segregated equation

$$(\omega+3i)/4(\omega+i) * u_+\hat{}(\omega) - i/3\omega = 2i/[3(\omega-3i)] + (\omega-i)/(\omega-3i) * g_-\hat{}(\omega) \qquad \text{// page 320 F}$$

In **Step 5** he shows that $u_+\hat{}(\omega) \sim \{u_+(0)/i\omega\}$ for large ω and similarly for $g_-\hat{}(\omega)$. Thus we arrive at the point where we can set either side of the above equation to $E(\omega) = 0$, and that then gives us $u_+\hat{}(\omega)$ and $g_-\hat{}(\omega)$. Finally in **Step 6** we do the inverse FT and find that $u(x) = (4/9)\{1 + 2e^{-3x}\}$. In **Step 7** I then look up this integral equation in Polyanin, find it, and verify the solution.

Example 2 [p 321-326!] : Here we revisit our 2D Helmholtstatics problem with the source sitting distance a away from the start of our half line on which u=0. The W-H integral equation now comes from the integral equation method for solving this problem, where we integrate over that half-line and hence get a (0,∞) integral. The 2D Helmholtz propagator appears, called first E, and then written in the usual K_0 form where K_0 is the modified Bessel function Stak calls the McDonald function. Thus, our integral equation is 7.297 which we now treat blindly as a W-H integral equation to be solved. Our to-be-found function is I(ξ), the surface source on the half line (it was u(ξ) in the previous example).

As in the previous example, we do things step at a time. The first step is to find the FT of the K_0 function and I give full details of that, the result is 7.298. It turns out to be a simple elementary function. In Step 1 we find that c = k, the Helmholtz parameter, and d' and d" are also found. I draw the strip picture and the strip ranges -ik to +ik. The diagonalized equation is 7.301 at this point. We grind away on this first using a quotient split, but there is an unpleasant term to deal with in 7.304. A sum split is now performed, which is a bit messy and consumes 2 Stak pages. We are eventually able to write a+(ω) as an erfc shown p 325D, and again I do the gory details in my raw notes. Finally in p 326 A we achieve segregation. He then studies the decay nature of the various terms, and finally we arrive again at $E(\omega) = 0$. This leads to I+^(ω) as in 7.306 which we then FT to get final result 7.307 for I(ξ), an elementary function. My raw notes don't comment on whether this agrees with a result Stak previously obtained for this problem. Earlier in 7.207 we found the Helmholtz Green's Function for this problem with an arbitrary source location. I suppose then we could compute $\partial_n g$ at the surface (half line) to compute I(ξ) and see if that agrees with this W-H method. I don't see that in any of the final exam exercises.

I feel certain there is a direct connection between this method and the dual integral equation approach of Sneddon's book, but here we have the extra fact that we can diagonalize the combined integral equation of convolution form in ω-space.

Final Exam Part II: (four WH related problems)

Exercise 7.54 (p 330, read only) Compute the integral shown p 325 B.

The problem here is to do a certain integral. The result of the integral was used in the main line text on p 325 B. It is one of the sum-splitting integrals in **WH** Example 2.

Exercise 7.55 (p 330, read only) Convert the Example 1 integral equation to ODE and solve.

Another way to find the solution.

Exercise 7.56 (p 330, read only) Half line with plane wave from above, use WH method.

A Weiner-Hopf problem. Stak's Example 2 used a point source off the left end of a half line. Here we use a different source: a plane wave flowing perpendicular to the half line. Our task is again to compute the current on the half-line, and results are given.

Exercise 7.57 (p 331, read only) Solve a certain 2D Helmholtz problem using the WH method.

Another **Weiner-Hopf** problem. We are given a Helmholtz problem for the upper half x-y plane. The BC's are that $\partial_{\mathbf{n}}u = 0$ for the negative x axis, but $u = e^{-\varepsilon x}$ for the positive x axis. No physical motivation is provided for this problem, nor is the solution stated.