Stakgold Chapter 7 raw notes 2 PhL 4.18.11

These notes start on Stak page 243. He has finished is 40 pages on the heat conduction equation and is starting on the wave equation.

Section 7.6 Preliminaries for the Undamped Wave Equation (243) **2**
Uniqueness. 2
p 245: Big Theorem with two parts: 4
Section 7.7 Causal Green's for the Wave Equation (246) **4**
The Two Statements of the PDE system for g (7.113 and 7.114 are similar to 7.8 and 7.8a) (246) 5
Space Eigenfunction Expansion (247) 6
Time Laplace on 7.114 to get alternate derivation of 7.118 (249) 10
Section 7.8 Problems in one spatial dimension (249) **11**
The Infinite String (249) 11
Digression on Heaviside Math and Inequalities 13
Derive 7.122 15
The Semi-Infinite String (251) 17
The Finite String (252) 21
Image Method for Finite String 21
Bilinear Sum Method for Finite String 23
Section 7.9 Problems in more than one spatial dimension (253) **25**
Problems in 3D with no boundary: propagator is C_3 25
Deriving the C_2 result by integrating a line of C_3 results (Hadamard). (p 255) 27
The Method of Steepest Descent 27
Method of Stationary Phase. 28
Section 7.10 Wave Equation with External Damping (257) **29**
Section 7.11 Monochromatic Excitation and Principle of Limiting Absorption (259) **32**
Solving the U(x) Helmholtz Equation to get 7.143 33
The Principle of Limiting Absorption 36
Energy Considerations. 36
Exercises **38**
Exercise 7.19. Details concerning the time averaged current J shown on page 262. 38
Exercise 7.20. Use energy argument to show solution uniqueness for damped wave equation. 39
Exercise 7.21. Kirchhoff's Formula. 39
Exercise 7.22. Kirchhoff's Formula Application 43
Exercise 7.23. A 2D example 45
Exercise 7.24. Infinite String with Air Resistance using 5.169 47
Exercise 7.25. Method of Descent Application 47
Exercise 7.26. Redoing the end-wiggle half-string problem 3 alternative ways. 48
Exercise 7.27. Half-string with elastic support at x = 0. 48
Exercise 7.28. Solve the String Ring problem 50
Exercise 7.29. The internally damped string problem. 50
Exercise 7.30. Solving a certain 1D driven Helmholtz ODE 50
Section 7.12 Helmholtz Green's Functions and Applications (265) **51**

Free-space Fundamental Solutions (266) 52
Addition Theorem for Cylindrical Waves (268) 52
Riemann Surface E function defined on φ in (-∞,∞) 53
Another Poisson Formula Example (from next wedge section) 56
Green's Function for a 2D wedge. 57
Another Form for the Free-Space Green's Function (273) 59
(a) Bessel ODE Review. 59
(b) Bessel ODE in Vol I. 60
(c) KL Transform Notes (moved). 60
(d) Express the K-L transform and Derive p 273 A. 61
(e) Solve the ODE for $\widetilde{\widetilde{E}}$ etc. 62
(f) More on the $\widetilde{\widetilde{E}}$ formula p 273 C, and computing E 63
(g) Comments on the above expansion for Green's Function E 63
(h) Solve the "very thin wedge" (Knife edge problem) 66
* Verify that you can just add twiddles to get the γ space BC's: 66
* Solve for A and B 67
* Compute the right version of p 275 E for $\widetilde{\widetilde{v}}$ 67
* Compute v from $\widetilde{\widetilde{v}}$ 69
* Transform our integral for g into the sum shown in 7.183 70
* Series for the E piece of g. 72
* Series for the v piece of g and then add to get g = v + E 73
* Before continuing, we have to clarify the poles of 1/ sh(2γπ) 73
* The final steps 74
(i) Hankel Transform 75
(j) Source Ring in Free Space (276) 76
(k) Hankel Transform of Order Zero on the Radial Coordinate. 76
(l) Fourier Transform on z 78

Section 7.6 Preliminaries for the Undamped Wave Equation (243)

Uniqueness. The wave equation system definition in 7.109 is perfectly clear. He then wants to prove that the solution to this system is unique, so as usual we assume we have two solutions that are different, and then the difference solution v must also satisfy the system, but with all 0 BC's, by superposition. He then shows this difference solution v must vanish (see following) and therefore the solution is unique. Here are some of the steps of his proof on page 244.

Derive p 244 A

$$\nabla\bullet[(\partial_t v)\nabla v] = (\partial_t v)\,\nabla^2 v + \nabla(\partial_t v)\bullet\nabla v \qquad \text{// vector identity}$$

$$(1/2)\partial_t|\nabla v|^2 = (\partial_t\nabla v)\bullet\nabla v = \nabla(\partial_t v)\bullet\nabla v = \nabla\bullet[(\partial_t v)\nabla v] - (\partial_t v)\,\nabla^2 v \qquad \text{// from above}$$

or

$$(1/2)\partial_t|\nabla v|^2 = \nabla\bullet[(\partial_t v)\nabla v] - (\partial_t v)\,\nabla^2 v$$

or

$$(\partial_t v)\,\nabla^2 v = \nabla\bullet[(\partial_t v)\nabla v] - (1/2)\partial_t|\nabla v|^2$$

Now apply $\partial_t v$ to the PDE and we get

$$(\partial_t v)(\partial_t^2 v) - (\partial_t v)\,\nabla^2 v = 0 \qquad (*)$$

But

$$(\partial_t v)(\partial_t^2 v) = (1/2)\,\partial_t\,(\partial_t v)^2$$

so (*) says

$$(1/2)\,\partial_t\,(\partial_t v)^2 - [\nabla\bullet[(\partial_t v)\nabla v] - (1/2)\partial_t|\nabla v|^2] = 0$$

or

$$(1/2)\,\partial_t\,(\partial_t v)^2 - \nabla\bullet[(\partial_t v)\nabla v] + (1/2)\partial_t|\nabla v|^2] = 0 \qquad \text{// agrees with p 244 A}$$

<u>Derive p 244 B,C,D</u>.

$$(1/2)\,\partial_t\,\{\int_v [(\partial_t v)^2 + |\nabla v|^2\,]\,\} - \int_v \nabla\bullet[(\partial_t v)\nabla v] = 0$$

Use the divergence theorem so

$$\int_v \nabla\bullet[(\partial_t v)\nabla v] = \int_s dS\,\hat{\mathbf{n}}\bullet[(\partial_t v)\nabla v] = \int_s dS(\partial_t v)\,\hat{\mathbf{n}}\bullet[\nabla v] = \int_s dS(\partial_t v)\,(\partial_n v)$$

So we then have

$$(1/2)\,\partial_t\,\{\int_v [(\partial_t v)^2 + |\nabla v|^2\,]\,\} = \int_s dS(\partial_t v)\,(\partial_n v) \qquad \text{// agrees with p 244 B}$$

His interpretation of this equation in terms of left = KE + PE seems somehow right, then RHS must be energy passing through the boundary somehow, he calls it work done by the boundary. I am unable to derive this fact in a simple manner, but since it will be zero from here on, that is fine. Then we get

$$(1/2)\,\partial_t\,\{\int_v [(\partial_t v)^2 + |\nabla v|^2\,]\,\} = C = \text{total conserved energy} \qquad \text{// agrees with p 244 C}$$

We then evaluate this equation at t = 0 and we then get to use the 7.110 IC to deduce that C = 0. We then have

$$\partial_t\,\{\int_v [(\partial_t v)^2 + |\nabla v|^2\,]\,\} = 0$$

or

$$\{\int_V [(\partial_t v)^2 + |\nabla v|^2]\} = C_1$$

Again we evaluate at t = 0 to find $C_1 = 0$, so we then have

$$\{\int_V [(\partial_t v)^2 + |\nabla v|^2]\} = 0$$

at any time. But this is the integral of a positive definite integrand, so that integrand must be zero everywhere,

$$[(\partial_t v)^2 + |\nabla v|^2] = 0$$ // agrees with p 244 D

which means $v = C_2$. But at t=0 we find $C_2 = 0$, so v = 0 everywhere inside. This is a typical Stak careful proof. He then claims *without* proof that if we replace our assumed Dirichlet u = 0 BC with the Neumann or Radiative one, we get the same conclusion, but only if the RHS of 244B is non-negative. [this will no doubt show up as a problem later on.]

The idea of this proof is to show that if a non-trivial difference solution v existed with 0 on all the boundaries, then the energy density of v must be zero everywhere inside R as in p 244 D and if the sum of two positive terms is 0, each term must be zero so v = 0. So uniqueness has been shown.

p 245: Big Theorem with two parts:

(1) If you compare two wave equation problems,

BC = 0	IC = 0	IC' = f(x)	solution u_f
BC = 0	IC = f(x)	IC' = 0	solution v_f

then if you find solution u_f, you can determine the unique solution v simply as $v_f = \partial_t u_f$.

(2) If you find solution u above, you know solution v as stated, and then you can superpose to find that

BC = 0	IC = $f_1(x)$	IC' = $f_2(x)$	solution w = $u_{f2} + v_{f1} = u_{f2} + \partial_t u_{f1}$

Section 7.7 Causal Green's for the Wave Equation (246)

The time invariance claim works like this. We know that the PDE is true for ANY t and t_0. So for example select t = t'+a and t_0= 0. The PDE then reads

$$(\partial_{t'}^2 - \nabla^2)\ g(t'-a, 0) = \delta(x-x_0)\delta(t'-a-0)$$

Now select a = t_0 and then remove prime on t' to get

$(\partial_t^2 - \nabla^2)\ g(t-t_0, 0) = \delta(x-x_0)\delta(t-t_0)$	new PDE
$(\partial_t^2 - \nabla^2)\ g(t, t_0) = \delta(x-x_0)\delta(t-t_0)$	original PDE

Since we claim the solution is unique, we just have $g(t, t_0) = g(t-t_0, 0)$ which is p 246 A. I could imagine showing this in some kind of operator Hilbert Space sense as well.

The Two Statements of the PDE system for g (7.113 and 7.114 are similar to 7.8 and 7.8a) (246)

I copied the heat notes here and then edited them:

(1) First, we define g as the solution of this equation when $t > t_0$ and we say $g = 0$ otherwise

$$(\partial_t^2 - \nabla^2)g(x,t\,;\,x_0,t_0) = \delta(\mathbf{x}-\mathbf{x_0})\delta(t-t_0) \qquad \text{with BC=0 (whichever it is, Dirichlet etc)}$$

Thus, we might write g this way, to make the forward propagation idea explicit,

$$g(x,t\,;\,x_0,t_0) = \theta(t-t_0)\,h(x,t\,;\,x_0,t_0)$$

At this point we have no idea what the functions g (or h) are. Depends on shape of σ. Now consider:

$$(\partial_t^2 - \nabla^2)g(x,t\,;\,x_0,t_0) = (\partial_t^2 - \nabla^2)[\,\theta(t-t_0)\,h(x,t\,;\,x_0,t_0)] \qquad (*)$$

Now consider this first term

$$\partial_t^2[\,\theta(t-t_0)\,h(x,t\,;\,x_0,t_0)]$$

$$= \partial_t\,\partial_t\,[\theta(t-t_0)\,h(x,t\,;\,x_0,t_0)]$$

$$= \partial_t\,[\delta(t-t_0)\,h(x,t\,;\,x_0,t_0)] + \partial_t[\theta(t-t_0)\,\partial_t h(x,t\,;\,x_0,t_0)]$$

$$= [\delta'(t-t_0)\,h(x,t\,;\,x_0,t_0)] + [\delta(t-t_0)\,\partial_t h(x,t\,;\,x_0,t_0)] + [\delta(t-t_0)\,\partial_t h(x,t\,;\,x_0,t_0)] + [\theta(t-t_0)\,\partial_t^2 h(x,t\,;\,x_0,t_0)]$$

$$= [\delta'(t-t_0)\,h(x,t\,;\,x_0,t_0)] + 2[\delta(t-t_0)\,\partial_t h(x,t\,;\,x_0,t_0)] + [\theta(t-t_0)\,\partial_t^2 h(x,t\,;\,x_0,t_0)]$$

But now we can apply Stak Vol I p 39 E which says we can do "parts" on the first term above to get

$$= -[\delta(t-t_0)\,\partial h(x,t\,;\,x_0,t_0)] + 2[\delta(t-t_0)\,\partial_t h(x,t\,;\,x_0,t_0)] + [\theta(t-t_0)\,\partial_t^2 h(x,t\,;\,x_0,t_0)]$$

$$= [\delta(t-t_0)\,\partial_t h(x,t\,;\,x_0,t_0)] + [\theta(t-t_0)\,\partial_t^2 h(x,t\,;\,x_0,t_0)]$$

We now add this to our second term in (*) above to get

$$(\partial_t^2 - \nabla^2)g(x,t\,;\,x_0,t_0) = \theta(t-t_0)\,\partial_t^2 h(x,t\,;\,x_0,t_0) - \theta(t-t_0)\,\nabla^2\,h(x,t\,;\,x_0,t_0) + [\delta(t-t_0)\,\partial_t h(x,t\,;\,x_0,t_0)]$$

$$= \theta(t-t_0)\{ (\partial_t^2 - \nabla^2)\, h(x,t\,;\, x_0, t_0)\} \; + [\delta(t-t_0)\, \partial_t h(x,t\,;\, x_0, t_0)] \qquad (\#)$$

Suppose we could find a function $h(x,t\,;\, x_0, t_0)$ such that [this is the "strike" IC !]

$$(\partial_t^2 - \nabla^2)\, h(x,t\,;\, x_0, t_0) = 0 \qquad \partial_t h(x,t_0+\,;\, x_0, t_0) = \delta(x-x_0) \qquad \text{with BC=0} \qquad (**)$$
$$\text{and} \qquad h(x,t_0+\,;\, x_0, t_0) = 0$$

Then if we insert that h into (#) above we find that the red term vanishes and so

$$(\partial_t^2 - \nabla^2) g(x,t\,;\, x_0, t_0) = [\delta(t-t_0)\, \partial_t h(x,t\,;\, x_0, t_0)]$$

Now the usual delta rule says that $\delta(t-t_0)F(t) = \delta(t-t_0)F(t_0)$ so we then get

$$(\partial_t^2 - \nabla^2) g(x,t\,;\, x_0, t_0) = \delta(t-t_0)\, \partial_t h(x,t_0+\,;\, x_0, t_0) = \delta(t-t_0)\, \delta(x-x_0)$$

So this all goes pretty much like the heat conduction situation, but we have $\partial_t h(x,t_0+\,;\, x_0, t_0) = \delta(x-x_0)$ with the ∂_t sitting there. We have that extra IC $h(x,t_0+\,;\, x_0, t_0) = 0$ sitting in our h definition because the solution is not nailed down without the IC and IC=0 is the simple thing to do and lines up nicely with g = 0 when $t \to t_0$ from the wrong side where $g \equiv 0$. So think of $g(t|t_0) = 0$ for $t \le t_0$ whereas in the heat world we had $t<t_0$ only because we had a delta function $\delta(x)$ when $t = t_0$. Here that $\delta(x)$ shows up in $\partial_t g$ instead. Note that 7.113 says nothing about $g(x,t_0+\,;\, x_0, t_0) = 0$ so we really are adding this fact, without which there would not be a unique solution.

If we do Dirichlet, we will have h = 0 on σ and then also g = 0 on σ. If we have Neumann or Radiative, we get the corresponding thing on both h and g. So the text assumes Dirichlet.

Stak claims that, whereas our heat g's were smooth in t, our wave g's won't be. And whereas our heat g's were functions, our wave g's will often be distributions. We saw evidence of this back in Chapter 5 where the causal propagator was smooth for heat, but involved θ and δ functions and their derivatives for waves.

Equation 7.115 is just a statement that we have g(←) whichever time argument labels you use. He has not commented on the possible symmetry or lack of it for this causal g in the x variables (but see below). We know there is no symmetry in the t ones due to causality.

At this point, I went off and added a new section to my BV document. I had already derived the wave equation Green's equation (7.7) in my "N dim" doc, but there was still much to be done, the current J_μ is different for the wave equation, and we end up therefore with an extra term. I have fully derived equation 7.116 in the fully general Dirichlet/Neumann/Radiative case (I call it 7.116 gen). The extra term now is that ∂_{t0} term and it has f_1, so you would argue that f_2 term is also different in that f_2 is the $\partial_t u$ IC.

Space Eigenfunction Expansion (247)

We expand g(x|x0) as shown p 247 A with inverse as in B (orthonormal eigenfunctions φ_i). This is roughly the "partial eigenfunction expansion" method of potential theory, and we will get an ODE for the $g_k(t)$. In our 7.114 alternate form we have, from above (I replace h with g)

$$(\partial_t^2 - \nabla^2)\, g(x,t\,;\, x_0, t_0) = 0 \qquad \partial_t g(x,t_0+\,;\, x_0, t_0) = \delta(x-x_0) \qquad \text{with BC=0} \qquad (**)$$

and $\quad g(x,t_0+ ; x_0, t_0) = 0$

where h = g for $t \geq t_0$. So do what Stak says

$$\int dx\, \varphi_{k'}^*(x)(\partial_t^2 - \nabla_x^2)\, g(x,t ; x_0, t_0) = 0$$

$$g(x,t ; x_0, t_0) = \Sigma_k\, g_k(t ; x_0, t_0)\, \varphi_k(x) \qquad \text{// p 247 A}$$

$$g_k(t ; x_0, t_0) = \int dx\, \varphi_k^*(x)\, g(x,t ; x_0, t_0) \qquad \text{// p 247 B just the inverse of the above}$$

$$\partial_t g_k(t ; x_0, t_0) = \int dx\, \varphi_k^*(x)\, \partial_t g(x,t ; x_0, t_0) \qquad \text{// p 247 B' (I added this for use below)}$$

so we have

$$\int dx\, \varphi_{k'}^*(x)(\partial_t^2 - \nabla_x^2)\, \Sigma_k\, g_k(t ; x_0, t_0)\varphi_k(x) = 0$$

$$\int dx\, \varphi_{k'}^*(x)\partial_t^2\, \Sigma_k\, g_k(t ; x_0, t_0)\varphi_k(x) - \int dx\, \varphi_{k'}^*(x)\, \nabla_x^2\, g(x,t ; x_0, t_0) = 0$$

$$\partial_t^2\, \Sigma_k\, g_k(t ; x_0, t_0) \int dx\, \varphi_{k'}^*(x)\, \varphi_k(x) - \int dx\, \varphi_{k'}^*(x)\, \nabla_x^2\, g(x,t ; x_0, t_0) = 0$$

$$\partial_t^2\, g_k(t ; x_0, t_0) - \int dx\, \varphi_k^*(x)\, \nabla_x^2\, g(x,t ; x_0, t_0) = 0 \qquad \text{// agrees with p 247 C}$$

Now in the second term we want to swing ∇^2 over to the φ_k, but I will do it differently:

$$- \int dx\, \varphi_k^*(x)\, \nabla_x^2\, g(x,t ; x_0, t_0) = - \int dx\, \varphi_k^*(x)\, \nabla_x^2\, \Sigma_{k'}\, g_{k'}(t ; x_0, t_0)\, \varphi_{k'}(x)$$

$$= - \Sigma_{k'}\, g_{k'}(t ; x_0, t_0) \int dx\, \varphi_k^*(x)\, \nabla_x^2\, \varphi_{k'}(x) = + \Sigma_k\, g_k(t ; x_0, t_0) \int dx\, \varphi_k^*(x)\, \lambda_{k'}\, \varphi_{k'}(x)$$

$$= + g_k(t ; x_0, t_0)\, \lambda_k$$

so we then have [I could have gotten here faster, skipping p 247 C]

$$\partial_t^2\, g_k(t ; x_0, t_0) + \lambda_k\, g_k(t ; x_0, t_0) = 0 \qquad \text{// p 248 A item 1}$$

Now from p 247B above we have

$$g_k(t_0+ ; x_0, t_0) = 0 \qquad \text{// p 248 A item 2}$$

and from p 247B' (see above) and 7.114 we above we have

$$\partial_t g_k(t\to t_0\,;\,x_0,\,t_0) = \int dx\;\varphi_k^*(x)\;\partial_t g(x,t\to t_0;\,x_0,\,t_0)$$

$$= \int dx\;\varphi_k^*(x)\;\delta(x-x_0) = \varphi_k^*(x_0) \qquad \text{// page 248 A item 3}$$

Now can we solve our system p 248 A? Well, we know it is sin and cos and the form p 248 B clearly solves the ODE and meets both BC's, so it IS the solution. And so we obtain 7.118 which is the bilinear expansion for g. It is this (we are in n spatial dimensions!)

$$g(x,t\,;\,x_0,\,t_0) = \Sigma_k\,[\,\sin\{\sqrt{\lambda_k}\,(t-t_0)\}\,/\,\sqrt{\lambda_k}\,]\;\varphi_k(x)\varphi_k^*(x_0) \qquad \text{// wave 7.118}$$

which we can compare to previous bilinear results

$$g(x,t\,;\,x_0,\,t_0) = \Sigma_k[e^{-\lambda_k(t-t0)}]\;\varphi_k(x)\varphi_k^*(x_0) \qquad \text{// heat 7.57}$$

$$g(x\,;\,x_0) = \Sigma_k\,[\,1/\lambda_k]\;\varphi_k(x)\varphi_k^*(x_0) \qquad \text{// Laplace 6.108}$$

This shows, by the way, that all three Causal Green's functions are Hermitian under $x \leftrightarrow x_0$. To the extent you think of this as h rather than g, it is anti-symmetric under $t \leftrightarrow t_0$, but the real g is obviously not so since it is $\equiv 0$ one way.

Pause: how do we interpret sources in the wave equation theory?

	potential theory	heat conduction	wave equation	
u	potential	temperature	displacement	
q	charge	heat source	driving force	// since $\partial_t^2 u \sim a$
$\partial_n u$	surface charge	surface heat source	surface driving force (pressure?)	

Example: Suppose in Wave World we have q = 0 (no driving forces in the volume R) and h = 0 (no displacement at the boundary, assuming Dirichlet case, so like a drumhead) . So "all you have" are your initial conditions. Then my 7.116 from BV

$$u(t,\mathbf{x}) = \int_0^t dt_0 \int_{R_n} d^n x_0\;\; g(x|x_0)\;q(t_0,\mathbf{x}_0)$$

$$+ \int_{R_n} d^n x_0\;\; \{g(x,t|x_0,0)\partial_{t0}u(0,\mathbf{x}_0) - u(0,\mathbf{x}_0)\partial_{t0}g(x,t|x_0,0)\}$$

$$- \int_0^t dt_0 \int_{\sigma_n} dS_{n0}\,[\,u(t_0,\mathbf{x}_0)\partial_{n\mathbf{x}0}\,g(x|x_0) - g(x|x_0)\partial_{n0}u(t_0,\mathbf{x}_0)\,]$$

becomes

$$u(t,\mathbf{x}) = + \int_{\mathbf{R_n}} d^n x_0 \; g(x,t|x_0,0)\partial_{t0}u(0,\mathbf{x_0}) - \int_{\mathbf{R_n}} d^n x_0 \; u(0,\mathbf{x_0})\partial_{t0}g(x,t|x_0,0)$$

Now install the bilinear expansion we just found for g:

$$g(x,t\,;x_0,t_0) = \Sigma_k \left[\sin\{\sqrt{\lambda_k}\,(t-t_0)\}/\sqrt{\lambda_k} \right] \varphi_k(x)\varphi_k^*(x_0)$$
$$\partial_{t0}\; g(x,t\,;x_0,t_0) = -\,\Sigma_k \left[\cos\{\sqrt{\lambda_k}\,(t-t_0)\} \right] \varphi_k(x)\varphi_k^*(x_0)$$

$$g(x,t\,;x_0,0) = \Sigma_k \left[\sin\{\sqrt{\lambda_k}\,(t)\}/\sqrt{\lambda_k} \right] \varphi_k(x)\varphi_k^*(x_0)$$
$$\partial_{t0}\; g(x,t\,;x_0,0) = -\,\Sigma_k \left[\cos\{\sqrt{\lambda_k}\,(t)\} \right] \varphi_k(x)\varphi_k^*(x_0)$$

$$u(t,\mathbf{x}) = \int_{\mathbf{R_n}} d^n x_0 \; g(x,t|x_0,0)\partial_{t0}u(0,\mathbf{x_0})$$
$$- \int_{\mathbf{R_n}} d^n x_0 \; u(0,\mathbf{x_0})\partial_{t0}g(x,t|x_0,0)$$

$$= \int_{\mathbf{R_n}} d^n x_0 \; \Sigma_k \left[\sin\{\sqrt{\lambda_k}\,(t-t_0)\}/\sqrt{\lambda_k} \right] \varphi_k(x)\varphi_k^*(x_0)\partial_{t0}u(0,\mathbf{x_0})$$
$$+ \int_{\mathbf{R_n}} d^n x_0 \; u(0,\mathbf{x_0})\, \Sigma_k \left[\cos\{\sqrt{\lambda_k}\,(t)\} \right] \varphi_k(x)\varphi_k^*(x_0)$$

$$= \Sigma_k \left[\sin\{\sqrt{\lambda_k}\,(t)\}/\sqrt{\lambda_k} \right] \varphi_k(x) \int_{\mathbf{R_n}} d^n x_0 \; \partial_{t0}u(0,\mathbf{x_0}) \; \varphi_k^*(x_0)$$
$$+ \Sigma_k \left[\cos\{\sqrt{\lambda_k}\,(t)\} \right] \varphi_k(x) \int_{\mathbf{R_n}} d^n x_0 \; u(0,\mathbf{x_0})\, \varphi_k^*(x_0)$$

$$= \Sigma_k \left[\sin\{\sqrt{\lambda_k}\,(t)\}/\sqrt{\lambda_k} \right] \varphi_k(x) \int_{\mathbf{R_n}} d^n x_0 \; f_2(\mathbf{x_0}) \; \varphi_k^*(x_0)$$
$$+ \Sigma_k \left[\cos\{\sqrt{\lambda_k}\,(t)\} \right] \varphi_k(x) \int_{\mathbf{R_n}} d^n x_0 \; f_1(\mathbf{x_0})\, \varphi_k^*(x_0)$$

$$= \Sigma_k \left[\sin\{\sqrt{\lambda_k}\,(t)\}/\sqrt{\lambda_k} \right] \varphi_k(x)\, f_{2,k}$$
$$+ \Sigma_k \left[\cos\{\sqrt{\lambda_k}\,(t)\} \right] \varphi_k(x)\, f_{1,k}$$ // agrees with p 248 C but 1↔2

where I have used the definitions of f_1 and f_2 as stated on page 243 7.109 where f_2 goes with $\partial_t u$. So I think Stak has a typo in p 248 C which I have corrected in the text. This says: if you have a "string" in any number of dimensions with initial conditions f_1 and f_2, the above tells you how it will "move" as time carries on. You can see there are "modes" with frequency $\omega = \sqrt{\lambda_k}$ and they just run on forever. The amount in each mode is determined by the amount of each basis function contained in the boundary condition functions. We have no damping yet.

Time Laplace on 7.114 to get alternate derivation of 7.118 (249)

I am doing this in detail mainly due to my recent problems with Laplace on second derivatives! We start with our usual Laplace 2nd derivative rule

$$\int_0^\infty ds\, e^{-st}\, \partial_t^2 f(t) = s^2F(s)-sf(0)-f'(0)$$

$$\int_0^\infty ds\, e^{-st}\, \partial_t^2 g(x,t) = s^2\, G(x,s) -s\, g(x,0)- \partial_t g(x,0)$$

NOTE: As we learned the hard way doing Exercise 7.16, we will not be successful with this kind of Laplace transform involving a second derivative unless we know both f(0) and f'(0) which in this case are just g(x,0) and $\partial_t g(x,0)$. But in THIS problem we *do* in fact know that g(x,0)= 0 and $\partial_t g(x,0) = \delta(x-x_0)$, so at least we have a chance of success!

We are thinking 7.114 (alternate version) of the Dirichlet Green's definition, so g = 0 is the IC and $\partial_t g(x,0) = \delta(x-x_0)$ is IC', so we have no linear s term and we get

$$\int_0^\infty ds\, e^{-st}\, \partial_t^2 g(x,t) = s^2\, G(x,s) - \delta(x-x_0)$$

Then 7.114 line 1 becomes

$$s^2\, G(x,s) - \delta(x-x_0) - \nabla^2\, G(x,s) = 0$$

or

$$(-\nabla^2+ s^2)\, G(x,s) = \delta(x-x_0)$$ // agrees with p 249 B.

and this is Helmholtz except s^2 has the wrong sign. We know that G(x,s) is the Laplace transform of g(x,t) and in 7.114 we have g(x,t) = 0 for x on σ. Thus, we have G(x,s) = 0 for x on σ. So we enhance the last line above:

$$(-\nabla^2+ s^2)\, G(x,s) = \delta(x-x_0) \qquad G(x,s) = 0 \text{ for x on } \sigma$$ // agrees with p 249 B.

Notice that this is a "Green's function" type equation in the variable x, when the L operator here is the Helmholtz operator!

Comment A: In the *absence* of a BC, we solved exactly this Helmholtz equation 5.112 on pages 53 in Chapter 5 ($x_0 = 0$ and $s^2 = \mu^2$) for any n (number of spatial dimensions). There we got $G(x,s) = E_n(|x|,\lambda)$ where $s = \mu = -i\sqrt{\lambda}$ and $\lambda = -\mu^2 = -s^2$, and where E_n was given for all n. But here we do have a BC, so that solution does not apply.

Now expand G(x,s) this way

$G(x,s) = \Sigma_k G_k(s)\varphi_k(x)$ // like p 247 A

$G_k(s) = \int dx\ \varphi_k^*(x)\ G(x,s)$ // like p 247 B

Our "Helmholtz" is then

$$(-\nabla^2 + s^2)\ \Sigma_k G_k(s)\varphi_k(x) = \delta(x-x_0) = \Sigma_k \varphi_k(x)\ \varphi_k^*(x_0)$$

$$\Sigma_k (\lambda_k + s^2) G_k(s)\varphi_k(x) = \delta(x-x_0) = \Sigma_k \varphi_k(x)\ \varphi_k^*(x_0)$$

Completeness of the $\varphi_k(x)$ then says

$$(\lambda_k + s^2) G_k(s) = \varphi_k^*(x_0)$$

$$G_k(s) = \varphi_k^*(x_0)/\ (s^2+\lambda_k)$$

and then we have [Green's Function is just a function, even though ODE is distributional] ,

$$G(x,s) = \Sigma_k G_k(s)\varphi_k(x) = \Sigma_k\ \varphi_k^*(x_0)/\ (s^2+\lambda_k)\ *\varphi_k(x)$$

$$= \Sigma_k[\ 1/\ (s^2+\lambda_k)]\ \varphi_k(x)\ \varphi_k^*(x_0)$$ // agrees with p 249 C

Now we can take this G(x,s) and put it into the Laplace inversion formula p 249 D, add up all the pole residues, and the claim is that we arrive at 7.118, so this is a second method of deriving the bilinear expansion 7.118. The first method was using $\varphi_k(x)$ expansions to get $g_k(t\ ;\ x_0,\ t_0)$ with x removed and we solve for this g_k thing and installed it into the spatial expansion. The second method was using a time Laplace to get $G(x,s;\ x_0,t_0)$ with t removed and installed it into the Laplace expansion.

Note: at this point, I finally finished off Exercise 7.16 using the Fourier Sine Transform in the last part, and I well understand why Laplace could not cut the mustard.

Section 7.8 Problems in one spatial dimension (249)

The Infinite String (249)

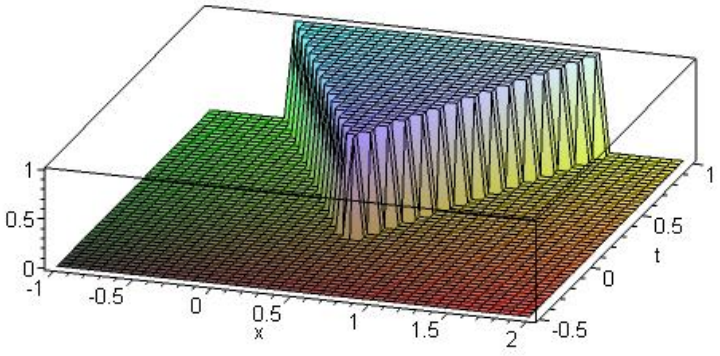
Comment B: In Chapter 5 pages 61-63 we exactly solved the "struck string" problem in n dimensions in the absence of BC's. The problem was presented in 5.142, but as usual we showed the "alternative form" p 61 A which has the IC being u=0, and IC' being $\partial_t u = \delta$ (that is, the "struck" IC set). For n = 2 this would be a strike at the center of an infinitely large drumhead membrane. We found the solution for any n, and there it was written $C_n(|x|,t)$. In particular, we found that $C_1(|x|,t) = (1/2)H(t-|x|)$. Now the method used to find $C_n(|x|,t)$ was time Laplace with ∂_t^2. The "Laplace danger" is avoided due to the IC's of p 61A, and we obtain the s-space equation 5.146. But Stak already solved this Helmholtz Green's equation back on page 53 (see Comment A above), where the solution was $c_n(|x|,s)$ as in 5.147 with |x| = r. This was easy to do ILT on and he then got general solutions 5.148 and 5.149 for odd and even n for $C_n(|x|,t)$.

There is no boundary, and we already solved this problem in Chapter 5 and we crib that result for 7.119. The problem we solved in Chapter 5 was p 61 A which has IC = 0 displacement, but IC' = $\delta(x-x_0)$ impulse. This is the notion of "striking" a metal string with a blade edge hammer. At t = 0+ it has a δ impulse velocity at the strike point but no displacement. Now the solution 7.119 says (t_0= 0)

C_1 = 1/2 when $|x-x_0| < t$ else 0

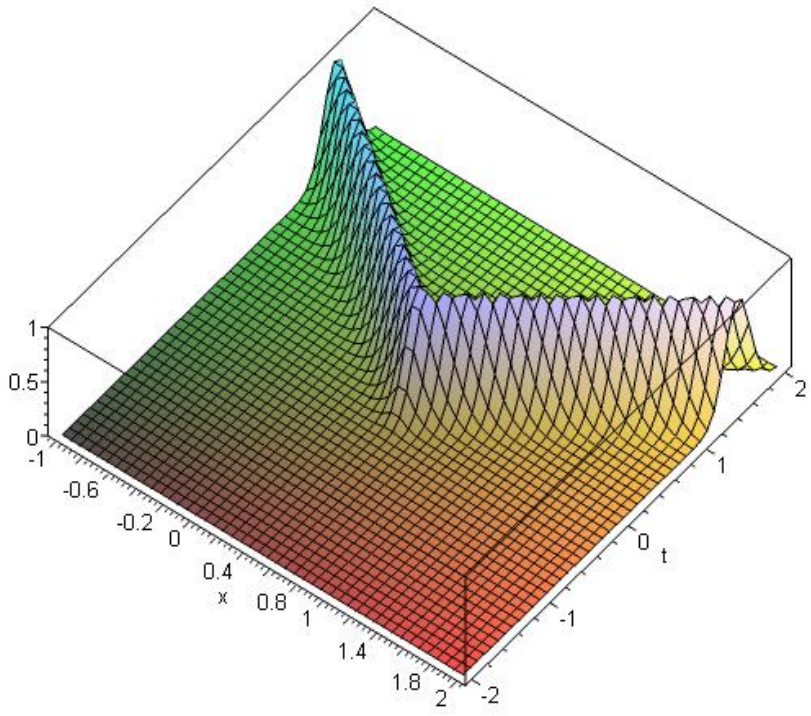
so this is an expanding positive square pulse whose edges are at $x_{edge} = |x-x_0| = t$, and $\partial_t x_{edge} = \pm 1$, so the edge expands at unit velocity. I drew this pulse on page 250. Here is a Maple plot of 7.119 in which the set of square pulses at different times form the axe wedge shown:

```
plot3d(Heaviside(t-abs(x-0.5)),x = -1..2, t=-0.5..1,
numpoints=2000);
```



where a slice at t = constant (after t = 0) will be a square pulse of some width, symmetric about $x=x_0$ = 1/2 in my plot (I have t_0= 0). If you look at this plot from the top, you get Stak's Fig 7.7 .

Now if we do a δ pluck instead of a δ strike, page 245 top says the solution is the ∂_t of the above solution, and this gives page 249 E. This is a bit hard to plot, but if I model the δ with an expo, we get



which shows the notion of starting with a δ displacement which then splits into two identical copies of itself so that a slice at constant t we see two delta functions, each one moving away from $x = x_0$ at unit velocity. The formula p 249 E for $t > 0$ would give a total area under the two delta functions as 1 since each has a 1/2 coefficient. But we know that our initial pluck deflection is a full δ. So it is a tricky limit taking the $t = t_0$ limit of p 249E, and the limit is $\delta(x-x_0)$.

Comment: So the infinite string wave propagator is simply $C_1(|x|,t) = (1/2)H(t-|x|)$. This is a much simpler function than the infinite rod heat propagator which was $C_1(|x|,t) = H(t)\exp(-|x|^2/4t) / \sqrt{4\pi t}$, but it is more singular. Again, for any finite t, the heat $C_1 \neq 0$ for all points x! Remember that in Wave World, the definition of the propagator depends on your choice of IC's, and the standard choice is IC=0 and IC' = δ as above.

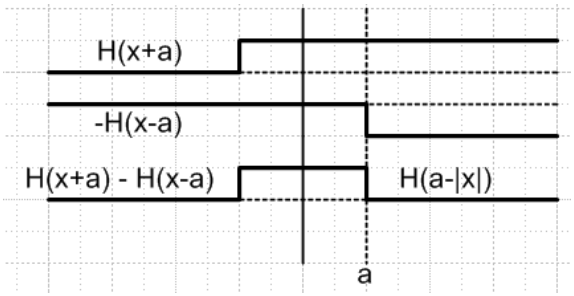
Digression on Heaviside Math and Inequalities

Theorem 0: Inequality $|x| < a$ is true in a disk of radius $a>0$ centered at the origin. If x and a are real, then this inequality is the same as requiring both $x < a$ and $x > -a$, which is to say, $-a < x < a$. Thus in our funny generalized Heaviside notation we could say

$$H(|x| < a) = H(-a < x < a) \qquad \text{or} \qquad |x| < a \quad \Leftrightarrow \quad -a < x < a$$

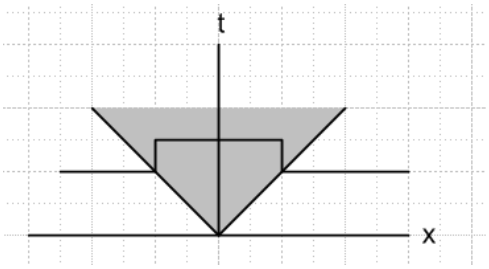
Theorem 1: If $a > 0$, then $H(a-|x|) = H(x+a) - H(x-a)$

Proof: Draw this picture:



H(a-|x|) = 1 when a > |x| which means |x| < a which means inside the pulse. Also, Maple plots confirm this result, in case I am being blocked.

Here is a plot of H(a-|x|) in the x-t plane:



Corollary 1: Assume t > 0, then (that is, we just let a = t here)

H(t-|x|) = H(x+t) - H(x-t)

Corollary 2: Assume Δt > 0 and replace x by Δx :

H(Δt -|Δx|) = H(Δx+Δt) - H(Δx-Δt)

Theorem 2: H(y) + H(-y) = 1 which seems pretty obvious.

Corollary 3: H(Δx-Δt) = 1 - H(Δt-Δx)

Fact: Combine corollary 2 and 3 to get

H(Δt -|Δx|) = H(Δx+Δt) - H(Δx-Δt) = H(Δt+Δx) - [1 - H(Δt-Δx)]

= H(Δt+Δx) + H(Δt-Δx) - 1

Application: Equation 7.119 says $C_1 = (1/2)H(\Delta t - |\Delta x|)$, so by the above Fact we have

$C_1 = (1/2)\{ H(\Delta t+\Delta x) + H(\Delta t-\Delta x) - 1 \}$ // which is 7.120

Theorem 3: $\partial_t H(t-|x|) = \delta(x+t) + \delta(x-t)$ but only one or the other δ can have a hit (dep on signx)

$$\partial_{\mathbf{x}} H(t-|x|) = \delta(x+t) - \delta(x-t)$$

Proof: Start with result of Theorem 1 with a = t > 0

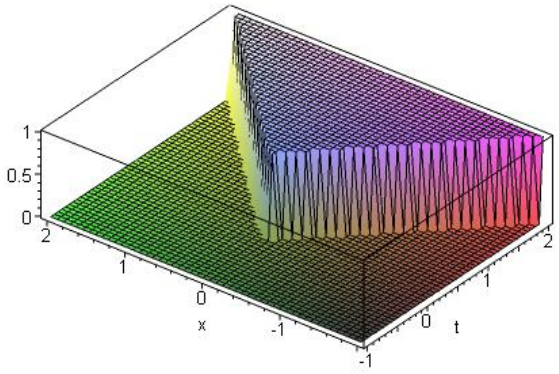
$$H(t-|x|) = H(x+t) - H(x-t)$$

Apply $\partial_{\mathbf{x}}$ and the second claim above is obvious.
Apply $\partial_{\mathbf{t}}$ to all three terms, and note that $\partial_{\mathbf{t}} = -\partial_{-\mathbf{t}}$:

$$\partial_{\mathbf{t}} H(t-|x|) = \partial_{\mathbf{t}} H(x+t) + \partial_{-\mathbf{t}} H(x-t) = \delta(x+t) + \delta(x-t)$$

The results of Theorem 3 are compatible with this picture of H(t-|x|)

plot3d(Heaviside(t-abs(x)),t=-1..2,x=-2..2, numpoints=2400, scaling=CONSTRAINED);



If we differentiate along the t axis, we get a positive δ no matter what x we are at. On the right side we will get δ(x-t) since x > 0, and on the left side we will get δ(x+t) since is at t = -x when x < 0:

$$\partial_{\mathbf{t}} H(t-|x|) = \delta(x+t) + \delta(x-t)$$ but only one or the other δ can have a hit

If we differentiate along the x axis, we get a positive δ at t = -x where x < 0 (so δ(t+x)) and we will get a negative δ at t = x when x > 0 (so - δ(t-x)) :

$$\partial_{\mathbf{x}} H(t-|x|) = \delta(x+t) - \delta(x-t)$$

Derive 7.122.

Start with 7.116 in the form with subscript 0 on the left side (this is directly from my BV doc)

$$u(t_0,\mathbf{x}_0) = \int_{\mathbf{0}}^{t_0} dt \int_{\mathbf{R_n}} d^{\mathbf{n}}x \; g(x_0|x)\, q(t,\mathbf{x})$$

$$+ \int_{\mathbf{R_n}} d^{\mathbf{n}}x \; \{g(x_0,t_0|x,0)\, f_2(x) + f_1(x)\partial_{t_0} g(x_0,t_0|x,0)\}$$

$$- \int_0^{t_0} dt \int_{\sigma_n} dS_n \, [\, h(t,\mathbf{x})\partial_{\mathbf{nx}} g(x_0|x) - g(x_0|x)\partial_{\mathbf{n}} h(t,\mathbf{x}) \,]$$

The infinite string has no boundary so the last line goes away. I put in the name C_1 for our infinite string propagator

$$u(t_0,\mathbf{x}_0) = \int_0^{t_0} dt \int_{\mathbf{R_n}} d^n x \; C_1(x_0,t_0\,|x,t)\, q(t,\mathbf{x})$$

$$+ \int_{\mathbf{R_n}} d^n x \; \{\, C_1(x_0,t_0\,|x,0)\, f_2(x) + f_1(x)\partial_{t0}\, C_1(x_0,t_0\,|x,0) \,\}$$

Now install from p 250 A,B and install 1D integration notation to get

$$u(t_0,\mathbf{x}_0) = \int_0^{t_0} dt \int_{-\infty}^{\infty} dx \; (1/2)\, H(t_0-t - |x-x_0|)\, q(t,\mathbf{x})$$

$$+ \int_{-\infty}^{\infty} dx \; \{\, (1/2)\, H(t_0 - |x-x_0|) \; f_2(x) + f_1(x)\partial_{t0}\, (1/2)\, H(t_0 - |x-x_0|) \;\}$$

$$2\, u(t_0,\mathbf{x}_0) = \int_0^{t_0} dt \int_{-\infty}^{\infty} dx \; H(t_0-t - |x-x_0|)\, q(t,\mathbf{x}) \qquad (*)$$

$$+ \int_{-\infty}^{\infty} dx \; \{\; H(t_0 - |x-x_0|) \; f_2(x) + f_1(x)\, \partial_{t0} H(t_0 - |x-x_0|) \;\}$$

Regarding the <u>first term</u> in (*) above, consider from Theorem 0 above that

$$|x| < a \quad \Leftrightarrow \quad -a < x < a$$
$$|x-x_0| < a \quad \Leftrightarrow \quad -a < (x-x_0) < a$$
$$|x-x_0| < (t_0-t) \quad \Leftrightarrow \quad -(t_0-t) < (x-x_0) < (t_0-t) \quad \Leftrightarrow \quad x_0-(t_0-t) \; < x < x_0+(t_0-t)$$

Thus our first term is

$$2u(t_0,\mathbf{x}_0)_{\texttt{first term}} = \int_0^{t_0} dt \int_{x_0-(t_0-t)}^{x_0+(t_0-t)} dx \; q(t,\mathbf{x}) \qquad \text{// 7.122, first term} \qquad (**)$$

which agrees with 7.122, the first term.

Now look at the <u>second term</u>:

$$\int_{-\infty}^{\infty} dx \; \{\; H(t_0 - |x-x_0|) \; f_2(x)$$

If we set t = 0 in our line (**) above, we get

$$|x-x_0| < (t_0) \Leftrightarrow \; -(t_0) < (x-x_0) < (t_0) \Leftrightarrow \quad x_0-(t_0) \; < x < x_0+(t_0) \quad \Leftrightarrow \quad x_0 - t_0 < x < x_0 + t_0$$

Thus the second term in (*) is this

$$\int_{x_0-t_0}^{x_0+t_0} dx \;\; f_2(x)$$

The third term is this

$$\int_{-\infty}^{\infty} dx\; f_1(x)\; \partial_{t_0} H(t_0 - |x-x_0|)$$

and according to Theorem 3 above we have

$$\partial_t H(t-|x|) \;= \delta(x+t) + \delta(x-t)$$

$$\partial_{t_0} H(t_0 - |x-x_0|) = \;\; \delta((x-x_0)+t_0) + \delta((x-x_0)-t_0) \;=\; \delta(\, x-(x_0-t_0)) + \;\; \delta(\, x-(x_0+t_0))$$

So we then have

$$\int_{-\infty}^{\infty} dx\; f_1(x)\; \partial_{t_0} H(t_0 - |x-x_0|) \;=\; \int_{-\infty}^{\infty} dx\; f_1(x)\; [\delta(\, x-(x_0-t_0)) + \;\; \delta(\, x-(x_0+t_0))\,]$$

$$= \;[\; f_1(x_0-t_0) + f_1(x_0+t_0)\;]$$

We assemble our three terms to get

$$2\; u(t_0,x_0) = \int_0^{t_0} dt \;\; \int_{x_0-(t_0-t)}^{x_0+(t_0-t)} dx\; q(t,x) \;+\; \int_{x_0-t_0}^{x_0+t_0} dx \;\; f_2(x) \;+[\; f_1(x_0-t_0) + f_1(x_0+t_0)\;] \qquad 7.122$$

and my f_2 sign disagree with Stak. Now in Stak 7.116 the f_2 term is plus and I agree, so that term could not come out being negative, so that f_2 sign in 7.122 is wrong. Apart from this sign, my result agrees with 7.122.

Comment: Equation 7.122 (despite its subscript 0 ugliness) is the complete solution to the problem posed in 7.121 : the most general wave problem one can talk about (in our class of problems) on the infinite string. Let me do the swap:

$$2\; u(t,x) = \int_0^{t} dt_0 \;\; \int_{x-(t-t_0)}^{x+(t-t_0)} dx_0\; q(t_0,x_0) \;+\; \int_{x-t}^{x+t} dx_0\; f_2(x_0) \;+[\; f_1(x+t) + f_1(x-t)\;]$$

If you have some kind of f_1 initial displacement, the third term component is just fixed copies of these things travelling in opposite directions! If $q = f_2 = 0$, this is all there is: this is what an infinite string does with a $t = 0$ shaped "pluck". The other terms are less obvious to me.

The Semi-Infinite String (251)

Using exactly the same image idea as we used with heat, we obtain 7.123 as the half-string propagator Green's, and note that $g = 0$ at the boundary of the half string. We can then install 7.123 into 7.116 to get

the complete solution to all (in class) half string problems. The only "novelty" as Stak says, compared to the infinite string, is that we now have a boundary, so the last line in 7.116 will be activated. If we set q and f_1 and f_2 all to zero (no sources and a perfectly undisturbed half string at t = 0) then this term is all there will be: (assuming Dirichlet BC on the g)

$$u(t_0,\mathbf{x_0}) \;=\; -\int_0^{t_0} dt \int_{\sigma_n} dS_n \,[\, h(t,\mathbf{x})\partial_{\mathbf{nx}}\, g(x_0|x) - g(x_0|x)\partial_{\mathbf{n}} h(t,\mathbf{x}) \,] \;=\; -\int_0^{t_0} dt\, h(t)\partial_{\mathbf{nx}}\, g(x_0|x)$$

where we have the subscripted labels first inside g.

On page 252 top Stak notes as I have before that $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$ so get a minus in p 252A. Next, we write

$$2C_1(x_0, t_0|\, x,t) = H(t_0 - t - |x-x_0|)$$

$$2C_{\mathbf{half}}(x_0, t_0|\, x,t) = 2C_1(x_0, t_0|\, x,t) - 2C_1(x_0, t_0|\, -x,t) \qquad \text{// as per 7.123}$$

$$= H(t_0 - t - |x-x_0|) - H(t_0 - t - |x+x_0|) \qquad \text{// as per p 250 A}$$

Then we apply our Theorem 1 above $H(a-|x|) = H(x+a) - H(x-a)$ two times to get

$$= [\, H((x-x_0) +(t_0-t)) - H((x-x_0)-(t_0-t))] \; - [H((x+x_0)+ (t_0-t)) - H((x+x_0)- (t_0-t)) \,]$$

We can then negate the arguments of the 2nd and 4th terms using Theorem 2 and the "1's" will cancel, and each of these terms then also changes sign since H(x) = -H(-x) + 1,

$$= [\, H((x-x_0) +(t_0-t)) + H(-(x-x_0)+(t_0-t))] \; - [H((x+x_0)+ (t_0-t)) + H(-(x+x_0)+ (t_0-t)) \,]$$

and now we change positions of x_0 and x keeping track of signs

$$= [\, H(-(x_0-x) +(t_0-t)) + H((x_0-x)+(t_0-t))] \; - [H((x_0+x)+ (t_0-t)) + H(-(x_0+x)+ (t_0-t)) \,]$$

and now put the t terms first and then remove the brackets

$$= [\, H(t_0-t \;-(x_0-x)) + H(t_0-t +(x_0-x)] \; - [H(t_0-t +(x_0+x)) + H(t_0-t -(x_0+x)) \,]$$

$$= H(t_0-t \;-(x_0-x)) + H(t_0-t +(x_0-x) - H(t_0-t +(x_0+x)) - H(t_0-t -(x_0+x))$$

Where terms are in order 1,2,3,4. If we reorder these as 1,2,4,3 we get

$$= H(t_0-t \;-(x_0-x)) + H(t_0-t +(x_0-x) - H(t_0-t -(x_0+x)) \; - H(t_0-t +(x_0+x)) \qquad \text{// agrees with p 252 B}$$

$$= 2C_{\mathbf{half}}(x_0, t_0|\, x,t)$$

Then the $\partial_{\mathbf{x}}$ is easy to do

$$2\partial_{\mathbf{x}}C_{\mathbf{half}}(x_0,t_0|\,x,t) = \delta(t_0\text{-}t\ \ \text{-}(x_0\text{-}x)) - \delta(t_0\text{-}t\ \text{+}(x_0\text{-}x) + \delta(t_0\text{-}t\ \text{-}(x_0\text{+}x))\ \ - \delta(t_0\text{-}t\ \text{+}(x_0\text{+}x))$$

Then set x = 0

$$2\partial_{\mathbf{x}}C_{\mathbf{half}}(x_0,t_0|\,x,t)|^{\mathbf{x=0}} = \delta(t_0\text{-}t\ \ \text{-}(x_0)) - \delta(t_0\text{-}t\ \text{+}(x_0) + \delta(t_0\text{-}t\ \text{-}(x_0))\ \ - \delta(t_0\text{-}t\ \text{+}(x_0))$$

$$= 2\delta(t_0\text{-}t\ \ \text{-}(x_0)) - 2\delta(t_0\text{-}t\ \text{+}(x_0))$$

$$\partial_{\mathbf{x}}C_{\mathbf{half}}(x_0,t_0|\,x,t)|^{\mathbf{x=0}} = \delta(t_0\text{-}t\ \ \text{-}x_0) - \delta(t_0\text{-}t\ \text{+}x_0)$$ // agrees with p 252 C

So continuing where we started above

$$u(t_0,\mathbf{x}_0) = -\int_0^{t_0} dt\ h(t)\partial_{\mathbf{nx}}\, g(x_0|x) = +\int_0^{t_0} dt\ h(t)\partial_{\mathbf{x}}\, g(x_0|x)$$

$$= \int_0^{t_0} dt\ h(t)\ [\delta(t_0\text{-}t\ \ \text{-}x_0) - \delta(t_0\text{-}t\ \text{+}x_0)]$$

$$= \int_0^{t_0} dt\ h(t)\ [\delta(t_0 - x_0\ \ \text{-}t\ ) - \delta(t_0 + x_0\ \text{-}t\ )]$$

$$= \int_0^{t_0} dt\ h(t)\ [\delta(t - (t_0\text{-}x_0)) - \delta(t - (t_0\text{+}\ x_0)]$$

But since $x_0 > 0$, the second δ does not hit ever. the first δ hits if $0 < t_0\text{-}x_0 < t_0$. The second part is fine, and the firsts part says $x_0 < t_0$. So our answer is

$u(t_0,\mathbf{x}_0) = h(t_0\text{-}x_0)$ if $x_0 < t_0$ we know that $x_0 > 0$ already
$= 0$ if $x_0 > t_0$

or

$u(t,\mathbf{x}) = h(t\text{-}x)$ if $x < t$ we know that $x > 0$ already
$= 0$ if $x > t$

For example, if h(t) = cos(t) which is the wiggle schedule of the x=0 end of the half string, we get

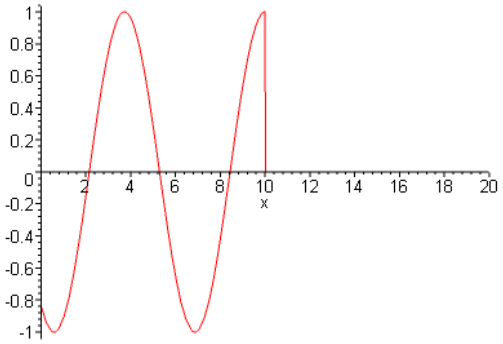
```
u := piecewise(x<t,cos(t-x),0);
```

$$u := \begin{cases} \cos(-10+x) & x < 10 \\ 0 & \text{otherwise} \end{cases}$$

```
t := 10;
```

$$t := 10$$

```
plot(u,x=0..20);
```



which shows that the string just "carries off to the right" the wave made at the left end. Now in this case, we have h(0) = 1 which is inconsistent with u(0,0) = 0 by 7.124 and u(0,0) = h(0) = 1. I think this is OK, because we have u(x,0) = 0 almost everywhere. In my example, u is discontinuous at x = 10. The issue here is that the wave disturbance travels at a certain finite speed and has not yet reached the string beyond this point. [Recall that the ω = 0 A-heated half-rod has the erfc solution so small influence "travels" instantaneously to all points on the rod. In contrast, the ω=0 A-displaced at the end string will have the influence travel in the form of a square wave at velocity 1.]

Stak claims we got this solution u(x,t) in Chapter 5 using the "method of characteristics", but I had trouble with that geometrically oriented section and would have to spend a lot of time on it, so I won't do that here.

Stak's Comment on Sine Transform in x.

He then says we could get the results using a "sine transform on the space coordinates". Having *finally* finished Exercise 7.16, I now understand why only this transform (of the three options sine, cosine or Laplace) is viable for this problem, because only u(x,0) is specified at t = 0.

You could treat this half-string problem differently.

You could imagine the whole string and then you insert a q source at the center at start it going at t = 0 (there is now no boundary so of course there is no boundary schedule to think about). Here is what the string would do:

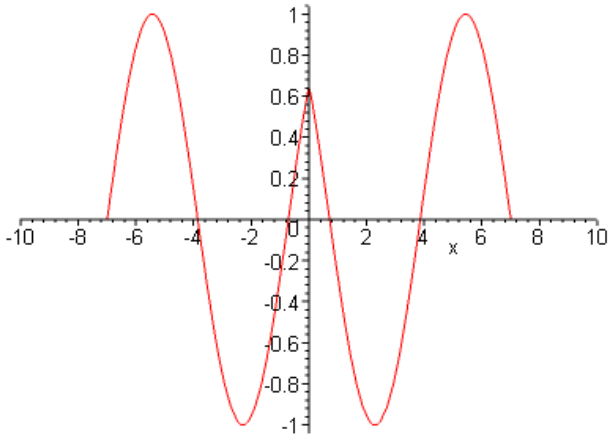
```
t := 7;
```

$$t := 7$$

```
u := piecewise(abs(x)<t,sin(t-abs(x)),0);
```

$$u := \begin{cases} \sin(7-|x|) & |x|<7 \\ 0 & \text{otherwise} \end{cases}$$

```
plot(u,x=-10..10);
```



The right side then replicates out previous half-string solution.

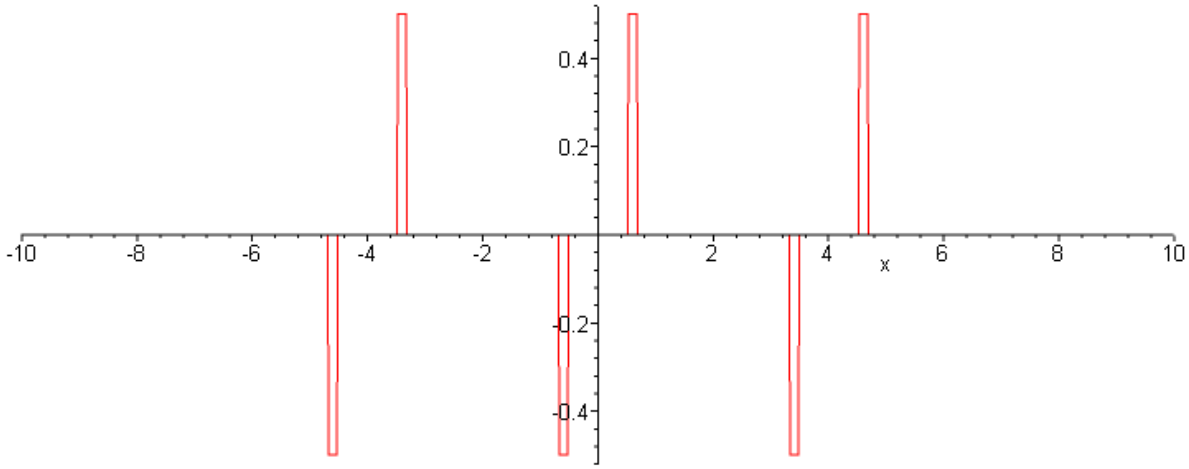
The Finite String (252)

Two ways to write the Green's for the finite string on (0,ℓ) with ends tied down, as in Chap 1 Vol I.

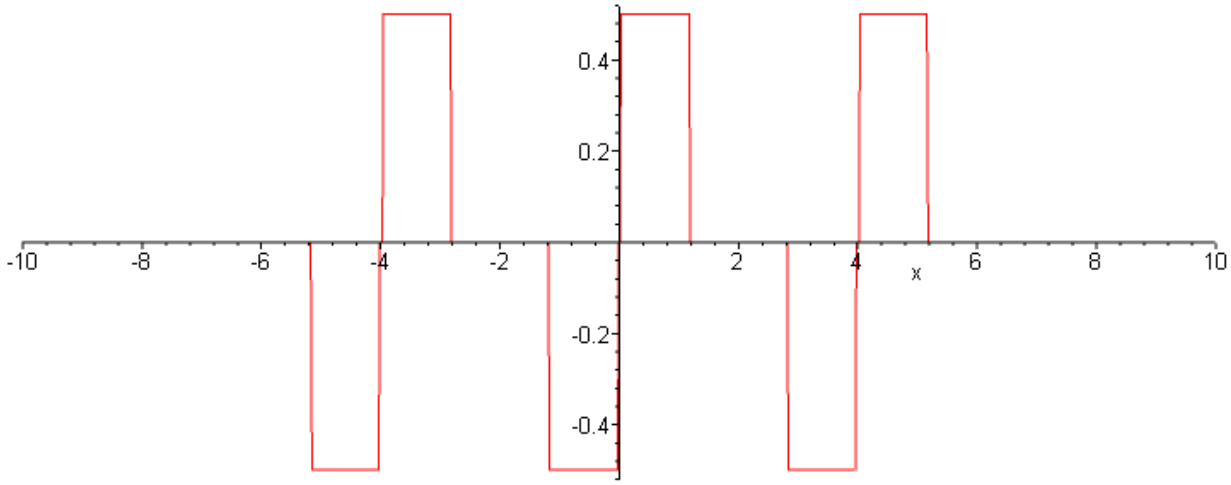
Image Method for Finite String

The first step is to place all the images as I show in pencil (dot positive, circle negative). As usual, by pairing these in two ways, you show that you must maintain u(0) = 0 and u(ℓ) = 0. This problem, with its infinite number of sources, is a legal wave solution, and it meets the BC's for our finite string, so by uniqueness it IS the solution. So by this method we get the sum of theta functions shown in 7.125. Each term is a little expanding box wave coming from its source point, with appropriate polarity. At t = t_0+ each box is just a little matchstick of height +1/2 or -1/2 at its image location. Remember that here we have "struck" the string simultaneously at all these image points with a $\delta(x-x_i)$ velocity impulse , and that in wave theory, the Green's Function g always has the "struck" IC's that g = 0 and $\partial_x g = \delta$.

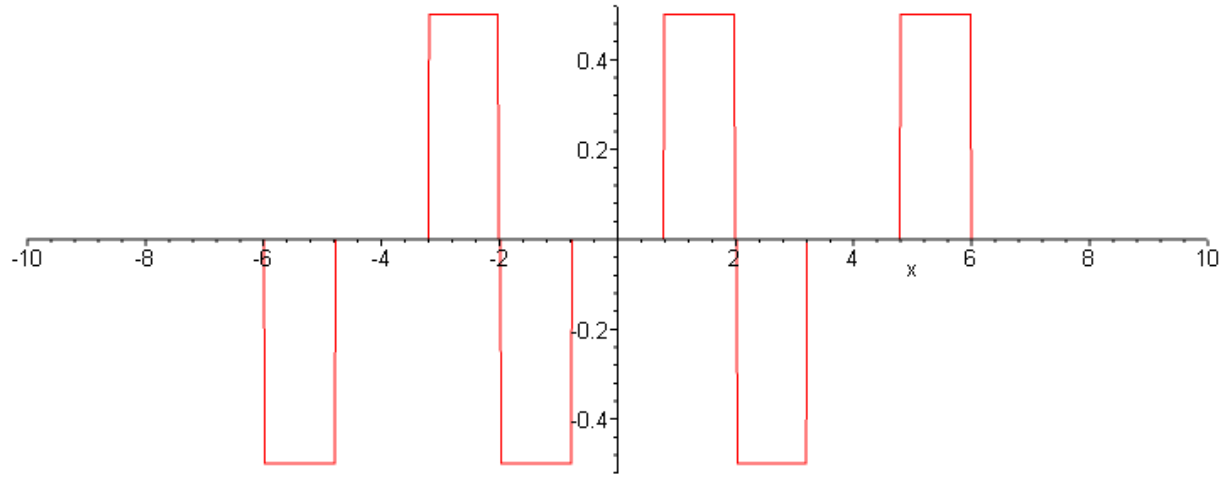
I first tried plotting this thing as plot3D, but soon found that an animation is MUCH clearer about what is going on! If you think only of a single source, yes, it makes a box that grows wider all the time. But if you think of pairs of sources (such as the pair at x=-x_0 and x=+x_0), the picture is completely different due to the cancellation between the Heaviside functions of opposite sign. I made the animation in Maple with just 3 pairs of sources, and here is what you see: as noted above, each source appears at first as a thin matchstick of appropriate polarity:



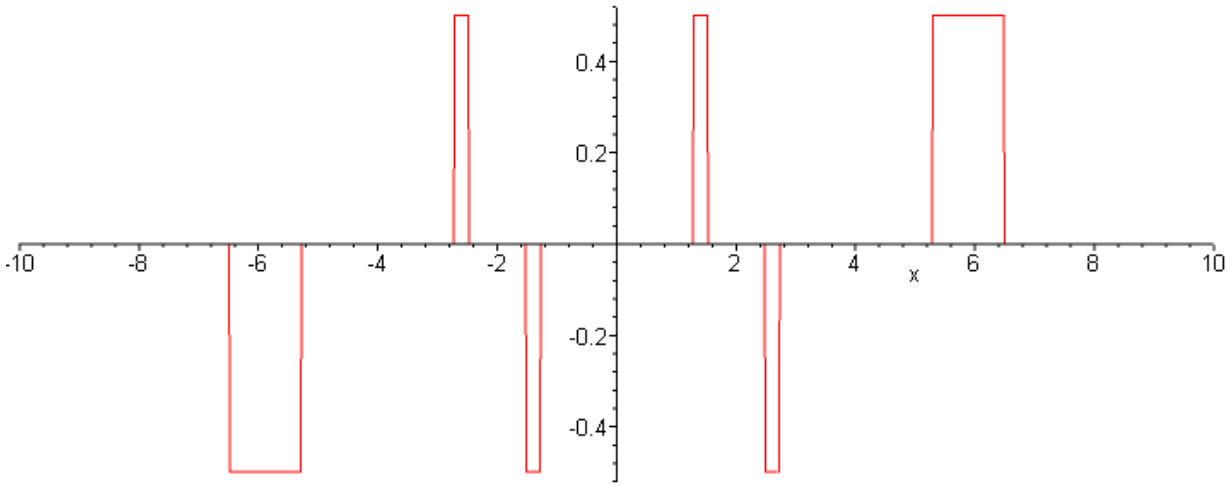
Then each matchstick symmetrically widens until it overlaps it's opposite polarity partner. This starts to happen when the width of each box is $2*x_0$ and in this example $x_0 = 0.6$ and also L = 2 so string is (0,2).



At this point, the pulses stop widening and the top ones translate to the right, the bottom ones to the left:



We now start to get cancellation of pulses pairwise



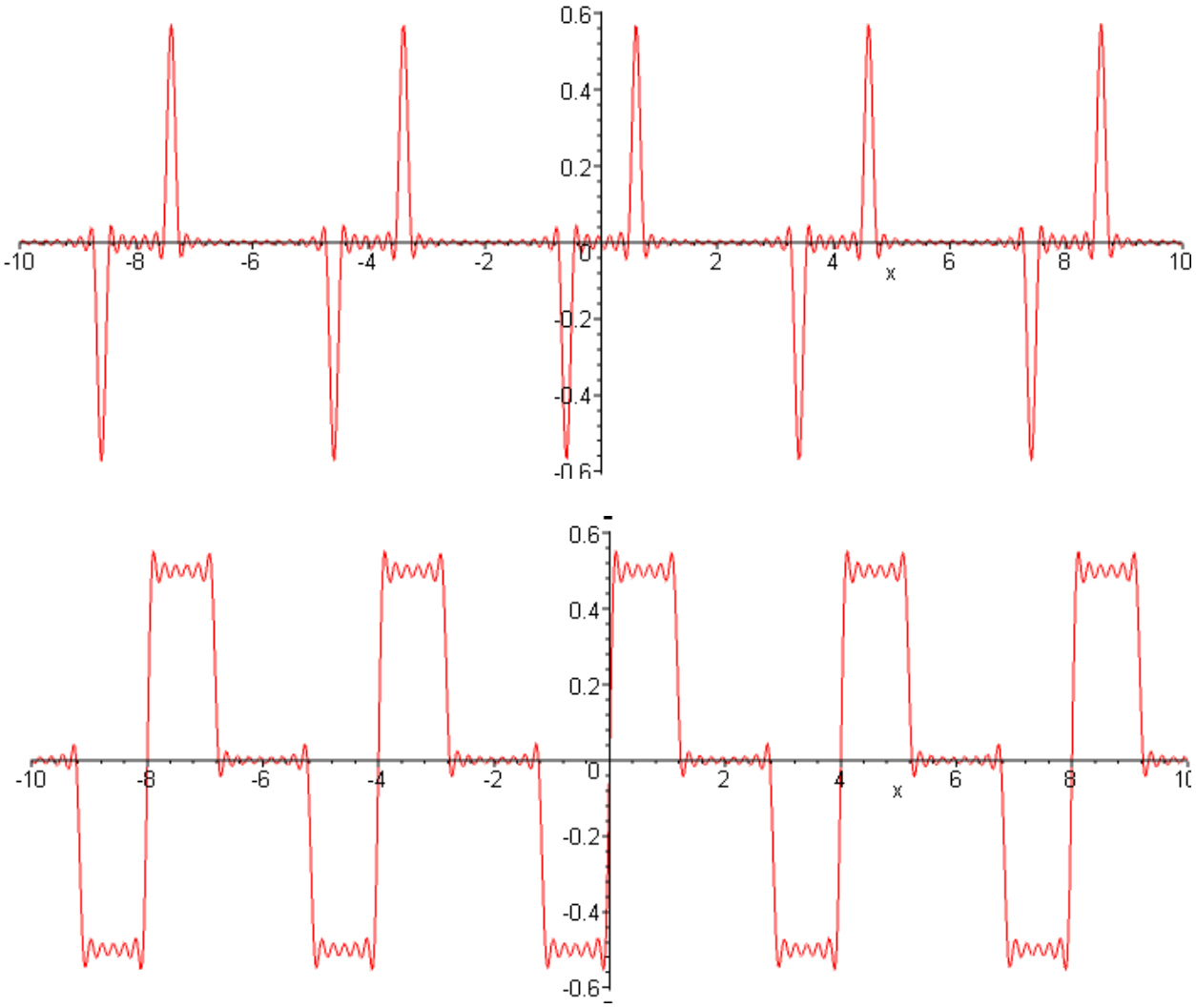
and this is how we keep u = 0 at x=0 and x = L all the time. The other pulses are not cancelled here because I only included 3 pairs of sources in the model. You need to stare at the animation which is located in "string green animation.mws".

With all the sources present, you have an initial "growth phase" where the initial matchsticks appear and then widen (and do not translate at all). Then after that the picture basically shows an infinite set of fixed width boxes on the top going to the right, and another infinite set on the bottom going to the left. Whenever two boxes overlap, you get the cancellation which keeps the solution 0 at x = 0,±2,±4, etc. The steady state solution is then a sum of two wavetrains of pulses going in opposite directions. Obviously each individual pulse itself is a wave solution of the PDE, and this superposition of solutions gives a solution that meets the BC's for our finite string with endpoints fixed at 0 and L.

If you remove all the negative images, you get a completely different solution as Maple shows.

Bilinear Sum Method for Finite String

In this method we just apply our generic bilinear formula 7.118 (for wave theory) and we install the known spatial eigenfunctions and eigenvalues quoted top of page 253, and we get 7.126. Since this is exactly the same as 7.125, we expect somehow to see that initial "growth phase" seen above. And sure enough, here are some plots of 7.126 showing the first 20 terms only of the sum. First, here are the matchsticks first appearing and after which they widen in place



After this things proceed just as shown above with the square wave, with the top square waves moving to the right and the bottom to the left.

General solution u(x,t) for the finite string

We take just the "pluck" term in 7.116 which is the $f_1(x)$ term and we have then

$$u(t_0,x_0) = +\partial_{t0}\{ \int_{R_n} d^n x\, f_1(x)\, g(x_0,t_0|x,0) \}$$ from my 7.116 gen

$$u(t,\mathbf{x}) = +\partial_t \{ \int_0^{\ell} dx_0\, f(x_0)\, g(x,t|x_0,0) \}$$

$$g(x,t|x_0,0) = 7.126 = \Sigma_{k=1}^{\infty}(2/\pi k)\sin(k\pi t/\ell)\sin(k\pi x/\ell)\sin(k\pi x_0/\ell)$$

so we get

$$u(t,\mathbf{x}) = +\partial_t \{ \int_0^{\ell} dx_0\, f(x_0) [\, \Sigma_{k=1}^{\infty}(2/\pi k)\sin(k\pi t/\ell)\sin(k\pi x/\ell)\sin(k\pi x_0/\ell)] \}$$

$$u(t,\mathbf{x}) = \{ \int_0^{\ell} dx_0\, f(x_0) [\, \Sigma_{k=1}^{\infty}(2/\pi k)(k\pi/\ell)\cos(k\pi t/\ell)\sin(k\pi x/\ell)\sin(k\pi x_0/\ell)] \} \quad \text{// if OK !}$$

$$= (2/\ell)\, \Sigma_{k=1}^{\infty} \cos(k\pi t/\ell)\sin(k\pi x/\ell) \int_0^{\ell} dx_0\, f(x_0)\sin(k\pi x_0/\ell)$$

$$= \Sigma_{k=1}^{\infty} \cos(k\pi t/\ell)\sin(k\pi x/\ell)\, f_k \qquad \text{// agrees with p 253 B and C}$$

Then p 253 D and C form what my transforms.doc calls the "Fourier Series Sine Transform" and thus I am happy with D (just got this stuff fixed up yesterday in transforms.doc!) Stak then talks about the same thing I do, the "oddized function" F as an extension of D to the $(-\ell,0)$ range. Stak wants to call this f* thing the odd periodic extension of f where you start with f just on $(0,\ell)$, and this is all just fine!

Question: What does this have to do with the string Green's Function of Chapter 1 Vol I ? THAT Green's is for the 1D Laplace equation, whereas THIS Green's is for the 1D wave equation, so not the same.

So if we look at p 253 F, we see our initial "pluck" function f(x) creating opposite moving periodic waves of that shape on an infinite string. This must then be the generalization of what we were seeing in the Green's Function with those pulse trains. I built a sawtooth but the result was not too enlightening. Learned how to make **custom periodic functions in Maple** just now.

Section 7.9 Problems in more than one spatial dimension (253)

Problems in 3D with no boundary: propagator is C_3

The causal wave free Green's for 3D is just $C_3 = \delta(t-R)/(4\pi R)$ where $R = |\mathbf{x}-\mathbf{x_0}|$. If we put $\mathbf{x_0}$ at the origin of a spherical coordinate system, we get $C_3 = \delta(t-r)/4\pi r$ which is an expanding δ shell with extra factor shown which happens to be the potential of a unit point charge in electrostatics!

At this point we might like to identify our generic wave equation problem 7.128 with Jackson page 180 where we have the wave equations for the vector and scalar potentials **A** and φ. We can interpret q in 7.128 as charge density ρ in Jackson 6.37. Then we can think then of "u" in our generic equation being the "scalar potential" φ which is not quite the potential of electrostatics because we now have time dependence. We shall come back to this interpretation in a moment.

We now want to compute our most general system solution to 7.128 and we know this is given by 7.116 *with no boundary σ*. Our 7.128 system is specifically without boundary, and that is why the

propagator is the C_3 we talk about above. Since it is just a delta function (and factor), the various integrals appearing in 7.116 are pretty easy to do. We have the q, f_2 and f_1 integrals which Stak calls I, J and K. He methodically computes each one (I followed all of it while reading the text), and the general result is 7.129.

Now f_1 is our "initial displacement" which we prescribe over all space, and f_2 is our "initial velocity" or ∂_t of displacement, again at all points in space. What shows up in the solution is f_1^* and f_2^* which are "averages of f_1 and f_2 taken over a spherical *shell* centered at x_0 having radius t_0", and (x_0,t_0) is our observation point at which we evaluate our solution. I think this is saying that the initial condition influence is due to all points on a sphere about our observation point of radius t_0, because it took time t_0 for that influence to get to us due to the wave velocity of 1. So there is "retardation" in the effect of these initial conditions. You are only affected now by an initial condition value then (ie, at t=0). I will definitely have to ponder this. The f_2^* works like this (velocity), while the f_1^* part of 7.129 is a little more complicated. You can of course trace it all in terms of the propagator C_3 and 7.116 if you want to generate an interpretation.

The q term in 7.129 is of most interest to me right now. Remember that q = ρ of the scalar potential wave equation. The q integral is over a ball of radius t_0 centered on the observer at x_0. We are being effected NOW by charge density AS IT WAS at the retarded time. But the charge density q = ρ is some distributed cloud, so each piece of it has a different retardation time and distance. So in a simple case we could say $\rho = \delta(x-x_1)$ and is a unit point charge. If this point charge is within our ball of points that can influence us, then the integral will give just $1/4\pi|x_0-x_1|$. But if this point charge is outside the ball, so it cannot influence us yet, then the integral gives 0. This is exactly those pictures drawn by Ed Purcell in the orange E&M book. If you turn on the point charge at t = 0, the potential is zero except inside a sphere of influence of radius t, and within that sphere V = q/r and outside it V = 0 because it hasn't gotten there yet! So we have the new notion here regarding electric potential V that it is something that cannot go faster than velocity v = 1 in our Stak framework. This is the speed of causality. Remember that in the heat equation we did not have such a well defined propagation speed notion. Now, if we wait a very long time, we have V = q/r everywhere, we are in a static state, and V becomes the electrostatic potential which solves the Potential Theory Laplace equation and is $1/4\pi R$ in that world. So the appearance of $1/4\pi R$ in the causal 3D wave Green's Function C_3 is very definitely related to the $1/4\pi R$ in the Laplace Green's Function. This C_3 thing says "influence exists only *on* a δ shell centered on the source which is $\delta(x)\delta(t)$", and this seems quite natural.

Comment: In a separate doc called "chap 7 ex PL1.doc" I was able to directly show that

$$(\partial_t^2-\nabla^2)\,[\theta(t)\delta(t-R)/4\pi R] = \delta(t)\delta(R) \qquad (1)$$

thus verifying that $C_3 = \theta(t)\,\delta(t-R)/(4\pi R)$ is indeed the 3D causal Green's Function for the wave equation with no boundaries. This was *much harder* than I thought it would be, and required me to write up doc "Delta Function Rules" as a sort of technology handbook without which I could not have done it. Perhaps this is why Stak obtained C_3 using a time Laplace transform back on page 62.

Comments: I thought I was going to find a very simple way of extending $-\nabla^2(1/4\pi R) = \delta^3(\mathbf{x}-\mathbf{x_0})$ to result (1) above, but I did not discover such a simple way. The Green's function is a "delta shell" that radiates out from a spacetime disturbance at $\delta(t)\delta^3(x-x_0)$ but of course only for t > 0. There is no "wake"

behind this shell, and I remember this is true in general for an odd number of dimensions for space (see p 62-3). The "influence" of this disturbance exists at some point in space exists only when t = R. There is no influence whatsoever when t < R and t > R. In contrast, the 1D Green's function is $C_1 = \theta(t)\, H(t-R)/2$ and the "wake" in this case is permanent and is the same size as the wavefront. In 2D we get halfway between these two solutions. There is a wake, but it is maximal just behind the wavefront which is infinite but integrable. We have $C_2 = (1/2\pi)\, H(t-R)/\sqrt{t^2-R^2}$, and this looks like the charge density on a disk of radius t and coordinate $\rho = R$! I'll bet there is some connection here. Remember that all these Green's functions can be interpreted as "a wave" (a solution to the homo wave equation) that results from a $\delta(x-x_0)$ "velocity impulse" at t = 0, not a "displacement impulse".

Deriving the C_2 result by integrating a line of C_3 results (Hadamard). (p 255)

Stak says this stuff is the method of descent, but does not quite explain that phrase, so: (circa 1923)

3.5 The method of descent

The method of descent, also due to Hadamard, consists simply in thinking of any solution to the wave equation in even ($n = 2d$) dimensions as a solution in one more dimension which does not depend on one of the space variables. In two dimensions, in particular, we can write

$$u(x, y, t) = \tilde{u}(x, y, z, t)\,,$$

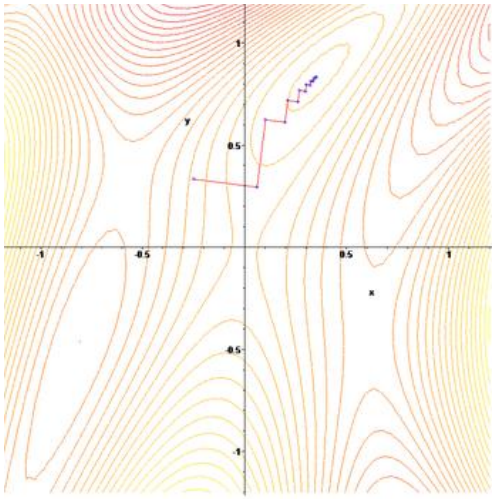
where $\tilde{u}$ is a solution to the three–dimensional wave equation with initial data that do not depend on z:

$$\tilde{u}(x, y, z, 0) = \tilde{g}(x, y, z) = g(x, y)\,, \quad \tilde{u}_t(x, y, z, 0) = \tilde{h}(x, y, z) = h(x, y)\,.$$

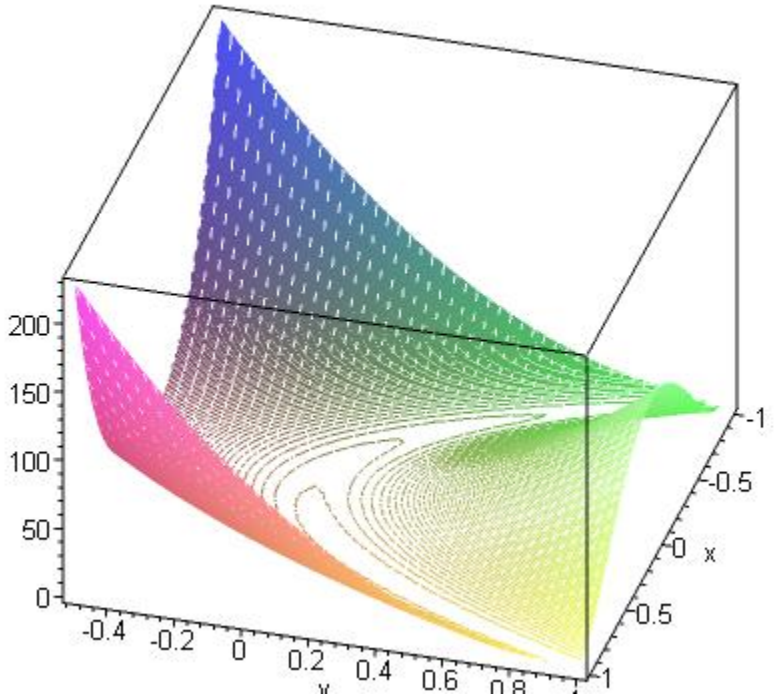
So imagine in 3D a continuous line of sources $q(z,y,z,t)dz = \delta(x)\delta(y)dz$ on the infinite z axis which drives the wave equation. We know C_3 for each point source on this line. Symmetry tells us that the solution to the wave equation driven by this line of sources q cannot be a function of z, just as it says above. Symmetry also tells us that the solution to this problem is C_2 if we restrict our interest to the z = 0 plane, and of course in this plane the source appears as a point source and the solution is obviously symmetric about the origin of the x,y plane. So the idea is just to do superposition of the C_3's to get C_2, again just as the quote above says. In our example we "descend" from n = 3 to n = 2.

Comment: Two other "methods" come to mind when we hear "method of descent" which have nothing to do with the above!

The Method of Steepest Descent (aka method of gradient descent). Suppose you want to find a place where some real f(x,y) where has a local minimum. An obvious method is to start at some random location (x,y), compute the gradient, go a small step in the negative gradient direction, and keep doing that till f stops getting lower. This should take you on some kind of zig-zag voyage down into a valley minimum and hopefully you won't hit one of the "walls" of the limited space you are working with. Here is a great picture:



So certainly the phrase "steepest descent" is quite clear. You might end up at a saddle point where the gradient happens to be horizontal in all directions. The Rosenbrock function shows how you might have trouble doing this (wiki) because you will go along the riverbed shown before you will finally "get down" to the minimum.



Method of Stationary Phase. This is a way to estimate say a 2D integral with factor $e^{if(x,y)}$ where you expect most of the integral to come from saddle points in f(x,y), which is to say, from regions where the phase is relatively stationary so the phase winding doesn't kill things off ! I have a Math binder writeup on this. These integrals of course arise in optics a lot. Now, you might USE the method of steepest descent to try to locate such saddle point regions! There is more to this, however. In general this is

presented in terms of a contour integral in a z = x+iy plane and you want to deform the contour so it passes through a nice saddle region I think. So stationary phase and steepest descent are connected somehow in this context, but I don't care right now.

So, don't confuse the Method of Steepest Descent with Hadamard's Method of Descent, since these two concepts are totally unconnected with each other!

Now back to Stak page 255. He cleanly shows how you superpose the C_3 solutions with a continuum of sources along the z axis to derive an expression for C_2 , including the Heaviside part of it.

[256] We have a little reminder of the notion of a "wake". When you have only a delta function shell of influence, which happens for n = 3,5,7..., he says that we have "Huygen's Principle" which is associated with the notion of influence by wavefronts. When n = 1, we don't have HP for a velocity source (strike), but we do for a displacement source (pluck).

He continues then and obtains C_1 from C_3 by this time superposing not just a line of charge along z, but an infinite number of such lines which then fill the x=0 plane with sources. If you restrict your interest just to points on the x axis (perp to the plane of sources), you can add things up in rings of C_3 sources to find C_1. Symmetry shows that the superposed wave solution can only depend on x, so in this case we are "descending" from x,y,z all the way down to x. As Hadamard says, "if you know more, then you also know less". Both of Stak's superpositions are completely clear to me.

Section 7.10 Wave Equation with External Damping (257)

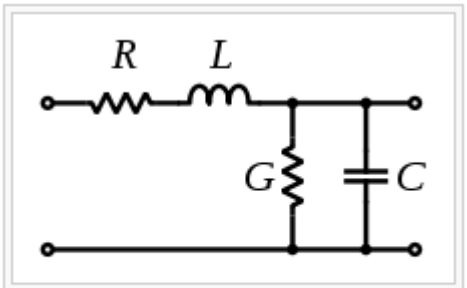
The extra term here is "perfectly clear" based on Vol I Appendix A. You have a damping force linear in velocity which opposite to its direction, hence $-2\gamma\partial_t u$ when put on the RHS where forces live, and $\partial_t^2 u$ is the famous "ma". So the question is: how does the presence of this $2\gamma\partial_t u$ term affect the analysis of things? Well, the first order of business is to state the isolated "strike" (u) and "pluck" (v) problems as in 7.131,132. Then in analogy with p 245, we find that $v = \partial_t u + 2\gamma u$ where we now have $2\gamma u$ as an "extra term" that was not present in the p 245 result. So again, we need only solve the strike problem. This extra term is necessary to make p 257 B work which in turn makes C work on the next page. So we have the analog of the Big Theorem discussed above in the $\gamma = 0$ case.

So how does the presence of $\gamma \neq 0$ change the analysis? On page 247 we solved for a g function by doing a spatial φ_i expansion and we got little time ODE p 248A which we solved by inspection to get 7.118. which was our "generic bilinear form for the wave world". Here, we do something similar. As we did in Chapter 5, we first change from u to w where $u = we^{-\gamma t}$ where we expose the major decay factor, then we rewrite our system for u (with only $f_2 = f$) and we get a system for w as in 7.135. [resumed below]

Aside: The key idea is that the u to w transform removes the $\partial_t u$ term which resulted from adding γ to the problem, and then you once again have a *wavelike* PDE except there is an extra γ^2 "constant term", and this PDE has the name **"the equation of telegraphy"** since it was used to analyze lossy transmission on phone lines (for n=1 dimension). This equation was studied on page 65 (n dimensions) where Stak used time Laplace to convert it to a Helmholtz Green's equation 5.160 where in place of s^2 we have $s^2-\gamma^2$ as our $-k^2$ Helmholtz constant. This *isotropic* BC Helmholtz Green PDE in turn was solved on page 53

with result E_n given in 5.116 in terms of an $H^{(1)}_{(n/2)-1}(kr)$. However, in our current section we *do* have a BC to worry about which is the f_2 strike function (for Green's causal this would be a δ) , and moreover we are not solving for a Green's function, but rather a homo solution with this BC, so this H is not the solution, and here we will only arrive at a bilinear form.

Aside: Here is wiki on "telegrapher's equation":



Schematic representation of the elementary components of a transmission line.

// model for each dz of transmission line length

You obtain two "telegrapher's equations" which you then combine into a single equation, either one:

$$\frac{\partial^2}{\partial x^2}V = LC\frac{\partial^2}{\partial t^2}V + (RC+GL)\frac{\partial}{\partial t}V + GRV$$
$$\frac{\partial^2}{\partial x^2}I = LC\frac{\partial^2}{\partial t^2}I + (RC+GL)\frac{\partial}{\partial t}I + GRI$$

Note that these equations resemble the homogeneous wave equation with extra terms in *V* and *I* and their first derivatives. These extra terms cause the signal to decay and spread out with time and distance. If the transmission line is only slightly lossy (small *R* and *G* = 0), signal strength will decay over distance as $e^{-\alpha x}$, where $\alpha = R/2Z_0$

So yes, if you set G = 0, you have only the velocity damping term $RC\partial_t u$ and this then matches Stak. Wiki talks about how all the parameters vary with frequency, a model is needed for each parameter.

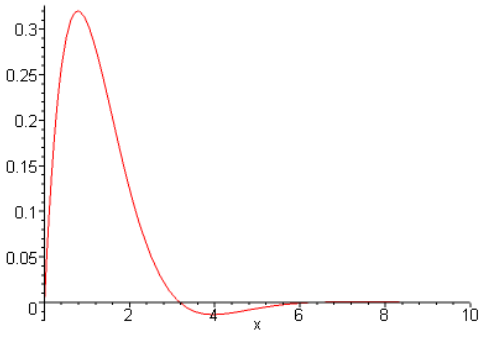
[resume Stak thread] So we have now our system for w which is 7.135 where f(x) is a strike function IC and with u and w both = 0 on whatever σ we have. Think string tied down at ends, or drumhead tied down around the drum perimeter. Now we do the same $\varphi_i(x)$ spatial expansion, but we allow time dependent coefficients (a partial EF expansion method; perturbation theory WM method; Stak has used this before) and we obtain then an ODE system for the coefficients $w_k(t)$ in 7.136 [in the partial EF method of finding Laplace Green's functions, we ended up with a 1D ODE Green's function equation] Here our ODE is not a Green's one, just a standard issue one. The $w_k(t) = 0$ at t=0 due to our IC on w, and the $w_k'(t)$ condition is the strike IC with now f_k instead of f. Once again, we can solve this little ODE "by inspection" and the solution is p 258 D where the "nature" of the solution depends on the size of γ relative to the spatial eigenvalue λ_k of one of our "modes". Recall from somewhere earlier that all $\lambda_k > 0$ (for some reason I forget right now).

Now, when we install our just-found solutions p 258D for $w_k(t)$ into p 258 A we get our final solution for u which is 7.137, and this is then the complete exact solution to our problem 7.131. The damping factor $e^{-\gamma t}$ is strong enough to overcome even the top case of w_k (since $\lambda_k > 0$) and we see that, as expected, the solution damps away eventually. The solution 7.137 looks a lot like the generic bilinear form for the heat equation 7.57, but in crude form only!

So this was the solution of 7.131 and we know from the Big Theorem that we can then solve any problem like this with f_1 and f_2 as the IC's.

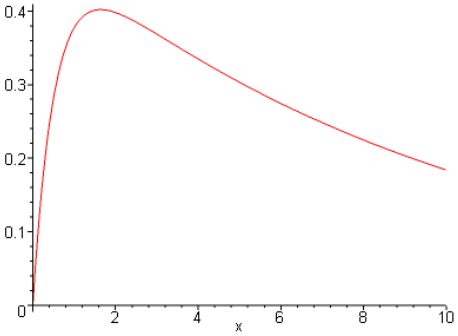
Now look at the $w_k(t)$ three cases in p 258 D. For a given γ size, modes with large $\lambda_k > \gamma^2$ will be oscillating functions against the envelope time decay. These modes are underdamped and Stak comments that they can fatigue materials.

```
plot( exp(-x)*sin(.99*x),x=0..10);
```



In contrast, the smaller λ_k modes are "monotonic" in shape (the damping part of the curve is monotonically decreasing instead of oscillating), for example

```
plot( exp(-x)*sinh(.9*x),x=0..10);
```



One suspects that normal struck drum response is underdamped when we think of the surface as oscillating. The middle case shows a linear rise in time, typical of resonance, but this is less dramatic than the upper case which is an expo rise. Both of course are tamed by the $e^{-\gamma t}$ in 7.137.

It would be good to carry out a string and drumhead version of this theory. We shall see what the exercises later have to offer.

Comment: In the section above we considered the homo damped wave equation, no driving function, just some IC's and BC = 0 and we saw the basically damped transient response that results. The main presentation dealt with the strike IC and I guess that solution really is the drum solution. I guess the only question for a real drum is whether all the modes are underdamped or not. Damping comes from air

friction and from internal friction in the membrane medium. So we really have solved a very general drumhead problem. Now, if we hit the drum with a drumstick at some point x_0 and think of $f(x) = \delta(x-x_0)$, then $f_k = \varphi_k(x_0)^*$ according to p 258B. We activate all the modes of the drumhead by this amount. Then our solution 7.137 looks even more like the heat or wave bilinear form. In fact, if we define w'_k to not have the f_k factors, we get

$$u(x,t;\, x_0,t) = \Sigma_{k=1}^{\infty}[e^{-\gamma t} w'_k(t;\, \lambda_k,\gamma)]\;\; \varphi_k(x_0)^*\, \varphi_k(x)$$

and this must be then the general bilinear form for the Green's function g for the damped wave equation. It seems odd that he did not consider this special case. I like to constantly compare all these forms

$$g(x\,;\, x_0) \;= \Sigma_k\, [\; 1/\lambda_k]\; \varphi_k(x)\varphi_k^*(x_0) \qquad \text{// Laplace 6.108}$$

$$g(x,t\,;\, x_0,\, t_0) \;= \Sigma_k [e^{-\lambda_k (t-t0)}]\; \varphi_k(x)\varphi_k^*(x_0) \qquad \text{// heat 7.57}$$

$$g(x,t\,;\, x_0,\, t_0) \;= \Sigma_k\, [\; \sin\, \{\sqrt{\lambda_k}\,(t-t_0)\}\, /\, \sqrt{\lambda_k}\;]\;\; \varphi_k(x)\varphi_k^*(x_0) \qquad \text{// wave 7.118}$$

$$g(x,t;\, x_0,t) = \Sigma_{k=1}^{\infty}[e^{-\gamma t} w'_k(t;\, \lambda_k,\gamma)]\; \varphi_k(x)\; \varphi_k(x_0)^* \qquad \text{// damped wave (PL)}$$

As $\gamma \rightarrow 0$. only the lowest w_k case exists (underdamped) which contains $\cos(\sqrt{\lambda_k}\, t\,)/\, \sqrt{\lambda_k}$. But why is this cos instead of sin ?? I think Stak has made a **big mistake**!! I think he should have sin and sinh in p 258 D!!! Only then do we get $w_k(t=0) = 0$ as 7.136 requires! And then the limit works right! I think in Stak's notes he had an ongoing confusion regarding f_1 and f_2 and we have seen this kind of error before! So I will just fix it right now, more things for my errata list. [this led me to search a bit on Stak's stuff, he is alive and well and I started a doc on him and his various books and their editions.]

So fixing this error, the damped wave equation bilinear solution matches the undamped when $\gamma \rightarrow 0$.

Note: Google has some of Vol 1 on line, and indeed this last problem has been fixed!

Therefore

$$w_k = \begin{cases} f_k\, \dfrac{\sinh\,(\gamma^2 - \lambda_k)^{1/2} t}{(\gamma^2 - \lambda_k)^{1/2}} & \text{for } k \text{ such that } \lambda_k < \gamma^2; \\ f_k t & \text{for } k \text{ such that } \lambda_k = \gamma^2; \\ f_k\, \dfrac{\sin\,(\lambda_k - \gamma^2)^{1/2} t}{(\lambda_k - \gamma^2)^{1/2}} & \text{for } k \text{ such that } \lambda_k > \gamma^2. \end{cases}$$

Probably all my "errata" have been fixed, and that is why there is no on line list.

Section 7.11 Monochromatic Excitation and Principle of Limiting Absorption (259)

Now we are going to add frequency ω drive for both the q function and the boundary schedule h, as we did elsewhere earlier in this chapter. This is set up on page 259. We then assume a solution of the FORM shown in 7.139 where we now have two functions U and v (I suspect v will be a transient response). This

correctly results in A,B,C on page 260. If we then look for a solution where U satisfies 7.140 (a priori we don't know this will even work) , then v must satisfy 7.141. So things are now all set up.

Now, the system for U is a driven spatial Helmholtz equation with a complex k^2 and with h(x) as the BC. Time is not involved of course in dealing with U. We can presumably solve this using a Helmholtz Green's Function by our usual methods (see below). We think of ω and γ as just some parameters.

The system for v is exactly our damped wave equation as studied in the last section [he has a typo here! He should have ∇^2 instead of ∂_x^2 because we are still in general n dimensions!] This v problem has both pluck and strike IC's as shown in 7.141, where we assume we know U(x) from solving the U problem. We know (he made a point of this earlier) that the solution v therefore damps away and is indeed just a transient solution, and then the steady-state solution is the $Ue^{-i\omega t}$ term!

Now he claims that the driven Helmholtz U problem has a unique solution. [resumed below]

Solving the U(x) Helmholtz Equation to get 7.143

Plan A. Start with [reader: skip to Plan B]

$(-\nabla^2-\lambda)U(x) = q(x) \qquad U(x) = h(x) \text{ on } \sigma \qquad (*)$

Expand in spatial eigenfunctions where we know $-\nabla^2\varphi_i = \lambda_i\varphi_i$ and $\varphi_i(x) = 0$ on σ. We note in passing that this means the solution U(x) is in some sense not uniformly continuous at the boundary. Ignoring this issue, we blindly plod ahead,

$U(x) = \Sigma_i u_i \varphi_i(x) \qquad u_i = \int_R dx\, u(x)\, \varphi_i^*(x)$

$q(x) = \Sigma_i q_i \varphi_i(x) \qquad q_i = \int_R dx\, q(x)\, \varphi_i^*(x)$

$(-\nabla^2-\lambda)\, \Sigma_i u_i \varphi_i(x) = \Sigma_i q_i \varphi_i(x)$
$\Sigma_i u_i (-\nabla^2-\lambda)\, \varphi_i(x) = \Sigma_i q_i \varphi_i(x)$ // assuming order interchange OK
$\Sigma_i u_i (\lambda_i -\lambda)\, \varphi_i(x) = \Sigma_i q_i \varphi_i(x)$

Use completeness of the φ_i to conclude that

$u_i (\lambda_i -\lambda) = q_i \qquad => \qquad u_i = q_i/ (\lambda_i -\lambda)$

Then our solution seems to be

$U(x) = \Sigma_i \varphi_i(x)[1/(\lambda_i -\lambda)]\, q_i$
$= \Sigma_i \varphi_i(x)[1/(\lambda_i -\lambda)] \left[\int_R dx\, q(x)\, \varphi_i^*(x) \right]$

This is a "particular" solution to (*). We need now to add a homo solution of (*) so that we can then match the BC that U(x) = h(x) on σ. Stak is claiming a certain result, but it involves a boundary integral, and that means we somehow have to get Green's Theorem involved here. In other words, with what we

have done above, we just don't have enough information to obtain that required homo solution we need to add in. So let's tack off a bit and then sail in from the right direction.

Plan B. What is the Green's Theorem (p 40) for the above Helmholtz operator L = $(-\nabla^2-\lambda)$? This is a case that Stak did not do back there, so I can try it now. Let's start with

$$\int_v dV\,[\,\psi\,\nabla^2\varphi - \varphi\,\nabla^2\psi\,] = \int_s dS\,[\,\psi(\,\partial\varphi/\partial n) - \varphi(\,\partial\psi/\partial n)\,] \qquad (6) \qquad \text{Green \#2}$$

Now I think we can alter the left side *for free* to get this:

$$\int_v dV\,[\,\psi\,(\nabla^2+\lambda)\,\varphi - \varphi\,(\nabla^2+\lambda)\,\psi\,] = \int_s dS\,[\,\psi(\,\partial\varphi/\partial n) - \varphi(\,\partial\psi/\partial n)\,]$$

which we write then as follows, where we now have L = $(-\nabla^2-\lambda)$ on the LHS

$$\int_v dV\,[\,\psi\,(-\nabla^2-\lambda)\,\varphi - \varphi\,(-\nabla^2-\lambda)\,\psi\,] = -\int_s dS\,[\,\psi(\,\partial\varphi/\partial n) - \varphi(\,\partial\psi/\partial n)\,]$$

The RHS is thus the same as it is for Laplace theory. Note that L* = L. I am now sort of following my BV problems doc here. Let's now set

$$\varphi = \varphi_i(x)^* \qquad\qquad \psi = U$$

to get

$$\int_v dV\,[U\,(-\nabla^2-\lambda)\,\varphi_i^* - \varphi_i^*\,(-\nabla^2-\lambda)\,U\,] = -\int_s dS\,[U\,(\,\partial\varphi_i^*/\partial n) - \varphi_i^*\,(\,\partial U/\partial n)\,]$$

But $\varphi_i^* = 0$ on the boundary, so this simplifies to

$$\int_v dV\,[U\,(-\nabla^2-\lambda)\,\varphi_i^* - \varphi_i^*\,(-\nabla^2-\lambda)\,U] = -\int_s dS\,U\,(\,\partial\varphi_i^*/\partial n)$$

Now install $-\nabla^2\,\varphi_i^* = \lambda_i\,\varphi_i^*$ and $(-\nabla^2-\lambda)\,U = q$ and we then have

$$\int_v dV\,[U\,(\lambda_i-\lambda)\,\varphi_i^* - \varphi_i^*\,q] = -\int_s dS\,U\,(\,\partial\varphi_i^*/\partial n)$$

$$\int_v dV\,U(x)\,(\lambda_i-\lambda)\,\varphi_i^*(x) = \int_v dV\,q(x)\,\varphi_i^*(x) - \int_s dS\,U\,(\,\partial\varphi_i^*/\partial n)$$

$$(\lambda_i-\lambda)\int_v dV\,U(x)\,\varphi_i^*(x) = \int_v dV\,q(x)\,\varphi_i^*(x) - \int_s dS\,U\,(\,\partial\varphi_i^*/\partial n)$$

$$(\lambda_i-\lambda)\, u_i = \int_V dV\, q(x)\, \varphi_i^*(x) - \int_S dS\, U\, (\partial\varphi_i^*/\partial n)$$

$$u_i = [1/(\lambda_i-\lambda)]\, [\int_V dV\, q(x)\, \varphi_i^*(x) - \int_S dS\, U(x)\, (\partial\varphi_i^*/\partial n)\,]$$

and we then insert this u_i into our Plan A expansion pair for U and we get

$$U(x) = \Sigma_i\, u_i\, \varphi_i(x)$$

$$= \Sigma_i\, \varphi_i(x)\, [1/(\lambda_i-\lambda)]\, [\int_V dx'\, q(x')\, \varphi_i^*(x') - \int_S dS'\, U(x')\, (\partial\varphi_i(x')^*/\partial n)\,] \qquad \text{// 7.143}$$

and we have done it! On the boundary we have U = h.

Now, suppose we select x to be a point on σ in the above equation. It would seem that we get

$$U(x \text{ on } \sigma) = 0 \qquad \text{since } \varphi_i(x) = 0 \text{ in all terms.}$$

But we know our series does not converge at the boundary from the very start, so maybe this can be ignored. No doubt this could be resolved using our limits to a surface discussion earlier in Stak.

Can I show now that the second term is a homo solution of the PDE?

$$U_h(x) = -\Sigma_i\, \varphi_i(x)\, [1/(\lambda_i-\lambda)] \int_S dS'\, U(x')\, (\partial\varphi_i^*/\partial n)$$

$$(-\nabla^2-\lambda)\, U_h(x) = -(-\nabla^2-\lambda)\Sigma_i\, \varphi_i(x)\, [1/(\lambda_i-\lambda)] \int_S dS'\, U(x')\, (\partial\varphi_i^*/\partial n)$$

$$= -\Sigma_i\, (-\nabla^2-\lambda)\varphi_i(x)\, [1/(\lambda_i-\lambda)] \int_S dS'\, U(x')\, (\partial\varphi_i^*/\partial n) \qquad \text{// assuming ...}$$

$$= -\Sigma_i\, (\lambda_i-\lambda)\varphi_i(x)\, [1/(\lambda_i-\lambda)] \int_S dS'\, U(x')\, (\partial\varphi_i^*/\partial n)$$

$$= -\Sigma_i\, \varphi_i(x) \int_S dS'\, U(x')\, (\partial\varphi_i(x')^*/\partial n)$$

$$= -\Sigma_i\, \varphi_i(x) \int_S dS'\, U(x')\, (\partial_n\varphi_i(x')^*)$$

$$= -\int_S dS'\, U(x')\, \partial_n\, [\,\Sigma_i\, \varphi_i(x)\, \varphi_i(x')^*\,] \qquad \text{// assuming ...}$$

$$= -\int_S dS'\, U(x')\, \partial_n\, [\delta(x-x')\,]$$

Here then is the slightly slippery argument. We know that $\delta(x-x') = 0$ unless x and x' are "infinitesimally close together". Whatever $\partial_{\mathbf{nx}'}\,\delta(x-x')$ is, we know it is zero except in the same circumstance. But in our equation above, x is supposed to be a point *inside* our volume, and x' is a point *on the boundary*, so they are not "infinitesimally close to each other", so really $\partial_{\mathbf{n}}\delta(x-x') = 0$ for our x and x', so this integral is zero, and we have shown that this term is a homo solution.

One possible way to make this a little clearer would be to invent a flat Cartesian coordinate system at a point of contact with z being the normal coordinate. Then

$$\partial_{\mathbf{n}}\,\delta(x-x') = \partial_{\mathbf{z}'}[\,\delta(z-z')\,\delta^{(\mathbf{n-1})}(\xi-\xi')\,] \qquad \text{where } x = (z,\xi) \text{ and } x' = (z',\xi')$$

$$= [\,\partial_{\mathbf{z}'}\delta(z-z')]\,\delta^{(\mathbf{n-1})}(\xi-\xi')$$

where ξ are all the other coordinates. But since $x \neq x'$, we know that $\xi \neq \xi'$ so $\delta^{(\mathbf{n-1})}(\xi-\xi') = 0$ and therefore $\partial_{\mathbf{n}}\,\delta(x-x') = 0$.

[resuming] Now as we were saying, Stak claims that the U system has a unique solution. Since I have now derived 7.143, I know that it exists as a solution. It exists because we have constructed it. Our solution would be dubious if we had $\lambda = \lambda_{\mathbf{i}}$ for some eigenvalue, but our λ is complex, so this is not a problem. I could show uniqueness I think "in the usual manner".

The Principle of Limiting Absorption

[261] **Now what happens if we take $\gamma \to 0$** so our λ then becomes $\lambda = \omega^{\mathbf{2}}$ which is then on the real axis and could coincide with one of our $\lambda_{\mathbf{i}}$. Stak's first comment is that we will imagine $\lambda = \omega^{\mathbf{2}}+i\varepsilon$ with a very slight $\varepsilon = 2\gamma\omega > 0$ so we imagine that the transient solution still "goes away" when we get to the steady state at large t.

But what if you drive at ω which hits an eigenvalue so $\omega^{\mathbf{2}} = \lambda = \lambda_{\mathbf{m}}$? Then you hit a resonance and the series (7.143) blows up, it is no longer valid, 7.142 is not valid, "the membrane breaks" because we get out of our small oscillation limit. Stak is just commenting, not studying the nature of this resonant solution (probably he will do that later).

Finally, if we brutishly set $\gamma = 0$ and we have an unbounded region R, it is very hard to "find" the solution to the system. The path to finding it is to start with $\gamma > 0$, perhaps very small, find the solution as outlined above (such solutions are damped at ∞), and then take the limit of page 261 A, which is the same as 7.144. So you are supposed to find a solution for ω just above the imaginary ω axis, and then take the limit of that solution as ω approaches the axis, and then THAT will be your $\gamma = 0$ solution. Yes it might not be damped at ∞, but it is the solution we want. In other words, we want to think there is always a very slight damping force $\gamma > 0$ to "regulate" our solution. We do NOT want $\gamma < 0$ in the limit because that will no doubt be in some region of instability and infinite nasty stuff (my imagination).

This idea of taking this careful limit as shown in 7.144 is what Stak means by **The Principle of Limiting Absorption**. The word absorption of course means "loss of energy" due to γ which I think he is about to consider next.

Energy Considerations.

All is well to mid page 262 where we have the energy conservation equation and the γ term is then the rate at which energy is consumed in the frictional γ effect.

Math detail.

If we had a real solution u, then $\mathbf{J} = -(\partial_t u)\nabla u$. But suppose $u = \mathrm{Re}(Ue^{-i\omega t})$ and we allow U to be a complex solution to our problem. How then do we write J ? Well

$$\mathbf{J} = -(\partial_t u)\nabla u = -(\partial_t \mathrm{Re}(U(x)e^{-i\omega t}))\nabla \mathrm{Re}(U(x)\, e^{-i\omega t})$$

Question: Can derivatives pass through the Re operator? I think so. Let f(x) = a(x) + ib(x)

$$\partial_x \mathrm{Re}[\, f(x)] = \partial_x a(x)$$

$$\mathrm{Re}[\, \partial_x f(x)] = \mathrm{Re}[\, \partial_x a(x) + i\, \partial_x b(x)] = \mathrm{Re}[\, \partial_x a(x)]$$

Therefore the answer is yes. So we have

$$J = -\ \mathrm{Re}(U(x)\, \partial_t\, e^{-i\omega t})\, \mathrm{Re}(e^{-i\omega t}\, \nabla U(x))$$

$$= -\ \mathrm{Re}(U(x)\, -i\omega\, e^{-i\omega t})\, \mathrm{Re}(e^{-i\omega t}\, \nabla U(x))$$

$$= \omega\ \mathrm{Re}(U(x)\, i\, e^{-i\omega t})\, \mathrm{Re}(e^{-i\omega t}\, \nabla U(x))$$

Now we have to write it all out

$$\mathrm{Re}(U(x)\, i\, e^{-i\omega t}) = (1/2)\ \{\, U(x)\, i\, e^{-i\omega t} - U^*(x)\, i\, e^{+i\omega t}\,\}$$

$$\mathrm{Re}(e^{-i\omega t}\, \nabla U(x)) = (1/2)\ \{\, e^{-i\omega t}\, \nabla U(x) + e^{i\omega t}\, \nabla U^*(x)\,\}$$

Therefore we have

$$\mathbf{J} = \omega\, (1/4)\ \{\, U(x)\, i\, e^{-i\omega t} - U^*(x)\, i\, e^{+i\omega t}\,\}\ \{\, e^{-i\omega t}\, \nabla U(x) + e^{i\omega t}\, \nabla U^*(x)\,\}$$

But now we imagine doing a time average over a single cycle of ω (or any integral number of cycles) so we throw out two of the terms and we get only the two cross terms

$$\mathbf{J} = \omega\, (1/4)\ \{\, U(x)\, i\ \nabla U^*(x) - U^*(x)\, i\ \nabla U(x)\}$$

$$= \omega\, (1/4)\ \{i\, [\, U(x)\, \nabla U^*(x) - U^*(x)\, \nabla U(x)]\,\} \qquad (*)$$

$$= \omega\, (1/4)\ \{i\, [2i\, \mathrm{Im}\, (U(x)\, \nabla U^*(x)\,)]\}$$

$$= -\, (\omega/2)\, \mathrm{Im}\, (U(x)\, \nabla U^*(x)\,) \qquad \text{// agrees with 7.148 left}$$

and continuing from line (*) above

$= \omega\ (1/4i)\ \{\ [\ -\ U(x)\ \nabla U^*(x)\ \ +\ U^*(x)\ \nabla U(x)]\ \}$

$= (\omega/4i)\ \{\ U^*(x)\ \nabla U(x) -\ U(x)\ \nabla U^*(x)\ \}$ // agrees with 7.148 right

In pencil on the bottom of page 262 I derive 7.149 and you can interpret the LHS as

$$\int_\sigma \mathbf{J_a} \bullet \mathbf{n}\ dS \qquad\qquad \int_\sigma \mathbf{J_a} \bullet \mathbf{n}\ dS = 0 \qquad 7.150$$

due to 7.148. If there is no damping $\gamma = 0$, we find that the above is 0 and this says there is no energy flux across the boundary on the average. This would seem to imply that $dE_a/dt = 0$ in 7.147 where E_a is the time averaged total energy in our volume. I think in the microscale there might be small t changes in E due to activity close to the boundary, but when averaged these details wash out.

I recall that this complex U business was used by Jackson and others in playing with Poynting vectors where we assumed complex E and B fields so we could use $e^{-i\omega t}$ time dependence. You do this anytime you have monochrome time dependence and something that is bilinear in the field of interest.

Status: We are now 20 pages into the wave part of Chapter 7, and we are facing a formidable looking list of "exercises".

Exercises

Exercise 7.19. Details concerning the time averaged current J shown on page 262.

I already did this above using Re. If I had taken the Im part, I can tap into the above presentation,

$$\mathbf{J} = \omega\ \ \mathrm{Re}(U(x)\ i\ e^{-i\omega t})\ \mathrm{Re}(e^{-i\omega t}\ \nabla U(x))$$

$$\rightarrow\ \omega\ \ \mathrm{Im}(U(x)\ i\ e^{-i\omega t})\ \mathrm{Im}(e^{-i\omega t}\ \nabla U(x))$$

Then the next lines would be

$$\mathrm{Im}(U(x)\ i\ e^{-i\omega t}) = (1/2i)\ \{\ U(x)\ i\ e^{-i\omega t}\ + U^*(x)\ i\ e^{+i\omega t}\ \}$$
$$\mathrm{Im}(e^{-i\omega t}\ \nabla U(x)) = (1/2i)\ \{\ e^{-i\omega t}\ \nabla U(x) - e^{i\omega t}\ \nabla U^*(x)\ \}$$

$$\mathbf{J} = -\ \omega\ (1/4)\ \{\ U(x)\ i\ e^{-i\omega t}\ + U^*(x)\ i\ e^{+i\omega t}\ \}\ \{\ e^{-i\omega t}\ \nabla U(x) - e^{i\omega t}\ \nabla U^*(x)\ \}$$

Again keeping only the cross terms we get

$$\mathbf{J} = -\ \omega\ (1/4)\ \{\ -\ U(x)\ i\ \ \nabla U^*(x) + U^*(x)\ i\ \ \nabla U(x)\ \ \}$$

$$= \omega\ (1/4)\ \{\ \ U(x)\ \ \nabla U^*(x) -\ \ U^*(x)\ i\ \ \nabla U(x)\ \ \}$$

$$= \omega\,(1/4)\,i\,\{\;U(x)\,\nabla U^*(x) - U^*(x)\,\nabla U(x)\;\}$$

But this is the same as (*) above, so everything will then come out the same, meaning 7.148 and 7.150.

Exercise 7.20. Use energy argument to show solution uniqueness for damped wave equation.

On page 244 Stak derived uniqueness of the wave equation solution by considering the system for the difference function v. The question is to see how this little proof goes if we add our γ term. Well, equation A would have a RHS that, instead of being 0, was the RHS of 7.145,so that B would have a new term added to the RHS which was $\int dx\;[\,-2\gamma\,(\partial_t v)^2 = -2\gamma \int dx(\partial_t v)^2$. We would then have

$$\partial_t\,\{\text{ stuff }\} = -2\gamma \int_R dx(\partial_t v)^2 = -\;f(t) \qquad \text{where } f(t) > 0, \text{ whatever it is}$$

Integrating from time t=0 to t then gives

$$\{\text{ stuff (t)}\} - \{\text{ stuff (0)}\} = -\int_0^t dt'\,f(t')$$

But we know that stuff(0) = 0 from the IC's, so we get

$$\{\text{ stuff (t)}\} = -\int_0^t dt'\,f(t')$$

This is a contradiction because the LHS is positive and the RHS is negative! Thus, our assumption that some solution v exists for the system shown is false, and no such v exists, and we then have uniqueness.

Exercise 7.21. Kirchhoff's Formula. I thought we already did this. Formula 7.116 is our "general formula" for solution u. Here we are specializing to 3D space, and we do have a boundary σ. Therefore, we don't know what g is, since it depends on σ. Thus, 7.116 is true, *but* not what is sought here.

On the other hand, 7.7 is our general "wave" cylinder Green's Theorem result, but the bottom of the cylinder is supposed to be at large τ < 0 rather than at t = 0. I derived 7.7 in Part III of my support doc "BV problems". I just read through that, and I can replace t=0 by t=τ << 0 in four places and I get this Green's Result

$$\int_\tau^T dt \int_{R_n} d^n x\;[\,v(t,\mathbf{x})\,(\partial_t^2 - \nabla^2)u(t,\mathbf{x}) - u(t,\mathbf{x})\,(\partial_t^2 - \nabla^2)v(t,\mathbf{x})\,]$$

$$= \int d^n x\,[\;\{\,v(T,\mathbf{x})\partial_t u(T,\mathbf{x}) - u(T,\mathbf{x})\partial_t v(T,\mathbf{x})\,\} - \{\,v(\tau,\mathbf{x})\partial_t u(\tau,\mathbf{x}) - u(\tau,\mathbf{x})\partial_t v(\tau,\mathbf{x})\,\}\;]$$

$$+ \int_\tau^T dt \int_{\sigma_n} dS_n\,[\,u(t,\mathbf{x})\partial_n v(t,\mathbf{x}) - v(t,\mathbf{x})\partial_n u(t,\mathbf{x})\,] \qquad (7.7)$$

I show in red the 6 places I replaced t=0 by t=τ << 0. Now the normal next step is to set $v(t,\mathbf{x}) = g(x_0,t_0|x,t)$ where g is the full propagator including σ. But Stak tells us instead to use the free-space 3D

propagator for v. Of course the theorem is true for *any* functions u and v you want! So this is taking a slightly different path.

Stak says the ODE is true in ALL of 3D space. On the other hand, he does mention a boundary σ. In my BV doc I set g equal to the Green's Function which vanishes on that boundary, and that then allows me to knock out one of the surface integral terms. I guess if we instead use the free space Green's Function g = C_3, then everything still "goes through", but you cannot then knock out one of the surface terms so you will end up with two surface terms. In my BV doc, these two facts will still be true (this time g = C_3)

$(\partial_t^2 - \nabla^2)u(t,\mathbf{x}) = q(t,\mathbf{x})$ // from 7.109 line 1

$(\partial_t^2 - \nabla^2)\, g(x_0|x) = \delta(x-x_0)\delta(t-t_0)$ // from 7.113 line 1

Repeat: This problem represents a deviation from the plan outlined in my support doc "BC problems....". In that doc, I set v = g where g is the g which vanishes on boundary σ, and this knocks out one of the surface integrals. Alternatively we might use Neumann or Radiative BC on σ for g, and I have done all that. But in THIS problem, we are supposed to use v = g_0 which is the free space propagator. If I do that, everything "goes through" as before, but we have to maintain both surface terms. But I did that anyway in my doc, so if I replace t=0 with t=τ << 0 everywhere, I get this result:

$$\int_\tau^T dt \int_{\mathbf{R_n}} d^n x \; g(x_0|x)\, q(t,\mathbf{x}) \qquad - u(t_0,\mathbf{x}_0)$$

$$= \int d^n x \; [\; \{g(x_0,t_0|x,T)\partial_t u(T,\mathbf{x}) - u(T,\mathbf{x})\partial_t g(x_0,t_0|x,T)\}$$
$$- \{g(x_0,t_0|x,\tau)\partial_t u(\tau,\mathbf{x}) - u(\tau,\mathbf{x})\partial_t g(x_0,t_0|x,\tau)\}\;]$$
$$+ \int_\tau^T dt \int_{\sigma_n} dS_n \; [\; u(t,\mathbf{x})\partial_n g(x_0,t_0|x,t) - g(x_0,t_0|x,t)\partial_n u(t,\mathbf{x})\;] \qquad (7.7)$$

where now g = C_3 = δ/4πr as in 7.127. Specifically, letting r = $|x_0-x|$, we have

$$g(x_0|x) = g(x_0,t_0|x,t) = \delta(t_0-t-r)/4\pi r$$
$$g(x_0,t_0|x,T) = \delta(t_0-T-r)/4\pi r$$
$$g(x_0,t_0|x,\tau) = \delta(t_0-\tau-r)/4\pi r$$

[Note: I have omitted the time θ function from C_3 but that seems not to affect things below.]

Thus the above becomes

$$\int_\tau^T dt \int_{\mathbf{R_n}} d^n x \; \delta(t_0-t-r)/4\pi r\, q(t,\mathbf{x}) \qquad - u(t_0,\mathbf{x}_0)$$

$$= \int d^n x \; [\; \{\delta(t_0-T-r)/4\pi r\, \partial_t u(T,\mathbf{x}) - u(T,\mathbf{x})\partial_t\, \delta(t_0-T-r)/4\pi r\}$$
$$- \{\delta(t_0-\tau-r)/4\pi r\, \partial_t u(\tau,\mathbf{x}) - u(\tau,\mathbf{x})\partial_t\, \delta(t_0-\tau-r)/4\pi r\}\;]$$
$$+ \int_\tau^T dt \int_{\sigma_n} dS_n \; [\; u(t,\mathbf{x})\partial_n\, \delta(t_0-t-r)/4\pi r - \delta(t_0-t-r)/4\pi r\, \partial_n u(t,\mathbf{x})\;] \qquad (7.7)$$

Now let's assume that $\tau << 0$ and $T >> 0$ so we are guaranteed to get a "hit" on the δ in the first term, so it becomes then

$$\int_{\tau}^{T} dt \int_{\mathbf{R_n}} d^n x \ \delta(t_0 - t - r)/4\pi r * q(t,\mathbf{x}) \quad = \quad \int_{\mathbf{R_n}} d^n x /4\pi r * q(t_0 - r)] \ = \ \int_{\mathbf{R}} dx \ q(t_0 - r)/4\pi r$$

and we then have obtained the correct first term on the RHS of 7.151. Changing signs of all terms, we have at this point:

$$u(t_0,\mathbf{x}_0) \ = \ \int_{\mathbf{R}} dx \ q(t_0 - r)/4\pi r$$

$$= - \int d^n x \ [\ \{ \ \delta(t_0 - T - r)/4\pi r \ \partial_t u(T,\mathbf{x}) - u(T,\mathbf{x}) \partial_t \ \delta(t_0 - T - r)/4\pi r \ \}$$

$$- \{ \ \delta(t_0 - \tau - r)/4\pi r \ \partial_t u(\tau,\mathbf{x}) - u(\tau,\mathbf{x}) \partial_t \ \delta(t_0 - \tau - r)/4\pi r \ \} \ ]$$

$$- \int_{\tau}^{T} dt \int_{\sigma_n} dS_n \ [\ u(t,\mathbf{x}) \partial_n \ \delta(t_0 - t - r)/4\pi r - \delta(t_0 - t - r)/4\pi r \ \partial_n u(t,\mathbf{x}) \ ]$$

and we now have to reduce our 6 RHS terms to the three Stak shows in 7.151. Note the sloppy notation on the various derivatives in these 6 terms, but I know what they mean.

Now here is a sloppy argument as well. In the first four terms on the RHS above, the time δ includes either T or τ, and if we say $T >>> 0$ and $\tau <<< 0$, perhaps we can say that none of these delta functions gets a hit, so then we would have only the surface integral terms, and thus

$$u(t_0,\mathbf{x}_0) \ = \ \int_{\mathbf{R}} dx \ q(t_0 - r)/4\pi r$$

$$- \int_{\tau}^{T} dt \int_{\sigma_n} dS_n \ [\ u(t,\mathbf{x}) \partial_n \ \{\delta(t_0 - t - r)/4\pi r\} - \delta(t_0 - t - r)/4\pi r \ \partial_n u(t,\mathbf{x}) \ ]$$

Let's now look at the last term only, since it is the easiest. It gets a hit when $t = t_0 - r$ so it gives

$$+ (1/4\pi) \int_{\sigma_n} dS_n \ \partial_n u(t_0 - r,\mathbf{x}) / r$$

which matches the 2nd term on the RHS of 7.151. So we now have

$$u(t_0,\mathbf{x}_0) \ = \ \int_{\mathbf{R}} dx \ q(t_0 - r)/4\pi r \ + \quad (1/4\pi) \int_{\sigma_n} dS_n \ \partial_n u(t_0 - r,\mathbf{x}) / r$$

$$- \int_{\tau}^{T} dt \int_{\sigma_n} dS_n \ [\ u(t,\mathbf{x}) \partial_n \{\delta(t_0 - t - r)/4\pi r\}$$

The last term here can be written

$$- (1/4\pi) \int_{\tau}^{T} dt \int_{\sigma_n} dS_n \ [\ u(t,\mathbf{x}) \ \mathbf{n} \bullet \nabla \{\delta(t_0 - t - r)/r\} \qquad (*)$$

and then according to $\nabla(\varphi v) = \varphi\nabla\psi + \psi\nabla\varphi$ we get

$$\nabla\{\delta(t_0\text{-}t - r)/r\} = \delta(t_0\text{-}t - r)\,\nabla(1/r) + (1/r)\,\nabla\,\delta(t_0\text{-}t - r)$$

Now

$$\nabla(1/r) = \nabla_{\mathbf{x}}(1/|x\text{-}x_0|) = \nabla_{\mathbf{y}}(1/|y|) \qquad y = x\text{-}x_0 \qquad |y| = r$$

Go to spherical coordinates in this y system and then continue

$$= \nabla(1/r) = -\hat{\mathbf{r}}/r^2$$

Similarly we have

$$\nabla\,\delta(t_0\text{-}t - r) = \nabla_{\mathbf{x}}\,\delta(t_0\text{-}t - |x\text{-}x_0|) = \nabla_{\mathbf{y}}\,\delta(t_0\text{-}t - |y|) = \nabla_{3D}\delta(t_0\text{-}t\text{-}r)$$

$$= \partial_{\mathbf{r}}\,[\delta(t_0\text{-}t\text{-}r)])\,\hat{\mathbf{r}} = \partial_{\mathbf{t}}\,[\,\delta(t_0\text{-}t\text{-}r)\,\hat{\mathbf{r}}]$$

So we have now shown that

$$\nabla\{\delta(t_0\text{-}t - r)/r\} = \delta(t_0\text{-}t - r)\,\nabla(1/r) + (1/r)\,\nabla\,\delta(t_0\text{-}t - r)$$

$$= \delta(t_0\text{-}t - r)\,[-\hat{\mathbf{r}}/r^2] + (1/r)\,\partial_{\mathbf{t}}\,[\,\delta(t_0\text{-}t\text{-}r)\,\hat{\mathbf{r}}]$$

Then our surface integrals in (*) have become

$$-(1/4\pi)\int_{\tau}^{T} dt \int_{\sigma_{\mathbf{n}}} dS_{\mathbf{n}}\,[\,u(t,\mathbf{x})\,\mathbf{n}\bullet\nabla\{\delta(t_0\text{-}t - r)/r\}$$

$$= -(1/4\pi)\int_{\tau}^{T} dt \int_{\sigma_{\mathbf{n}}} dS_{\mathbf{n}}\,[\,u(t,\mathbf{x})\,\mathbf{n}\bullet\delta(t_0\text{-}t - r)\,[-\hat{\mathbf{r}}/r^2]$$

$$-(1/4\pi)\int_{\tau}^{T} dt \int_{\sigma_{\mathbf{n}}} dS_{\mathbf{n}}\,[\,u(t,\mathbf{x})\,\mathbf{n}\bullet\,(1/r)\,\partial_{\mathbf{t}}\,[\,\delta(t_0\text{-}t\text{-}r)\,\hat{\mathbf{r}}]$$

What do we do with $\mathbf{n}\bullet\hat{\mathbf{r}} = n_{\mathbf{r}}$ at this point? Is this connected with $\partial_{\mathbf{n}}r$? Well

$$\partial_{\mathbf{n}}r = \mathbf{n}\bullet\nabla(r) = \mathbf{n}\bullet\hat{\mathbf{r}}$$

so they are in fact the same and then our two terms above become

$$= +(1/4\pi)\int_{\tau}^{T} dt \int_{\sigma_{\mathbf{n}}} dS_{\mathbf{n}}\,[\,u(t,\mathbf{x})\,\partial_{\mathbf{n}}r\,(1/r^2)\,\delta(t_0\text{-}t - r)$$

$$-(1/4\pi)\int_{\tau}^{T} dt \int_{\sigma_{\mathbf{n}}} dS_{\mathbf{n}}\,[\,u(t,\mathbf{x})\,\partial_{\mathbf{n}}r\,(1/r)\,\partial_{\mathbf{t}}\,[\,\delta(t_0\text{-}t\text{-}r)\,]$$

The first term gets a hit at $t = t_0 - r$ and thus gives

$$+ (1/4\pi)\, u(t_0-r,\mathbf{x})\, \partial_n r / r^2$$

and this matches the 3rd term on the RHS of 7.151. Now consider the residual term

$$- (1/4\pi) \int_\tau^T dt \int_{\sigma_n} dS_n \, [\, u(t,\mathbf{x})\, \partial_t \, [\, \delta(t_0-t-r)\,]$$

$$= - (1/4\pi) \int_{\sigma_n} dS_n \; \partial_n r\, (1/r) \int_\tau^T dt \, [\, u(t,\mathbf{x})\, \partial_t \, [\, \delta(t_0-t-r)\,]$$

If we do a parts on the time integration, we argue as we did for the other thrown out terms that we get no hit on the δ, so parts = 0 and we throw over the time derivative to get

$$= + (1/4\pi) \int_{\sigma_n} dS_n \; \partial_n r\, (1/r) \int_\tau^T dt \, [\partial_t u(t,\mathbf{x}) \, [\, \delta(t_0-t-r)\,]$$

$$= + (1/4\pi) \int_{\sigma_n} dS_n \; \partial_n r\, (1/r)\, [\partial_t u(t,\mathbf{x})]^{\,t=t0-r}$$

and lo and behold, this is the required last term. So to make all this work, we have to assume that T and τ are both very large compared to t_0 and r. Our final result is then:

$$u(t_0,\mathbf{x}_0) = \int_{\mathbf{R}} dx\; q(t_0-r)/4\pi r + (1/4\pi) \int_{\sigma_n} dS_n\, \partial_n u(t_0-r,\mathbf{x}) / r$$

$$+ (1/4\pi)\, u(t_0-r,\mathbf{x})\, \partial_n r / r^2 + (1/4\pi) \int_{\sigma_n} dS_n \; \partial_n r\, (1/r)\, [\partial_t u(t,\mathbf{x})]^{\,t=t0-r}$$

or

$$u(t_0,\mathbf{x}_0) = \int_{\mathbf{R}} dx\; q(t_0-r,\, x)/4\pi r + (1/4\pi) \int_{\sigma} dS$$

$$\{\, r^{-1}\; \partial_n u(t_0-r,\mathbf{x}) + r^{-2}\, (\partial_n r)\; u(t_0-r,\mathbf{x}) + r^{-1}\, \partial_n r\, [\partial_t u(t,\mathbf{x})]^{\,t=t0-r} \,\}$$

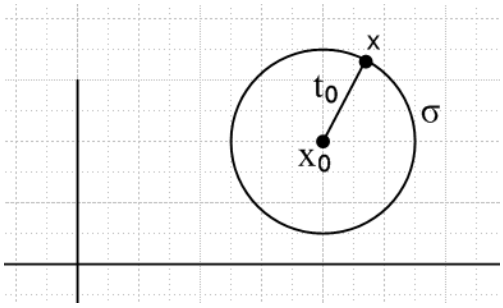
which agrees with 7.151. This is called **Kirchhoff's Formula**. You have to know $\partial_n u$, u, and $\partial_t u$ on the boundary and then retard all three of them. This hardly seems like a useful formula since you are unlikely to have all this BC data, but let's see what the next examples can drum up. Notice that T and τ do not appear in the Formula. A web paper takes note that this Formula provides the solution to a problem which has no time IC's! That is quite different from the class of problems we have generally been looking at.

Exercise 7.22. Kirchhoff's Formula Application

We are supposed to derive p 255 7.129 from the K formula. Solution 7.129 was for a system that HAS initial conditions, but has no boundary σ, whereas the K formula does not involve IC's, and *does* have a boundary σ which, however, is arbitrary.

Plan A: Suppose we assume that q(t,x) and u(t,x) are both 0 for $t < 0$, and that u(t,x) has the initial conditions shown in 7.128. Then in our first K term we must have $t_0 > r$, which is to say $r < t_0$, and then our first K term becomes exactly the first term in our K formula. Fine.

Now, since boundary σ in the K formula is arbitrary, let's choose σ to be a sphere of radius t_0 with center at x_0, since that is the sphere that appears in text top of page 255.



Now again using $\mathbf{y} = \mathbf{r} = \mathbf{x}\text{-}\mathbf{x}_0$ we have $\mathbf{n} = \hat{\mathbf{r}}$ so $\partial_{\mathbf{n}} r = \mathbf{n} \bullet \hat{\mathbf{r}} = 1$ ($\mathbf{n}$ is always a unit vector here). Then our K formula three surface terms are: (here we set $\partial_{\mathbf{n}} r = 1$)

$$(1/4\pi) \int_{\sigma} dS \{ \{ r^{-1} \partial_{\mathbf{n}} u(t_0\text{-}r,\mathbf{x}) + r^{-2} u(t_0\text{-}r,\mathbf{x}) + r^{-1} [\partial_t u(t,\mathbf{x})]^{t=t0-r} \}$$

and here we set $r = t_0$ and $\partial_{\mathbf{n}} = \partial_{\mathbf{r}}$

$$= (1/4\pi) \int_{\sigma} dS \{ \{ t_0^{-1} \partial_{\mathbf{r}} u(0,\mathbf{x}) + t_0^{-2} u(0,\mathbf{x}) + t_0^{-1} [\partial_t u(t,\mathbf{x})]^{t=0} \}$$

Now we can install our initial conditions from 7.128

$$= (1/4\pi) \int_{\sigma} dS \{ \{ t_0^{-1} \partial_{\mathbf{r}} f_1(x) + t_0^{-2} f_1(x) + t_0^{-1} f_2(x) \}$$

Now we can write $dS = r^2 d\Omega = t_0^2 \, d\Omega$ and then we have

$$= (1/4\pi) \int_{\sigma} d\Omega \{ t_0 \partial_{\mathbf{r}} f_1(x) + f_1(x) + t_0 f_2(x) \}$$

But again, $r = t_0$ so $\partial_{\mathbf{r}} = \partial_{t0}$ and we can combine the first two terms

$$t_0 \partial_{t0} f_1(x) + f_1(x) = \partial_{t0} [t_0 f_1(x)]$$

where be careful to note that $\mathbf{x} = \mathbf{x}_0 + \mathbf{r} = \mathbf{x}_0 + t_0 \hat{\mathbf{r}}$ and thus x is a function of t_0 in this little setup. Thus, we cannot just say $\partial_{t0} f_1(x) = 0$, So we then have

$$= (1/4\pi) \int_{\sigma} d\Omega \{ \partial_{t0} [t_0 f_1(x)] + t_0 f_2(x) \}$$

$$= (1/4\pi) \{ \partial_{t_0} [t_0 \int_\sigma d\Omega\, f_1(x)] + t_0 \int_\sigma d\Omega\, f_2(x) \}$$

where $d\Omega$ refers to the direction of $\hat{\mathbf{r}}$ and can be moved through t_0 stuff. Continuing,

$$= \{ \partial_{t_0} [t_0 (1/4\pi) \int_\sigma d\Omega\, f_1(x)] + t_0 (1/4\pi) \int_\sigma d\Omega\, f_2(x) \}$$

$$= \{ \partial_{t_0} [t_0 f_1^*] + t_0 f_2^* \}$$

where we have

$$f_i^* \equiv (1/4\pi) \int_\sigma d\Omega\, f_i(x)$$

Now our little sphere is centered at x_0 and has radius t_0, so σ is a function of these two parameters, so f_i^* are also functions of these parameters. Thus we write $f_i^*(x_0,t_0)$ and we have then shown 7.129 as requested. This is certainly a strange formula since the sphere on which you average varies with t_0 which is the time of the observation point in spacetime.

Exercise 7.23. A 2D example.

The 2D free propagator is C_2 as shown in 7.130 which agrees with former 5.151 computation. That is to say,

$$C_2(x,t) = (1/2\pi)\, H(t-r) / \sqrt{t^2-r^2}$$

where r is a spherical coordinate of point x and the source is at the origin at t=0. Replace t by $t-t_0$ to get

$$C_2(x, t-t_0) = (1/2\pi)\, H(t-t_0-r) / \sqrt{(t-t_0)^2-r^2}$$

and then shift origin to x_0 to get

$$C_2(x, t-t_0;\ x_0, 0) = (1/2\pi)\, H(t-t_0-r) / \sqrt{(t-t_0)^2-r^2} \qquad \text{where now } r = |x-x_0|$$

As usual we have the 7.9 idea page 198 which says add same t to both sides, so add t_0 to get

$$C_2(\mathbf{x}, t;\ \mathbf{x}_0, t_0) = (1/2\pi)\, H(t-t_0-r) / \sqrt{(t-t_0)^2-r^2} \qquad \text{where now } r = |\mathbf{x}-\mathbf{x}_0|$$

Now finally swap the labels to get

$$C_2(\mathbf{x}_0, t_0;\ \mathbf{x}, t) = (1/2\pi)\, H(t_0-t-r) / \sqrt{(t-t_0)^2-r^2} \qquad \text{where now } r = |\mathbf{x}-\mathbf{x}_0|$$

and we finally have our desired form for g. Note that $t_0 > t+r$ or $t < t_0 - r$. This says you only get influence at some location x_0 away from x up to this time $t = t_0 - r$. There is a wavefront and a wake.

Now 7.109 is a statement of the "full problem" with q, f_1, f_2 and h on σ, but we are supposed to take only the case where R = all 2D space so no σ. But I think we know that the solution to 7.109 is our famous 7.116 with no σ integral then, which is this (from BV)

$$u(t_0,\mathbf{x}_0) = \int_0^{t_0} dt \int_{\mathbf{R_n}} d^n x \; g(x_0,t_0|x,t)\, q(t,\mathbf{x})$$
$$+ \int_{\mathbf{R_n}} d^n x \; g(x_0,t_0|x,0)\, f_2(x) + \partial_{t_0} \{ \int_{\mathbf{R_n}} d^n x \, f_1(x)\, g(x_0,t_0|x,0) \}$$
$$- \int_0^{t_0} dt \int_{\sigma_n} dS_n \, [\, h(t,\mathbf{x}) \partial_{\mathbf{nx}} g(x_0|x) - g(x_0|x) \partial_{\mathbf{n}} h(t,\mathbf{x}) \,] \qquad (7.116,\ \text{full})$$

$$u(t_0,\mathbf{x}_0) = \int_0^{t_0} dt \int_{\mathbf{R_n}} d^n x \; g(x_0,t_0|x,t)\, q(t,\mathbf{x})$$
$$+ \int_{\mathbf{R_n}} d^n x \; g(x_0,t_0|x,0)\, f_2(x) + \partial_{t_0} \{ \int_{\mathbf{R_n}} d^n x \, f_1(x)\, g(x_0,t_0|x,0) \} \qquad (7.116,\ \text{no } \sigma)$$

So our task is to insert $g = C_2$ and try to do any integrals we can do, given that R_n is all of 2D space. It is not obvious to me that I can do any integrals, it might just be sub in and stop.

The first term on RHS becomes

$$(1/2\pi) \int_0^{t_0} dt \int_{\mathbf{R_n}} d^n x \; H(t_0-t-r) / \sqrt{(t-t_0)^2 - r^2} \; q(t,\mathbf{x})$$

The H says $t_0 > t+r$ or $r < t_0 - t$ so we could write this as

$$= (1/2\pi) \int_0^{t_0} dt \int_{\mathbf{r<t_0-t}} d^2 x \; / \sqrt{(t-t_0)^2 - r^2} \; q(t,\mathbf{x})$$

But since we know nothing about the distribution of q(t,x), there is nothing more to do. For each value of t in the range shown, we integrate in space over a disk of radius $r = t-t_0$. Another way to write this term is to think of $t < t_0 - r$ and then we get instead

$$(1/2\pi) \int_{\mathbf{R_n}} d^n x \int_0^{t_0-r} dt \; / \sqrt{(t-t_0)^2 - r^2} \; q(t,\mathbf{x}) = (1/2\pi) \int_{\mathbf{r<t_0}} d^2 x \int_0^{t_0-r} dt \; / \sqrt{(t-t_0)^2 - r^2} \; q(t,\mathbf{x})$$

The second term on RHS becomes

$$(1/2\pi) \int_{\mathbf{R_n}} d^n x \; H(t_0-r) / \sqrt{t_0^2 - r^2} \; f_2(x) = (1/2\pi) \int_{\mathbf{r<t_0}} d^2 x \, f_2(x) / \sqrt{t_0^2 - r^2}$$

and this is another disk integration over disk of radius t_0 which has a boost at the edge from last factor.

The third term on RHS becomes just

$$\partial_{t0}\ \{\ \int_{R_n} d^n x\ f_1(x)\ H(t_0-r)/\sqrt{t_0^2-r^2}\ )\ \}\ =\ (1/2\pi)\ \partial_{t0}\ \{\ \int_{r<t_0} d^2x\ f_1(x)/\sqrt{t_0^2-r^2}\ \}$$

So all I can do here is write out all these terms and say that is the result:

$$2\pi\ u(t_0,\mathbf{x}_0)\ =\ \int_{r<t_0} d^2x\ \int_0^{t_0-r} dt\ q(t,\mathbf{x})/\sqrt{(t-t_0)^2-r^2}$$
$$+\ \int_{r<t_0} d^2x\ f_2(x)/\sqrt{t_0^2-r^2}\ +\ \partial_{t0}\ \{\ \int_{r<t_0} d^2x\ f_1(x)/\sqrt{t_0^2-r^2}\ \}$$

I might add some comments. For n = 2, there is a "wake" in C_2 and that is why the two initial condition terms are integrals over disks and not just circles. Similarly, for given r, the effect of q(x,t) is limited such that $t_0 - r > 0$ which says $t_0 > r$.

Exercise 7.24. Infinite String with Air Resistance using 5.169

I presume air resistance falls into our velocity γ model, so we are talking the 1D damped wave equation. Back in Chapter 5 we considered this equation in nD and got C_n , and for n=1 we found this

$$C_1(r,t;\gamma) = (1/2)e^{-\gamma t} I_0(\gamma\sqrt{t^2-r^2}\ )\ H(t-r)$$

and again we have a wavefront at r = t and a wake behind it. If $\gamma\to 0$, $I_0(0) = 1$ according to Jackson page 73-5, so get (1/2) H(t-r) which is correct.

Now we are asked to solve the "general initial value problem". Does that mean q is present or not? I guess "general" means it is present, and this q problem was NOT solved in the text p 257. We only did the 7.131 problem with no q, but we know how to superpose the 7.132 problem. But with q present this is not going to work, and this then becomes a long problem.

Another approach would be to process Green's Theorem with this γ term present as in our BV document (and then the N dim doc), something I could do and which I don't think Stak did. This takes us way back to finding the appropriate current J_μ, something we never did back near page 41. Or maybe we do this with the telegrapher's equation. But sticking with the damped wave, page 41 5.81 would have to have this added to both sides of p 41 A:

$$v\ [\ 2\gamma\partial_t u] - u\ [\ 2\gamma\partial_t v]\ =\ 2\gamma\ [\ v\partial_t u - u\partial_t v]$$

But the next step does not work. We try to write this as $\nabla\bullet[\ \hat{t}\ (uv-vu)\ ]\ = \partial_t(uv-vu)\ = \partial_t 0 = 0$, so we would then have to add $J_\mu = 0$ to the existing current?? Or what do we do? I fear this would take me off on another very long side trip. The fact is that we just don't have a very complete theory for the damped wave equation and this Exercise is asking us to form such a theory.

So I declare this problem to be "too hard" to do right now because I don't want a 2 week sideshow at this time.

Exercise 7.25. Method of Descent Application

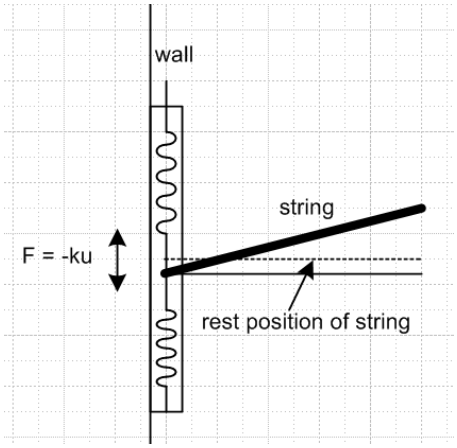
In 5.168 we have C_3 which is the damped wave equation free propagator with $\gamma = 1$, and it involves the function $I_0'(\sqrt{t^2-r^2}\ )$. Our task here is to first superpose a line of sources along the z axis, say, and from this superposition obtain a formula for C_2 (Hadamard's Method of Descent). Then we consider a whole sheet of sources and we are supposed to find C_1. We know C_1 from 5.167, and we could get C_2 using methods of Chapter 5 (iteration idea shown top page 58 in 5.131). We know from p 68 C how to reinstall $\gamma \neq 1$ into C_n. So I think I could mechanically do this problem, but I am going to skip it. Unlike the previous problem, I know HOW to do this problem and it is just a matter of doing it. In the previous problem I am facing uncharted waters.

Exercise 7.26. Redoing the end-wiggle half-string problem 3 alternative ways.

System 7.124 is for the half string with $f_1 = f_2 = q = 0$ and we just have an h driving the left end x=0. The text solved this problem on page 251 by using the surviving σ term in general formula 7.116 for the wave equation and the solution is the wiggle solution for which I did a Maple animation. In this exercise, we are supposed to solve this same problem by three alternate methods. Those methods are time Laplace, space Fourier Sine, and a difference function method. The first two ideas are similar no doubt to the general idea of Exercise 7.16. So this is really 3 problems in one, each would take me at least an hour. So this goes into the rainy day bin. Right now I am just trying to understand what the problems are!

Exercise 7.27. Half-string with elastic support at x = 0.

I explain this situation in my meta notes Comment around page 20 of those notes, and here is my picture



The idea is that the perpendicular (vertical in this picture) downward force applied to the string by the spring mount is on the one hand $F = \partial_x u$ (> 0 for my picture) as explained in those notes. On the other hand, the Hooke's Law says the downward force applied to the string is F = -ku. For my picture, u < 0 and again we have F > 0. Setting F = F we find that $\partial_x u = -ku$ or $\partial_x u + ku = 0$. But Stak shows $k = -\theta$, and I think this might be an errata. But the new edition does not show the sign changed.

Now in the heat world, this kind of boundary condition is just the Radiative BC, so this is nothing really "new" in a conceptual sense.

So we are supposed to find the Causal Green's Function for the half string with this BC at x = 0. A good problem I think and since it is really something new for the half string.

On page 209 Stak solved this same radiative BC problem in the heat world (the half rod), and he noted there that the image method did not work so he rolled his own little method which I will try to mimic here. The first step is to set $v = \partial_x g - \theta g$ and then write the system for v. I can at least do that much. I will copy and paste from the p 209 solution and edit as needed: In these next 3 lines, I just replaced ∂_t by ∂_t^2 everywhere:

$v = (\partial_x g - \theta g)$

$\partial_t^2 v - \partial_x^2 v = \partial_t^2 (\partial_x g - \theta g) - \partial_x^2(\partial_x g - \theta g) = \partial_t^2\partial_x g - \theta\partial_t^2 g - \partial_x^3 g + \theta\partial_x^2 g$

$= \partial_x [\partial_t^2 g - \partial_x^2 g] - \theta[\partial_t^2 g - \partial_x^2 g] = (\partial_x - \theta) [\partial_t^2 g - \partial_x^2 g] = (\partial_x - \theta) [0] = 0$

Now the wave system for g has IC's $g(t=0) = 0$ and $\partial_t g(t=0) = \delta(x-x_0)$, and of course the radiative BC is going to be $\partial_x g + \theta g = 0$ at x=0. So we want to translate all this stuff from g to v.

$v(t=0) = (\partial_x g(t=0) - \theta g(t=0)) = 0 - 0$ since $g(t=0) = 0$ for all x IC1

$\partial_t v(t=0) = \partial_x\partial_t g(t=0) - \theta\, \partial_t g(t=0) = \partial_x\, \delta(x-x_0) - \theta\, \delta(x-x_0)$ IC2

$v(x=0) = \partial_x g - \theta g = 0$ BC

So things are fairly similar to the page 209 situation. We then have this system for Green's Fctn v:

$(\partial_t^2 - \partial_x^2)v = 0 \qquad v(t=0) = 0 \qquad \partial_t v(t=0) = \partial_x\, \delta(x-x_0) - \theta\, \delta(x-x_0) \qquad v(x=0) = 0$

This v problem is one for a semi-infinite string with a peculiar f(x) IC function as in 7.111. Obviously due to this $\partial_t v(t=0) \neq \delta(x-x_0)$, we cannot directly identify v with $C_{half} = g$ for the half string. But C_{half} is the correct free-space propagator for this system, and 7.123 shows this as the difference between two C_1's where C_1 is the full string propagator.

Comment: I often mention in the past notes that a Green's function can be Dirichlet, Neumann or Radiative based on the condition for g on σ. But in all those cases, I had in mind $\partial_t g(t=0) = \delta(x-x_0)$ in the wave world. So this system for v really is "something different". We are Dirichlet in terms of σ, but we are Radiative in terms of the impulse which makes up $\partial_t v(t=0)$.

So, this v problem is the problem stated in 7.111 (p245) with $f(x) = \partial_x\, \delta(x-x_0) - \theta\, \delta(x-x_0)$. On page 251 Stak finds the Causal Green's for the half-string Green's system which corresponds with 7.111 and the result is 7.123. We would then insert this Green's g into 7.116 (which is now written for v instead of u) , and we would set h = 0 so no last term, and we would set $f_1 = 0$ since v(t=0,x) = 0, and f_2 = our f(x) above. So we would have this result: (there is of course no q either)

$v(x_0,t_0) = \int dx\, g_{7.123}\, [\partial_x\, \delta(x-x_0) - \theta\, \delta(x-x_0)]$

Again, this is very similar to what happened in the page 209 effort, and this g is the difference of two C_1 terms as shown in 7.123, and C_1 is the full-string propagator in 7.119. So we do parts twice in the above integral just as we did on page 209 and we end up with four terms for $v(x_0,t_0)$. Then once we have v, we could probably find g by the same method shown page 210 E, and eventually we would get an answer. Once again, this answer would be the Causal Green's function for a half-string which is connected "elastically" at x = 0. If you "strike" this string at some location x_0 with your unit knife edge, the response of the half-string is what this Causal Green's tells you! This radiative BC is much harder to handle than either the Dirichlet or Neumann one because you cannot add a simple image impulse.

I think I could do this problem in full, but I would do it in a separate doc. Since Stak does not tell us the answer, I would not know whether I got my answer right. Maybe this problem appears with an answer in Stak's later combined book. // But I have that as a djvu and I don't see it mentioned! It would take a while to find a statement of this result! Marshall mentions it in his Green's Function pdf, and states a way to do it, but does not state the answer.

Exercise 7.28. Solve the String Ring problem.

Recall that on page 211 we did "the wire ring" in heat world with BC's like those stated here. We took the ring and unwrapped it to get an infinite wire with two sets of image sources. The ring then corresponded to the piece of wire $(0,\ell)$. We can do the same thing here, but each source has a response that is the infinite string wave propagator C_1 instead of the infinite string heat propagator C_1. This will produce a result that we will then want to transform into another sum by the Poisson sum formula. Recall that we also did this "finite rod" problem using the eigenfunction method on page 216, so I presume we could also use that method here with the "finite string" in Wave World.

Exercise 7.29. The internally damped string problem.

Another heavy duty problem. Up to now we have assumed damping was a $\gamma\, \partial_t u$ velocity type term as might apply with external air friction. But here Stak attempts to model the notion of "internal damping" of an infinite string by considering as a string a series of equal masses connected by equal springs with frictional dashpots of some sort. Sadly Stak does not have a picture, so we don't quite know what he means. Perhaps these dampers are velocity dampers one of which parallels each spring. The distinction here would then be that these dampers are affected by displacement along the string, not displacement perp to the string. Whatever he has in mind, if we take a continuum limit somehow, he claims we end up with a PDE which is the wave equation with the addition of a new and ugly looking $\partial_t \partial_x^2 u$ term. So this is an entirely different L operator, neither heat nor wave nor damped wave as discussed in the text. Our problem is first to derive this PDE, and then to find the normal modes of the PDE, presumably in a fashion similar to what the text did with the externally damped wave equation (with only a striking function f(x) as in 7.131). This could be a tough problem, perhaps you have to come up with some intermediate step like the telegraphy equation. Or maybe you do a time Laplace transform. Probably various ways to handle it. I understand the statement of this problem except for the picture, but I don't want to undertake doing a solution.

Exercise 7.30. Solving a certain 1D driven Helmholtz ODE

The k^2 constant is λ, more or less, and we have a specific and simple driving function. On page 265 B Stak shows that this ODE has a homo solution which is the first two terms, which will be quite easy to show, but then we have to find the particular solution shown in C which we then add to the homo solution. I think this gets tied in with the Weyl stuff of Chapter 4, and he claims there is only one finite-s-norm solution relative to $|x|\to\infty$, so maybe limit-point. This is the case if λ is off the positive real axis. But if λ is on this real axis, then solutions only exist if $a = n\pi/\sqrt{\lambda}$. I guess we could interpret these as EV's for $L = -\partial_x^2 - f_a(x)$ where we then study $Lu = \lambda u$. After all this work, we see that he is concocting an example of the **limiting absorption principle**, but that was in terms of the damped wave equation. Maybe this $f_a(x)$ acts in a damping fashion here. Perhaps there is an issue of which direction you select to approach the real positive λ axis. So doing this problem would take me on a long review voyage of the Weyl world, and then I would have to get the solution, and then somehow fit this peg into the limiting absorption hole. This last part is very unobvious to me right now. The real axis solutions are only stable at the eigenvalue points, so I suppose the issue is how we approach these stable solutions in the λ plane. But I don't think there is any λ cut on the right side, so any approach ought to be OK. But maybe you only have stable solutions above these points and not below them?

OK, so I actually did 5 of Stak's 12 problems here. He probably spent 30 years getting these problems developed, I cannot do all that work in my passing rowboat. But I do think it is important to understand what all the problems are asking, and I have done that above.

At this point I have covered about 72 pages of this Chapter 7 and there are still 66 pages to go! Ouch!

Section 7.12 Helmholtz Green's Functions and Applications (265)

Up to this point in Chapter 7 we did Heat World, then Wave World, then some Wave World with damping. All that stuff involved spatial coordinate(s) x and time t, parabolic and hyperbolic.

Now suddenly we are starting over on a new topic which involves only the spatial coordinates x. One might think of this as a generalization of the "potential theory" of Chapter 6 where we just add a constant to the Laplace operator. Time is going to reappear on page 287 after we do about 23 pages of this Helmholtz Potential Theory. No doubt all this Helmholtz digression is going to be necessary in order to move ahead with our Wave World work, and I am sure the reason is that when we do time Laplace, we always end up with a Helmholtz spatial equation where s is just a parameter (replacing t). We have already done some work on the Helmholtz PDE and I guess here we are going to firm that up with much more detail.

The section opens with a list of places we have recently seen the H equation (H = Helmholtz) appear. One place for sure is when we do time Laplace on the undamped wave equation to replace t with s, as noted.

The λ parameter is given the same sign as the Laplacian, so the H equation is stated as $(\nabla^2 + \lambda)u = 0$, while the official H operator is stated as $L_\lambda = -\nabla^2 - \lambda$ and in Chap 4 language we would write $L_\lambda u = 0$. This is a standard potential theory eigenvalue problem affected by the BC's, and λ has some "spectrum" which might include discrete real positive eigenvalues λ_k for a string in 1D. We learned in Ch 4 about the "spectrum" of a 2nd order differential operator like $L = -\nabla^2$, but only in one spatial dimension! I don't think Stak has said anything to this point about the spectrum of operators L in > 1 dimension!

Something I failed to pay attention to much before is the sign of λ in the various applications. Certainly in 1D if λ is positive, our (homo H equation) solutions have the form $u = e^{ivx}$ where $v = \sqrt{\lambda}$,

since then $\partial_x^2 e^{ivx} = -v^2 e^{ivx} = -\lambda u$. So perhaps we should associate $\lambda > 0$ with oscillatory and $\lambda < 0$ as expo. This is born out in the next section. In Case 1 we consider $\lambda = -k^2 < 0$ in 3,2,1 dimensions, so $\lambda < 0$, and in 153, 155 and p 267A we see expo decay for large R. This is for the no-boundary homo H equation except at the spatial point the δ function hits. That is to say, away from the δ point, the H Green's Function is an H homo solution.

Free-space Fundamental Solutions (266)

We did all this back in Ch 5 p 53-55 for general dim=n for the H "fundy" Green's, so here we are just quoting those results. But here Stak is being very careful about the sign of λ, and in general the location of λ in the λ plane. I think we can assume there is some n-dim spectral theory that is similar to what we learned in Ch 4 for n=1, and that the positive real λ axis plays a special role in things.

Stak distinguishes three "cases", the third of which is $\lambda = 0$, which of course means regular old potential theory for our operator L. In each of the first two cases (λ *not* on pos real axis, λ *on* pos real axis), he writes the explicit fundy solutions for n = 3,2,1. In Case 1, the λ not on pos real axis case, he also considers the special case $\lambda = -k^2$ where then λ is on the neg real axis. In Case 2 he sets $\lambda = \omega^2 > 0$ and basically he takes the three Case 1 results and replaces $\sqrt{\lambda} = \omega$. The Case 1 solutions are the expo ones I commented on above, and the Case 2 solutions are all oscillatory.

For n=3 we get the very famous outgoing spherical wave $[e^{i\omega R}/4\pi R] e^{-i\omega t} = \exp[i\omega(R-t)]/4\pi R$, where we momentarily assume that our H equation is coming from time Laplace of the wave equation in 3D. I think we are missing the usual distinction of "k and ω" because we have velocity = 1.

For n=2 we have an $H_0^{(1)}$ Hankel function, but for large R we write the asymptotic form which has the general form $\sim [e^{i\omega R}/\sqrt{R}] e^{-i\omega t}$ and this is regarded as an outgoing cylindrical wave. That is to say, in a 3D problem with no variation in z , we can reduce the problem to a 2D problem. The term outgoing cylindrical wave then of course applies in the 3D sense in this case.

For n=1 we just get $\sim [e^{i\omega R}]$ which is a "plane wave" in 3D if you like in the same reduction sense noted for 2D above, and $R = |x-x_0|$ where x and x_0 are scalar numbers.

I have shown the exact constant $1/4\pi$ in the n=3 case, while the other two cases have constants I have not shown, which depend in fact on ω. These constants are determined by the unit scaling of the δ RHS in the Green's defining ODE.

Finally, if we set $\lambda \rightarrow 0$ we should recover that potential theory fundy solutions, but this only works for n=3 and fails for n = 1,2 !! But it can be done Stak claims if you compute $\partial_R E$, take the limit, and then integrate the result. In any event, the three $\lambda = 0$ limits are stated top of page 268 and they are all "old friends" from Ch 5.

Addition Theorem for Cylindrical Waves (268)

This is the first of several what I would call "expansions". I have already written docs describing how you can expand 1/R in various coordinate systems, where 1/R is $4\pi E_3$ for $\lambda = 0$. Here we are going to find fancy expansions for the Helmholtz E_2 function (n=2 only). As usual, there is group theory floating around behind the scenes (no Stak mention), and the term "addition theorem" is appropriate therefore. It will interest me greatly to see how Stak derives his expansions.

I will just describe what is happening without doing the details because I am familiar with all this stuff by now. We know the E_2 Green's Function and can write it as in 7.159 in polar coordinates r,φ, but we want to find another expansion way to write it. So we are going to use the "partial eigenfunction

expansion method" of earlier Stak. We write the homo H equation (λ is in there) as in p 268 A, and we try a separated form with separation constant μ. As with any azimuthal coordinate system with a full range for φ, we know that μ = integers and we have our angular atoms as just $e^{in\varphi}$. He uses three symbols for the separation constant, μ and ν and n, and we have $\mu = -\nu = n^2$.

Now, I think 7.161 is Bessel's equation with atomic solutions of the form $J_\nu(\sqrt{\lambda}\, r)$ and of course all other Bessel functions of this form as well, but Stak does not say that right here. So page 268 B is our "partial eigenfunction expansion" where we are able to remove the φ coordinate from the problem, and we then need to solve for the functions $a_n(r)$. From complete set in φ, p 269A is true.

Next, we write out our full Green's equation in r and φ, and notice we have a factor of 1/r on both sides, so 7.162, fine. Then we do exactly what he says, and the RHS becomes $1/2\pi\ \delta(r-r_0)$. On the LHS we use p 269 A three times so E is replaced by $a_n(r)$ and we end up with p 269 B which is a 1D Green's Equation for $a_n(r)$. This is the point of the method, to grind it down to a 1D Green's problem for which we use our old methods from volume 1. We are on $(0,\infty)$ so we select reasonable left and right side Bessel functions and obtain p 269 C. The point here is that away from $r = r_0$ equation B has these atoms, and this B homo part is the same as 7.161 where we now have to find constant A from the usual jump condition which is shown as D. Installing our candidate a_n C we get E, and then we can use the Wronskian equation B.8 from volume 1 and it turns out that A = i/4, very simple, so we know C and we end up then with our solution E written as 7.163 which is the desired expansion form! Then using the usual J_n and J_{-n} symmetry properties as stated in 7.164, he can fold over the negative part of the series and get result 7.165 which I would have written with the ε_n symbol.

So 7.163 is what our E_2 function looks like if we expand it on $e^{in\varphi}$ basis functions! As he says, throw in difference in φ and this thing is then $E_2(r,\varphi; r_0,\varphi_0; \lambda)$ where $r_<$ has the usual meaning. Very good!

Now in all of the above he assumed that λ was "not positive", but that seems pretty weak to me. In the solutions, we know everything will be just fine if $\lambda \geq 0$ since then we know what $\sqrt{\lambda}$ means. We can call upon the "principle of limiting absorption" to verify that we want the plus root, he says.

What happens if $\lambda < 0$ and we write it as $\lambda = -k^2$? We then have to pick a phase so $\sqrt{\lambda} = \pm ik$ and I imagine symmetry says either phase will do. In this case J and $H^{(1)}$ become (constants) I and K as shown Jackson p 75 and we end up with 7.166.

I feel no need to do more detail in this section, except to see if there are typos!

Riemann Surface E function defined on φ in $(-\infty,\infty)$

If we think of φ as a complex variable, our physical E_2 solution 7.159 corresponds to the strip let's say between $Re\varphi = -\pi$ and π. We know E_2 is periodic in φ as you move along the real axis certainly. [This whole section is still dealing only with n = 2.]

Now Stak wants to define a different version of E_2, which he calls E, where we replace the simple δ of φ with the sum shown in 7.167. I think if you do this, then E in any Riemann strip of width 2π can serve as E_2. He calls E an "extended" version of E_2. If we think of E then as $E(\varphi)$, a function of a complex variable φ, then I guess the strip $(-\pi,\pi)$ in the domain space gives you some range for E in the range space, for fixed r. It is not clear to me what this range looks like as φ takes all values in the strip, and that would take some work to determine. But suppose it is the entire complex E plane. Then as you move across a domain strip boundary, you would have to move to another Riemann sheet of E, so there would then be cuts in the domain along the strip boundaries. I did some of this somewhere looking at the function f(z) = cos(z), which in itself is quite complicated, then we have 7.159 as a next step to E.

So we have then branch points in the domain at $\varphi = \pm\pi, \pm 3\pi$ and so on, given our strip definitions. I think Stak has made a tiny error by saying these "singularities" are at $\varphi = 2\pi n$, because that really says the strips are like $(0,2\pi)$. I look at Google books and find that he did in fact make this goof, and here is what you see there (the corrected edition that I don't have)

fundamental solution. The name stems from the complex variable analogy of a Riemann surface having an infinite number of sheets obtained by letting φ range from $-\infty$ to ∞: each 2π interval in φ of the form $(2k-1)\pi < \varphi < (2k+1)\pi$ defining one sheet of the Riemann surface. The function E_s has a singularity on only one of these sheets (at $\varphi = 0$), whereas E has a singularity on every sheet (at $\varphi = 2k\pi$). Thus E_s satisfies

But I think he still has it wrong? No, I now think he did not make a goof, but people called him on it, so he has now clarified where his strips are. Maybe in addition to the sheet branch points there is in fact another set of singularities at the integers. Look back at 7.162 for the regular E_2 function. Since the RHS has $\delta(\varphi)$, we imagine that something strange might happen at $\varphi = 0$, but it is not obvious since we expanded on $e^{in\varphi}$. Had we instead expanded on the r atomic functions, perhaps we would see a discontinuity in the function E as a function of φ, in the sense that it might have a slope discontinuity at $\varphi = 0$ which I suppose is some kind of "singularity". After all, this is the angle at which we are placing our Green's point source (need $r = \varepsilon$ in this case).

So using definition of E as in 7.167 we can assume that we have these "discontinuities" at $\varphi = 2n\pi$, I agree. Now having gotten me to agree to that, he suddenly seeks a solution E that has ONLY a $\varphi = 0$ singularity of this nature. If we can find such a special solution, we will call it E_s. He wants E_s to have a singularity on only the principle sheet, while E has our singularity on all sheets. To get E_s to have this form, he then goes back and removes all those other δ terms and we get 7.168 for E_s which is the same as 7.162 that we had for $E = E_2$. OK, he is just doing a superposition method to find E and later in p 271 F he will add the E_s functions to get E by superposition.

I think Stak is being pretty clumsy here. He wants to say we get the same separation equations shown in 7.160,7.161 BUT we want to have the full infinite range for φ. We no longer have p 268 C being true since our new E_s is not periodic, so the separation constant is not quantized to integers n^2, and this means the separation constant is just a real number and he writes the basis functions as $e^{-in\alpha}$ as shown in p 271 B where then the coefficient function $E_s\hat{}(r,\alpha)$ is just the Fourier transform of $E_s(r,\varphi)$ in the φ variable, so φ and α are the FT conjugate pair of variables. So all he is saying is that we no longer have quantization so our S-L spectrum is all α, fine. [That is to say, discrete n is replaced by continuous $n = \alpha$.]

We then do the step which led to p 269 B and this gives us p 271 C which is the same thing where $n \to \alpha$. He has defined the Fourier projection so that the $1/2\pi$ factor does not appear on the right, so instead of an equation for $a_n(r)$, it is for $E_s\hat{}(r,\alpha)$. He is just repeating everything now. He constructs the r-space Green's Function as before, but needs $|\alpha|$ in place of n in order to get proper finite-s norm behavior at the two r endpoints. We find the jump constant as before and arrive then at p 271 D. We install this projection into expansion p 271 B and our final expansion form is then 7.169. He then folds things over as before and makes a huge typo error which was not corrected in the new edition, so I mark it as usual. I agree with p 271 E where he just puts in the difference angle.

Now in page 271 F we note that we can superpose E_s solutions to get our full E solution just because E has the δ sum as in 7.167, and this then gives us G. So we end up then with a new complicated form for

E which we can compare to 7.163 which had only the n sum, but now we have an n sum AND an integral on α. [too bad he used n for this sum, I would have maintained α=n and maybe used i .] OK, now as the last effort, Stak shows that if you apply our famous Poisson Sum Formula, this result G replicates 7.163 ! So we have come full circle. As before, we can set $\lambda = -k^2$ and get a version of the formula as in 7.171.

So this was certainly a bunbuster Stak section. I suspect he might have done this in a paper or lecture he wrote somewhere in his 30 year history, but Stak gives no references except general ones at the book end! His motivation here is doing the wedge in the next section (which I think I could do by other means perhaps).

Mental Block. The Poisson Sum Formula (see doc on that subject) says

$$\Sigma_{n=-\infty}^{\infty} f(n) = \Sigma_{n=-\infty}^{\infty} \hat{f}(2\pi n) \qquad \text{where } \hat{f}(k) = \int_{-\infty}^{\infty} d\alpha\, f(\alpha) e^{ik\alpha}$$

The block is that I cannot seem to apply this to p 272 A to get p 271 G. Suppose we define the J H product to be this K(|α|) just for compact notation. Then we have

272 A: $E(\varphi) = (i/4) \Sigma_n e^{-in\varphi} K(|n|)$

271 G: $E(\varphi) = (i/4)\Sigma_n \int d\alpha\, e^{i\alpha 2\pi n} e^{-i\alpha\varphi} K(|\alpha|)$

If these really are equal, that would say the following must be true for all φ,

$$\Sigma_n e^{-in\varphi} K(|n|) = \Sigma_n \int d\alpha\, e^{i\alpha 2\pi n} e^{-i\alpha\varphi} K(|\alpha|)$$

One approach is to assume the LHS here is $\Sigma_n f(n)$, so we would then have to identify

$$f(n) = e^{-in\varphi} K(|n|)$$

or, undoing the special integral case of the argument, we would have to identify

$$f(\alpha) = e^{-i\alpha\varphi} K(|\alpha|)$$

We could then go off and compute the FT of f(α) this way

$$\hat{f}(x) = \int d\alpha\, e^{ix\alpha} f(\alpha)$$

then the RHS of our Poisson sum thing would be this

$$\Sigma_n \hat{f}(x=2\pi n) = \Sigma_n \int d\alpha\, e^{i2\pi n\alpha} f(\alpha) = \Sigma_n \int d\alpha\, e^{i2\pi n\alpha} e^{-i\alpha\varphi} K(|\alpha|)$$

So our Poisson sum formula then says that

$$\Sigma_n\, e^{-in\varphi}\, K(|n|) \;=\; \Sigma_n \int d\alpha\, e^{i2\pi n\alpha}\, e^{-i\alpha\varphi}\, K(|\alpha|)$$

or

$$(i/4)\Sigma_n\, e^{-in\varphi}\, K(|n|) \;=\; (i/4)\Sigma_n \int d\alpha\, e^{i2\pi n\alpha}\, e^{-i\alpha\varphi}\, K(|\alpha|)$$

Now the RHS here is in fact p 271 G. And the LHS is in fact p 272 A. QED. My confusion I guess was about φ. You just have to think of it as a fixed number, like the r's.

Another Poisson Formula Example (from next wedge section)

We want to show that p 272 C is the same as 7.173. Using the same K object as above, we can say

$$E_s(\varphi_0) = (i/4) \int d\alpha\, e^{-i\alpha\varphi}\, e^{+i\alpha\varphi 0}\, K(|\alpha|)$$

Then the sum of p 272 C becomes this

$$g = \Sigma_n\, [E_s(\varphi_0 + 2n\psi) - E_s(-\varphi_0 + 2n\psi)\,]$$

$$= (i/4)\, \Sigma_n \int d\alpha\, e^{-i\alpha\varphi}\, [e^{+i\alpha(\varphi 0 + 2n\psi)} - e^{i\alpha(-\varphi 0 + 2n\psi)}]\, K(|\alpha|)$$

$$= (i/4)\, \Sigma_n \int d\alpha\, e^{-i\alpha\varphi}\, [e^{+i\alpha\varphi 0} - e^{-i\alpha\varphi 0}]\, e^{i\alpha 2n\psi}\, K(|\alpha|)$$

$$= 2i\, (i/4)\, \Sigma_n \int d\alpha\; e^{-i\alpha\varphi}\, e^{i\alpha 2n\psi}\, \sin(\alpha\varphi_0)\, K(|\alpha|)$$

$$= -\, (1/2)\, \Sigma_n \int d\alpha\; e^{-i\alpha\varphi}\, e^{i\alpha 2n\psi}\, \sin(\alpha\varphi_0)\, K(|\alpha|)$$

Now let's make a new definition of f(α) such that

$$f(\alpha) = -\, (1/2)\, e^{-i\alpha\varphi}\, \sin(\alpha\varphi_0)\, K(|\alpha|)$$

so that the above then says

$$g = \Sigma_n \int d\alpha\, e^{i\alpha 2n\psi}\; f(\alpha)$$

As before we define

$$f\hat{}(x) = \int d\alpha\, e^{ix\alpha}\, f(\alpha)$$

so that we recognize $\int d\alpha\, e^{i\alpha 2n\psi}\ f(\alpha) = f\hat{}(2n\psi)$. So we then have

$$g = \Sigma_n\ f\hat{}(2n\psi)$$

At this point, we can apply this version of the Poisson sum formula (this is a separate λ of course)

$$\Sigma_n\ f(n\lambda) \ = (1/|\lambda|)\ \Sigma_n\ f\hat{}(2\pi n/\lambda)$$

We need then to have $2n\psi = 2n\pi/\lambda$ so that $\psi = \pi/\lambda$ and $\lambda = \pi/\psi$. We then have

$$\Sigma_n\ f(n\pi/\psi) \ = (\psi/\pi)\ \Sigma_n\ f\hat{}(2n\psi)$$

Thus we have shown that

$$g = \Sigma_n\ f\hat{}(2n\psi) = (\pi/\psi)\ \Sigma_n\ f(n\pi/\psi)$$

and we can then fill in $f(\alpha)$ on the RHS to get

$$g = -\ (1/2)\ (\pi/\psi)\ \Sigma_n\ e^{-in\pi\varphi/\psi}\ \sin(n\pi\varphi_0/\psi)\ K(|n|\pi/\psi)$$

The n=0 term vanishes, so we now fold the negative series over to get

$$g = -(\pi/\psi)\ (1/2)\ \Sigma_{n=1}^{\infty}\ \{\ e^{-in\pi\varphi/\psi}\ \sin(n\pi\varphi_0/\psi)\ K(n\pi/\psi)\ -\ e^{+in\pi\varphi/\psi}\ \sin(n\pi\varphi_0/\psi)\ K(n\pi/\psi)\ \}$$

$$=(-\pi/2\psi)\ )\ \Sigma_{n=1}^{\infty}\ \sin(n\pi\varphi_0/\psi)\ K(n\pi/\psi)\ \{\ e^{-in\pi\varphi/\psi}\ -\ e^{+in\pi\varphi/\psi}\}$$

$$=(\ -\pi/2\psi)\ )\ \Sigma_{n=1}^{\infty}\ \sin(n\pi\varphi_0/\psi)\ K(n\pi/\psi)\ \{\ -2i\sin(n\pi\varphi/\psi)\}$$

$$=(\ i\pi/\psi)\ )\ \Sigma_{m=1}^{\infty}\ \sin(m\pi\varphi/\psi)\ \sin(m\pi\varphi_0/\psi)\ K(m\pi/\psi)$$

where $K(\alpha)$ is the $J_\alpha\ H_\alpha^{(1)}$ product, and we have thus arrived at 7.173 !

Green's Function for a 2D wedge (272)

How do we show that the proposed image sum in p 272 C is antisymmetric about $\varphi = 0$ and about $\varphi = \psi$? Well that means that

$$g(\varphi) = -\ g(-\varphi)$$
$$g(\psi+\varphi) = -\ g(\psi-\varphi)$$

If these are in fact both true, we have that $g(0) = 0$ and $g(\psi) = 0$, which are the desired wedge boundary conditions. Now why are they true? Consider inside the sum C

$$g(\varphi) = \Sigma_n \{ E_s(\varphi \mid \varphi_0+2n\psi) - E_s(\varphi \mid -\varphi_0+2n\psi) \}$$

Then

$$g(-\varphi) = \Sigma_n \{ E_s(-\varphi \mid \varphi_0+2n\psi) - E_s(-\varphi \mid -\varphi_0+2n\psi) \}$$

$$= \Sigma_n \{ E_s(\varphi \mid -\varphi_0-2n\psi) - E_s(\varphi \mid +\varphi_0-2n\psi) \} \qquad \text{// can negate both } \varphi\text{'s in } E_s$$

$$= -\Sigma_n \{ E_s(\varphi \mid +\varphi_0-2n\psi) - E_s(\varphi \mid -\varphi_0-2n\psi) \}$$

$$= g(\varphi) \qquad \text{// since sum over all integers n}$$

The next thing says

$$g(\varphi+\psi) = \Sigma_n \{ E_s(\varphi+\psi \mid \varphi_0+2n\psi) - E_s(\varphi+\psi \mid -\varphi_0+2n\psi) \}$$

$$g(-\varphi+\psi) = \Sigma_n \{ E_s(-\varphi+\psi \mid \varphi_0+2n\psi) - E_s(-\varphi+\psi \mid -\varphi_0+2n\psi) \}$$

$$= \Sigma_n \{ E_s(-\varphi-\psi \mid \varphi_0+2n\psi - 2\psi) - E_s(-\varphi-\psi \mid -\varphi_0+2n\psi - 2\psi) \} \qquad \text{// subtract } 2\psi \text{ all}$$

$$= \Sigma_n \{ E_s(\varphi+\psi \mid -\varphi_0-2n\psi + 2\psi) - E_s(\varphi+\psi \mid +\varphi_0-2n\psi + 2\psi) \} \qquad \text{// subtract } 2\psi \text{ all}$$

$$= -\Sigma_n \{ E_s(\varphi+\psi \mid +\varphi_0-2n\psi + 2\psi) - E_s(\varphi+\psi \mid -\varphi_0-2n\psi + 2\psi) \}$$

$$= -\Sigma_{n'} \{ E_s(\varphi+\psi \mid +\varphi_0-2n'\psi) - E_s(\varphi+\psi \mid -\varphi_0-2n'\psi) \} \qquad \text{// shift n by 1 unit}$$

$$= -g(\varphi+\psi)$$

QED. So this assembly of image charges does in fact cause g to vanish on the two terminal angles of the wedge which are $\varphi = 0$ and $\varphi = \psi$.

Review of the grounded parallel plates image method. The idea is to have one real charge at some position between the plates, and then two infinite sets of image charges, no charge of which is between the plates. Using something like what I just did above, or just pairing the charges in a drawing, you see that you must have u=0 (potential in this case) on both metal plates. In Chap 6 Stak and I did this problem in 2D where you can then think of the charges as line charges in 3D, or point charges in 2D. I realize now that Stak never did this problem in 3D as I have just stated it, though I did the 3D problem in cylindrical coordinates in my coordinate systems section and also in Cartesians. The point I wish to make is this: when you consider all these charges (real plus all image charges), if you *stay away* from the charges, you certainly have a function which satisfies the homo Laplace equation. This is because the potential of each point charge is such a solution and you just use superposition. So you are now thinking not of a Green's Function with a δ driver on the RHS, you are just thinking of the Laplace equation. Since you then have a Laplace solution which has V = 0 on both plates, that solution must BE the Green's function in the region between the plates which is our region of interest where we have a point charge.

The same idea applies here. We have the messy sum of E_s "displacements" shown in p 272 C. If we *stay away* from the sources (which are at $r = r_0$ and a double infinite set of φ values), this sum of E_s functions is a solution of the homo Helmholtz equation in 2D, because each term in the sum is such a solution. But this special sum also meets the boundary conditions of 0 displacement at $\varphi = 0$ and $\varphi = \psi$ which are the edges of our wedge. In the wedge region there is only one source, the images are all outside the wedge, just as in the previous paragraph. Thus, in the wedge, this Helmholtz equation solution must BE the Green's Function! I feel Stak did not make this point clear. On page 169 Stak tells us that this image method for doing the wedge of an arbitrary angle ψ is "an ingenious method of Sommerfeld", but he does not repeat that reference here!

Summary: So, we have shown that p 272 C is the correct Green's Function for a point source located inside a 2D wedge (which wedge goes out to infinity radially). It solves the homo H equation and meets the BC's. Earlier in these notes I then showed (Poisson Sum stuff) that 272 C implies 7.173 which is a more traditional form of the solution. Stak then points out that we COULD HAVE approached this problem by using φ eigenfunctions for the wedge. We would then be doing the "partial eigenfunction method" and we would come up with a 1D Green's equation for $a_n(r)$ and we would then find that to be the J $H^{(1)}$ combination. This was the approach we took on page 269 where we had the full 2π azimuth situation where those partial eigenfunctions were $e^{in\varphi}$, but in this wedge case would be $\sin(n\pi\varphi/\psi)$. The last result 1.174 on p 273 is just the usual I K rewrite in the case $\lambda = -k^2$ and I checked the details using Jackson, see pencil top p 273.

Comments. You can think of 7.173 in this manner:

$$g(x,; x_0) = \Sigma_m [\, f_m(x,x_0;\, \lambda)\,]\ \varphi_m(x)\varphi_m^*(x_0) \qquad x = (r,\varphi) \qquad \text{Helmholtz}$$

which we can compare, say, with the form for a wave equation or Laplace Green's Function:

$$g(x,t\,;\, x_0, t_0) = \Sigma_k [\, \sin\{\sqrt{\lambda_k}\,(t-t_0)\}\,/\,\sqrt{\lambda_k}\,]\ \varphi_k(x)\varphi_k^*(x_0) \qquad \text{wave}$$

$$g(x\,;\, x_0) = \Sigma_k [\, 1/\lambda_k]\ \varphi_k(x)\varphi_k^*(x_0) \qquad \text{Laplace}$$

Be careful not to confuse λ_k, which are the eigenvalues which go with $\varphi_k(x)$, with the "parameter" λ which just sits in the H equation. The function f_m in square brackets for Helmholtz is pretty complicated, being the J $H^{(1)}$ thing. One presumes that if we take $\lambda \to 0$ correctly, we would end up with the Laplace result and the λ_k are just those values (dependent on ψ and integer k) which make u=0 on the wedge boundaries. In fact, a solution of the Laplace limit infinite wedge is given in Exercise 6.43 p 169 (which I did not do) and you see how the J H complex function becomes just powers of $r_>$ and $r_<$. These powers must be part of the $\varphi_k(x)$, and the other part is the $\sin(\varphi..)$ factor. I think one can think of this in terms of Smythian Forms in polar coordinates.

Another Form for the Free-Space Green's Function (273)

Lots of review is going to be needed here before I can move forward.

(a) Bessel ODE Review. I was confused about the form of solutions to Bessel's equation with two parameters. I cleared this up in a separate doc "Bessel Functions.doc", summarized at the top. I will leave

a few old confusing notes below that I made before writing that doc. A major issue is that the same Greek symbols are used for different things in different sections of Stakgold's massive two volumes dealing with Bessel functions! Sometimes the same symbol is used in conflicting positions. Sometimes symbols are squared, sometimes terms differ in sign, and sometimes the "official λ" eigenvalue parameter is different from what you might think. This leads for example to the difference between the KL and Hankel transforms (see doc comparing them).

We start with Bessel's ODE of Schaum page 136 where the only parameter there is n, and solutions are like $J_n(x)$. This also appears in Stak Vol I app B as B.1. The next level of complication arises when we have not just one parameter like n, but two parameters. **Our case of interest is p 268 equation 7.161** where these parameters are called λ and $\nu = -\mu = -n^2$. We know the solutions have the form

$$-(rR')' - \lambda rR - \nu r^{-1}R = 0 \qquad \nu = -\mu = -n^2 \qquad J_n(\sqrt{\lambda}\, r) = J_{\sqrt{\mu}}(\sqrt{\lambda}\, r) = J_{\pm i\sqrt{\nu}}(\sqrt{\lambda}\, r)$$

Because Stak shuffles symbols around a lot, let's change the above to read

$$-(rR')' - arR - br^{-1}R = 0 \qquad b = -c = -n^2 \qquad J_n(\sqrt{a}\, r) = J_{\sqrt{c}}(\sqrt{a}\, r) = J_{\pm i\sqrt{b}}(\sqrt{a}\, r)$$

$$-(rR')' + drR - br^{-1}R = 0 \qquad b = -c = -n^2 \qquad K_n(\sqrt{d}\, r) = K_{\sqrt{c}}(\sqrt{d}\, r) = K_{\pm i\sqrt{b}}(\sqrt{d}\, r) \quad (*)$$

where here when we replace a = -d, we get $J_n(\sqrt{-d}\, r) = J_n(\pm i\sqrt{d}\, r) \sim K_n(\sqrt{d}\, r)$ where we ignore constants and are just showing the general nature of the solution type.

(b) Bessel ODE in Vol I. Now we go look at volume 1 p 317 where we see this Bessel ODE:

$$-(xw')' + \mu xw - \lambda x^{-1}w = 0 \qquad \text{// homo}$$

$$-(xg')' + \mu xg - \lambda x^{-1}g = \delta(x-\xi) \qquad \text{// Green's}$$

and we shall use (*) above to make these replacements

(*)	p317
R	w
r	x
d	μ
b	λ

Our solutions are then going to look like $K_{\pm i\sqrt{\lambda}}(\sqrt{\mu}\, r)$.

(c) KL Transform Notes (moved). At this point I made lots of notes here concerning the KL transform, but I have moved them all to a separate doc of that name in the Transforms folder.

Question: How does this relate to the Hankel Transform which seems to apply to a similar Bessel ODE with two constants on the same interval $(0,\infty)$? For an answer, see "Comparison of KL and Hankel transforms.doc" in the transforms folder.

(d) Express the K-L transform and Derive p 273 A. Now we want to *translate* this to the language of 7.161. We take $\mu \rightarrow -\lambda = k^2$ and $\lambda \rightarrow \nu$ to get

$$-(xg')' - \lambda x g - \nu x^{-1} g = \delta(x-\xi)$$

and then our transform becomes

$$F_k(\gamma) \equiv \int_0^\infty dr\, f(r)\, K_{i\gamma}(kr)\, /r$$

$$f(r) = (2/\pi^2) \int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kr)\, F_k(\gamma) \qquad \text{which is 4.139.}$$

where Stak uses $F_k(\gamma) = \tilde{f}(\gamma)$. Now recall from above that we had

$$-(rR')' + drR - br^{-1}R = 0 \qquad b = -c = -n^2 \qquad K_n(\sqrt{d}\, r) = K_{\sqrt{c}}(\sqrt{d}\, r) = K_{\pm i\sqrt{b}}(\sqrt{d}\, r) \quad (*)$$

and from 7.161 we get $d = -\lambda = k^2$ and $b = \nu$. If we set $\nu = \gamma^2 = b$ then the above becomes

$$-(rR')' - \lambda r R - \gamma^2 r^{-1} R = 0 \qquad K_{\pm i\gamma}(kr)$$

so yes, I agree that $K_{i\gamma}(kr)$ is a solution of 7.161 if we define γ to be $\nu = \gamma^2$.

Can we show p 273 A? We are supposed to "do parts" but ignore "the parts"

$$\int dr\; K(kr)[\, -\partial_r(r\partial_r E(r)) + k^2 r E(r)]$$

$$= \int dr\; \{\; \partial_r K(kr)\, (r\partial_r E(r)) + K(kr)\, k^2 r E(r)\,\}$$

$$= \int dr\; \{r\, \partial_r K(kr)\, \partial_r E(r) + K(kr)\, k^2 r E(r)\,\}$$

$$= \int dr\; \{-\partial_r[r\, \partial_r K(kr)]\, E(r) + K(kr)\, k^2 r E(r)\,\}$$

$$= \int dr\; E(r)\, \{-\partial_r[r\, \partial_r K(kr)] + K(kr)\, k^2 r\,\}$$

Now I guess we are supposed to use 7.161 which is solved by our R = K (with $\lambda = -k^2$) to get

$$= \int dr\; E(r)\, \{\gamma^2/r * K\} = \gamma^2 \int dr\; E(r)\, K / r = \gamma^2\; \tilde{E}(r) \qquad \textbf{which is p 273 A}$$

Stak refers to what I call "the parts" as "the integrated terms" of the parts integration.

(e) Solve the ODE for $\widetilde{E}$ etc. How then do we solve the distributional ODE p 273 A. Let's just show that C works. First, what about the BV's?

$f(|\varphi|) = f(|-\varphi|)$ // so left side of B is valid with $\varphi = \pi$

$\partial_\varphi f(|\varphi|) = f'(|\varphi|)\,\text{sign}(\varphi)$ // so the right side of B is NOT valid with $\varphi = \pi$.

except in our special case C, we have $f'(|\pm\pi|) = 0$ as we see below, so B is in fact then valid, $0 = 0$. Continuing on,

$\partial_\varphi \cosh[\gamma(\pi-|\varphi|)] = \sinh[\gamma(\pi-|\varphi|)] * (-\gamma\,\text{sign}(\varphi))$ // note here that $f'(|\pm\pi|) = 0$ since $\text{sh}(0)=0$

$\partial_\varphi^2 \cosh[\gamma(\pi-|\varphi|)] = \partial_\varphi \{ \sinh[\gamma(\pi-|\varphi|)] * (-\gamma\,\text{sign}(\varphi)) \}$

$= \cosh[\gamma(\pi-|\varphi|)]\,(-\gamma\,\text{sign}(\varphi))^2 = \gamma^2 \cosh[\gamma(\pi-|\varphi|)]$

So away from $\varphi = 0$ our solution satisfies A. We then have to verify the jump condition. I think the jump difference gives $-2\gamma \sinh(\gamma\pi)$, so we have

$A\,[-2\gamma \sinh(\gamma\pi)\,] = K/a = K/(-1) \quad \Rightarrow \quad A = [2\gamma \sinh(\gamma\pi)\,]^{-1} K$

which agrees with p 273 C. This in turn quickly leads to page 274 A.

Now we have to remember that the function E is our n=2 Green's Function for free space which is the solution of 7.162 and which is also given by 7.159. If $\lambda = -k^2$ we then have

$E = (i/4)H_0^{(1)}(ik\sqrt{\dots}\,) \quad \Rightarrow$

$2\pi E = 2\pi(i/4)H_0^{(1)}(ik\sqrt{\dots}\,) = (i\pi/2) * H_0^{(1)}(ik\sqrt{\dots}\,) = K_0(ik\sqrt{\dots}\,)$ // Jackson p 75

which says that $E = K_0/(2\pi)$, and this gives us the amazing result 7.179 which is then one of those fancy addition theorem gizmos like 7.166 and like what we have for Legendre functions and spherical harmonics etc. The group theory is hiding in here somewhere! The result is *similar to* a result I have in my cylindrical 1/R doc which says

$K_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')}\,) = \Sigma_{m=0}^{\infty} \varepsilon_m I_m(k\rho_<) K_m(k\rho_>) \cos[m(\varphi-\varphi')]$ // addition theorem

but it is an "integral addition theorem" thing. I found verification of 7.179 on p 750 of GR7

6.794

1. $$\int_0^\infty K_{ix}(a)\,K_{ix}(b)\cosh[(\pi-\varphi)x]\,dx = \frac{\pi}{2} K_0\left(\sqrt{a^2+b^2-2ab\cos\varphi}\right)$$ EH II 55(42)

and indeed this does appear on page 55 of EH II as claimed. Bateman provides a different path for derivation.

(f) More on the $\widetilde{E}$ formula p 273 C, and computing E

Where is the "source" for this formula? Clearly it is at $\varphi = 0$ since we have $\delta(\varphi)$ sitting in 7.177. If we wanted to adjust this formula to represent a source at φ_0 we would replace $\varphi \to \varphi-\varphi_0$ in the usual way, and then it would read:

$$\widetilde{E}(\gamma, \varphi \mid r_0, \varphi_0) = [\, 2\gamma \sinh(\pi\gamma)\,]^{-1} \cosh[\gamma(\pi - |\varphi-\varphi_0|)]\, K_{i\gamma}(kr_0)$$

If we were to insert this into 7.176 we would get this

$$E(r,\varphi \mid r_0, \varphi_0) = (2/\pi^2) \int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kr)\, \widetilde{E}(\gamma, \varphi \mid r_0, \varphi_0)$$

$$= (2/\pi^2) \int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kr)\ \{[\, 2\gamma \sinh(\pi\gamma)\,]^{-1} \cosh[\gamma(\pi - |\varphi-\varphi_0|)]\, K_{i\gamma}(kr_0)\,\}$$

$$= (1/\pi^2) \int_0^\infty d\gamma\ \{\cosh[\gamma(\pi - |\varphi-\varphi_0|)]\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\,\}$$

$$= (1/2\pi)\, K_0(\, k\sqrt{\ldots}\,) \qquad \text{// as shown in 7.180}$$

$$= E(r,\varphi \mid r_0, \varphi_0) \qquad \text{// as expected}$$

In the knife-edge problem coming up next, the picture is on page 274 and we have our source at $\varphi_0 = \pi$, so our correct $\widetilde{E}$ function is this

$$\widetilde{E}(\gamma, \varphi \mid r_0, \varphi_0=\pi) = [\, 2\gamma \sinh(\pi\gamma)\,]^{-1} \cosh[\gamma(\pi - |\varphi-\pi|)]\, K_{i\gamma}(kr_0)$$

If we evaluate this at $\varphi = 0$ and $\varphi = 2\pi$, here is what we get

$$\widetilde{E}(\gamma, \varphi=0 \mid r_0, \varphi_0=\pi) = [\, 2\gamma \sinh(\pi\gamma)\,]^{-1}\ 1\ \ K_{i\gamma}(kr_0)$$

$$\widetilde{E}(\gamma, \varphi=2\pi \mid r_0, \varphi_0=\pi) = [\, 2\gamma \sinh(\pi\gamma)\,]^{-1}\ 1\ K_{i\gamma}(kr_0)$$

(g) Comments on the above expansion for Green's Function E

Equation p 268 A is the homo H equation in n=2 in polar coordinates r and φ. If you try to find a solution in separated form $R(r)\Phi(\varphi)$, you obtain the pair of equations shown in 7.160 and 7.161 where ν is the "separation constant".

1. [Starting with φ.] If we require Φ and Φ' to be single-valued in a problem with full azimuth, the separation constant is then forced to be $\nu = -n^2$ where n is an integer. In this case we find "atoms" of this generic form

$$\text{atom} = e^{in\varphi}\ Z_n(\sqrt{\lambda}\, r) \qquad \text{// Z = J or Y or H type Bessel function}$$

If we put δ(x-ξ) on the RHS of p 268 A, we find the Green's Function expressible in these atoms,

$$E(r,\varphi;\, r_0,0) = (i/4)\ \Sigma_n\ e^{in\varphi}\ J_n(\sqrt{\lambda}\, r_<)\ H_n^{(1)}(\sqrt{\lambda}\, r_>) \qquad \text{// as in 7.163}$$

Notice that $\nu = -n^2$ are the eigenvalues and spectrum here, λ is just a fixed parameter that is in the H equation.

2. [Starting with r.] With ν again as the eigenfunction spectrum parameter (which anywhere else we would call λ, but not here!), we note that 7.161 is of the KL form, not the Hankel form (see transforms folder, comparison of). We can interpret 7.161 as an eigenfunction problem with eigenvalue ν. We know that there is a complete set of eigenfunctions. From our Hankel/KL comparison we know that (I use κ because k will soon be occupied),

$$-(xg')' + \nu^2 xg - \lambda g/x = \delta(x-\xi) \quad \Rightarrow \quad \varphi_\kappa(x) = (\sqrt{2}/\pi)\ \sqrt{\kappa\, \mathrm{sh}(\pi\kappa)}\ K_{i\kappa}(\nu x) \qquad \kappa \equiv \sqrt{\lambda}$$

We have to carefully "translate" this to the parameters seen in 7.161 with δ added on the RHS which reads

$$-(rR')' - \lambda rR - \nu\, R/r = \delta(r-r_0)$$

$$-(xg')' - \lambda xg - \nu\, g/x = \delta(x-\xi)$$

So we take our $\varphi_k(x)$ and make these changes:

$$\nu^2 \to -\lambda \quad \Rightarrow \nu = \sqrt{-\lambda} \qquad\qquad \lambda \to \nu \ \text{ which } \Rightarrow\ \kappa \to \sqrt{\nu}$$

so we then obtain

$$\varphi_\kappa(x) = (\sqrt{2}/\pi)\ \sqrt{\kappa\, \mathrm{sh}(\pi\kappa)}\ K_{i\kappa}(\sqrt{-\lambda}\, x) \qquad \kappa \equiv \sqrt{\nu}$$

Stak then sets the Helmholtz parameter $\lambda = -k^2$ so we can write this as

$$\varphi_\kappa(x) = (\sqrt{2}/\pi)\ \sqrt{\kappa\, \mathrm{sh}(\pi\kappa)}\ K_{i\kappa}(kx) \qquad \kappa \equiv \sqrt{\nu} \quad k = \sqrt{-\lambda}$$

where now κ is the spectrum parameter taking values in (0,∞). Notice that $\nu = \kappa^2$ is not quantized, it is just a continuous parameter. We can regard these $\varphi_\kappa(x)$ as the eigenfunctions of this eigenvalue equation: {the eigenvalue is ν, not λ }

$[-(x \varphi_\kappa ')' - \lambda \varphi_\kappa] = \nu (1/x) \varphi_\kappa$ // no BC's !

Meanwhile, in the φ world we just have from 7.160 that $\Phi'' = \nu\Phi = \kappa^2\Phi$ so that $\Phi(\varphi) = e^{\kappa\varphi}$. Our atoms are then

$\text{atom} = e^{\kappa\varphi}\, \varphi_\kappa(r) \sim e^{\kappa\varphi}\, K_{i\kappa}(kr)$ $\kappa = -\infty,\infty$

and we then expect to write the solution of the 2D Helmholtz Green's function problem as

$E(r,\varphi \mid r_0,0) = \int_{-\infty}^{\infty} d\kappa\, e^{\kappa\varphi}\, K_{i\kappa}(kr)\, b(\kappa; r_0)$ b = some coefficient function

But we imagine somehow that $g = E \sim \int_{-\infty}^{\infty} d\kappa$ something(κ) $[e^{\kappa\varphi}\, \varphi_\kappa(r)]\, [e^{\kappa 0}\, \varphi_\kappa(r_0)]$ which then suggests that we expect

$$E \sim \int_{-\infty}^{\infty} d\kappa \text{ something}(\kappa) [e^{\kappa\varphi} K_{i\kappa}(kr)] [K_{i\kappa}(kr_0)]$$

Another argument is that we expect E to be symmetric somehow and thus $b(\kappa; r_0) \sim K_{i\kappa}(kr)$.

Both of these arguments then lend "support" for p 274 A where $\kappa = \gamma$. Of course I have derived that exactly in section (f) above.

3. So what is the point of this "comment"?

The point is that if you "start with" the R equation with ν as your spectrum variable [$\kappa \equiv \sqrt{\nu}$] , you find that your r-eigenfunctions are $\varphi_\kappa(r) \sim K_{i\kappa}(kr)$ with a continuous κ (or ν) spectrum, you realize that you have the KL form of the ODE in r, and it seems reasonable that the transform involving these $K_{i\kappa}(kr)$ functions is going to help you out -- the $K_{i\kappa}(kr)$ are going to appear in an expansion for the Green's Function E (see p 274 A)

On the other hand, if you "start with" the Φ equation with ν as your spectrum variable, you find that $\nu = -n^2$ gives a discrete spectrum and you find that your φ-eigenfunctions are $\varphi_n(\varphi) = e^{in\varphi}$, you have the Fourier Series form of ODE in φ, and you would transform your PDE with these functions -- the $e^{in\varphi}$ are going to appear in the expansion for the Green's Function E (see 7.163). This is what we did p 269 to get B from the H PDE. B has the Hankel ODE form so we are not surprised to see the Hankel type Green's Function appear in p 269 C!

4. You might ask: how can the ν spectrum be both discrete (φ first method) and continuous (r first method) ?

(wrong answer) I am reminded of the 1D Schrodinger equation and the E spectrum for a potential well. We have discrete states for $E < 0$, and we have continuum states for $E > 0$. In our φ first problem above, we had $\nu = -n^2$ so we had discrete spectral points for $\nu < 0$, but in the r first method we got a continuous spectrum for $\nu > 0$. But this is a red herring! In this QM problem, there really is a mixed spectrum, and there is only one dimension, there is only one 1D ODE in play.

(right answer). In the φ-first approach, the ODE in play is the φ ODE, and it has the discrete spectrum (for the problems we considered; this is true for example in the full azimuth problem like the fundy Green's function, and in a partial azimuth problem like the wedge problem including the knife-edge problem). In the r-first approach, the ODE in play is the r ODE and it has a continuous spectrum. In each case, we deal with a 1D ODE spectral problem, and then the spectral values are induced over to the other variable's function. The reason the spectrum can be both discrete and continuous is that these spectra are for *different* 1D ODE problems!!!!! We can analyze our 2D problem in terms of one or the other of these 1D ODE systems, and they happen to have different spectra! In our application here, it is the finite range $(0,\psi)$ of φ that causes that system to have a discrete spectrum, whereas it is the infinite range $(0,\infty)$ of r that causes that system to have the continuous spectrum.

(h) Solve the "very thin wedge" (Knife edge problem) .

There are a lot of steps involved here, so I put little subheadings below. Stak has 3 errors in this section which made my life more difficult, but I got it all done in the end.

We set up so that we want our "field" (as he calls it suddenly, it is the solution of the Helmholtz equation) to vanish on wedge surfaces $\varphi = 0$ and $\varphi = 2\pi$, with appropriate ε limits. If we set up $g = E + v$ as shown in 7.181, then of course we must have $v = -E$ on the boundaries. Since E satisfies the δ, we know that v is just a homo solution of the H equation and thus we get all parts of 7.182.

Now on page 273 we took the K-L transform of 7.162 and got p 273 A. Our ODE in 7.182 is exactly the same ODE but we have no δ term, so we get p 273 A with RHS = 0: this is then p 275 A. I am getting the idea now that it you transform the 2D polar coordinates H equation using the K-L transform, which in effect replaces r by γ, you get a very simple ODE in the transformed space, as p 275 A or p 273 A shows. Then by trivial inspection we can express $\tilde{v}$ as shown in p 275 B.

Now, we can apply the K-L transform to both boundary conditions in 7.182, and all this does is put a twiddle over the v and over the E. This gives us the first equalities in p 275 C and D.

*** Verify that you can just add twiddles to get the γ space BC's:**

We start with

$$v(r,\varphi=0|r_0,\pi) = -E(r,\varphi=0|r_0,\pi)$$
$$v(r,\varphi=2\pi|r_0,\pi) = -E(r,\varphi=2\pi|r_0,\pi)$$

Now what do these BC's look like in the KL γ space? We know from the KL (such as 7.178) that

$$\tilde{v}(\gamma,\varphi=0|r_0,\pi) = \int_0^\infty dr\, K_{i\gamma}(kr)\, v(\gamma,\varphi=0|r_0,\pi)/r$$

$$= -\int_0^\infty dr\, K_{i\gamma}(kr)\, E(r,\varphi=0|r_0,\pi)/r$$

$$= -\tilde{E}(\gamma,\varphi=0|r_0,\pi)$$

So this confirms that the BC's in the two spaces have the same form, you just add twiddles. In other words, this confirms the left equalities in C and D.

*** Solve for A and B**

$$A + B = - [\, 2\gamma\, sh(\gamma\pi)]^{-1} K$$

$$Ae^{-2\gamma\pi} + Be^{2\gamma\pi} = - [\, 2\gamma\, sh(\gamma\pi)]^{-1} K$$

Multiply first equation by $e^{-2\gamma\pi}$

$$A\, e^{-2\gamma\pi} + B\, e^{-2\gamma\pi} = - [\, 2\gamma\, sh(\gamma\pi)]^{-1}\, e^{-2\gamma\pi} K$$

$$Ae^{-2\gamma\pi} + Be^{2\gamma\pi} = - [\, 2\gamma\, sh(\gamma\pi)]^{-1} K$$

Subtract these last two equations:

$$B\,(e^{-2\gamma\pi} - e^{2\gamma\pi}) = - [\, 2\gamma\, sh(\gamma\pi)]^{-1} K\,(e^{-2\gamma\pi} - 1) = - [\, 2\gamma\, sh(\gamma\pi)]^{-1} K\, e^{-\gamma\pi}\,(e^{-\gamma\pi} - e^{\gamma\pi})$$

Convert to sinh functions

$$B\,[\, -2sh(2\gamma\pi)] = - [\, 2\gamma\, sh(\gamma\pi)]^{-1} K\,(e^{-2\gamma\pi} - 1) = - [\, 2\gamma\, sh(\gamma\pi)]^{-1} K\, e^{-\gamma\pi}\,[\, -2sh(\gamma\pi)]$$

$$= [\, \gamma]^{-1} K\, e^{-\gamma\pi} = K\, e^{-\gamma\pi}/\, \gamma$$

$$B = [\, K\, e^{-\gamma\pi}/\, \gamma\,] / [\, -2sh(2\gamma\pi)] = - K\, e^{-\gamma\pi} / [2\gamma\, sh(2\gamma\pi)]$$ // agrees with earlier

Now doing A↔B is the same as doing γ↔-γ, so we know that

$$A = - K\, e^{\gamma\pi} / [2\gamma\, sh(2\gamma\pi)]$$

and therefore we obtain our previous results,

$$A = - K\, e^{+\gamma\pi} / [2\gamma\, sh(2\gamma\pi)]$$
$$B = - K\, e^{-\gamma\pi} / [2\gamma\, sh(2\gamma\pi)]$$

*** Compute the right version of p 275 E for $\tilde{\nu}$**

We install these A and B above into p 275 B to get

$$\tilde{\nu}(\gamma,\varphi|r_0,\pi) = A\, e^{-\gamma\varphi} + B\, e^{+\gamma\varphi}$$

$$= -[\, e^{+\gamma\pi}\, e^{-\gamma\varphi} + e^{-\gamma\pi}\, e^{+\gamma\varphi}]\, K/[2\gamma\, sh(2\gamma\pi)]$$

$= -[\ 2\ ch[\gamma(\ \pi-\varphi)]]\ K/[2\gamma\ sh(2\gamma\pi)] \quad = \quad -\ ch[\gamma(\ \pi-\varphi)]\ K/[\ \gamma\ sh(2\gamma\pi)]$

which is to say

$\tilde{v}(\gamma,\varphi|r_0,\pi) = -\ ch[\gamma(\ \pi-\varphi)]\ K/[\ \gamma\ sh(2\gamma\pi)]$

So I have now explicitly double checked all three entries in our result above which I now quote:

$A = -\ K\ e^{\pi\gamma}\ /[2\gamma\ sh(2\pi\gamma)\ ]$
$B = -\ Ke^{-\pi\gamma}\ /[2\gamma\ sh(2\pi\gamma)]$
$\tilde{v}(\gamma,\varphi|r_0,\pi) = -K_{i\gamma}(kr_0)\ ch[\gamma(\varphi-\pi)]\ /\ [\gamma\ sh(2\pi\gamma)]$ // corrected p 175 E (2)

Now let's check this at the two φ points. We get

$\tilde{v}(\varphi=0) = -K\ ch[\pi\gamma]\ /\ [\gamma\ sh(2\pi\gamma)] = -K\ ch[\pi\gamma]\ /\ [\gamma\ 2sh(\pi\gamma)ch(\pi\gamma)] = -K\ /\ [\gamma\ 2sh(\pi\gamma)]$ = yes
$\tilde{v}(\varphi=2\pi) = -K\ ch[\pi\gamma]\ /\ [\gamma\ sh(2\pi\gamma)]$ = same = yes

so at least our proposed solution $\tilde{v}$ meets the BC's shown in C and D p 275. By the way, here is Stak in the newer edition, those two sin things should be sinh things!

At $\varphi = 0$ we have

$$A + B = -\tilde{E}(\gamma, 0 \mid r_0, \pi) = -\frac{1}{2\gamma \sin \gamma\pi} K_{i\gamma}(kr_0).$$

At $\varphi = 2\pi$,

$$Ae^{-2\gamma\pi} + Be^{2\gamma\pi} = -\tilde{E}(\gamma, 2\pi \mid r_0, \pi) = -\frac{1}{2\gamma \sin \gamma\pi} K_{i\gamma}(kr_0).$$

We deduce that

$$\tilde{v} = \frac{1}{\pi^2} \frac{\sinh \gamma(\varphi - 2\pi) - \sinh \gamma\varphi}{\sinh 2\gamma\pi} K_{i\gamma}(kr_0),$$

from which v can be calculated by (7.176) and then g from (7.181). By contour integration the result can be transformed to

$$g = \sum_{m=1}^{\infty} \frac{1}{\pi} \sin \frac{m\varphi}{2} \sin \frac{m\pi}{2} K_{m/2}(kr_>) I_{m/2}(kr_<), \qquad (7.183)$$

in agreement with (7.174), with $\psi = 2\pi$.

There are no less than 3 errors in my clip above, Stak missed all of them in his new edition! His answer is wrong on this simple grounds: evaluate it at φ = 0:

$\tilde{v}(\varphi=0)_{his} = (1/\pi)^2\ \{\ -sh[\gamma 2\pi)] - 0\ \}\ K/\ sh(2\pi\gamma) = -\ (K/\pi)^2$

which disagrees with C.

I think my answer is correct. I wonder if this problem is in the Lebedev book? Well, he mentions the general subject but does not seem to have this particular problem.

*** Compute v from $\tilde{v}$**

So let's take MY answer and try to compute v

$$f(r) = (2/\pi^2) \int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kr)\, F_k(\gamma)$$

$$v(r,\varphi) = (2/\pi^2) \int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kr)\, \tilde{v}(\gamma,\varphi)$$

$$= -(2/\pi^2) \int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, ch[\gamma(\varphi-\pi)] / [\gamma\, sh(2\pi\gamma)]$$

$$= -(2/\pi^2) \int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, ch[\gamma(\varphi-\pi)] / [\gamma\, 2sh(\pi\gamma)ch(\gamma\pi)]$$

$$= -(1/\pi^2) \int_0^\infty d\gamma\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, ch[\gamma(\varphi-\pi)] / ch(\gamma\pi)$$

But now this has a problem? At $\varphi = 0$ my answer then gives

$$v(r,\varphi=0) = -(1/\pi^2) \int_0^\infty d\gamma\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)$$

but this is supposed to be $-E(r,0|r_0,\pi)$. Well, if we look at 7.180 and set $\varphi_0 = \pi$ and $\varphi = 0$ and realize that the LHS is just E, and we are looking good. At $\varphi = 2\pi$ we get the same thing. So we can now form g in the following manner from 7.181

$$g(r,\varphi|r_0,\pi) = E + v = (1/\pi^2) \int_0^\infty d\gamma\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, ch[\gamma(\pi - |\varphi-\pi|)]$$
$$-(1/\pi^2) \int_0^\infty d\gamma\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, ch[\gamma(\varphi-\pi)] / ch(\gamma\pi)$$

$$= (1/\pi^2) \int_0^\infty d\gamma\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0) \{\ ch[\gamma(\pi - |\varphi-\pi|)] - ch[\gamma(\varphi-\pi)] / ch(\gamma\pi)\ \}$$

Notice that g = 0 at $\varphi = 0$ and 2π as required. Can we simplify the {..} somehow? We know that we have $(0,2\pi)$ for our range of φ. So we have

$$ch[\gamma(\pi - |\varphi-\pi|)] = ch[\gamma(\pi - (\varphi-\pi))] = ch[\gamma(2\pi - \varphi)] \qquad \varphi > \pi$$
$$ch[\gamma(\pi - |\varphi-\pi|)] = ch[\gamma(\pi + (\varphi-\pi))] = ch[\gamma\varphi] \qquad \varphi < \pi$$

I don't see any way to make this simpler really.

*** Verify that (7.183) is the correctly stated limit of 7.174**

We know that our "knife edge" problem is the same problem as the "wedge problem" where we set the wedge angle to $\psi = 2\pi$. The wedge solution is 7.174 which I know is correct, and if we set $\psi = 2\pi$ in that formula, the sines become $\sin(m\varphi/2)\sin(m\pi/2)$ and indeed this gives 7.183, so I do think 7.183 is in fact correct. It turns out that $\sin(m\pi/2) = 0$ when m = even integers, so in reality the sum on 7.183 is only over the values m = 1,3,5…. -- the sum has only odd terms. r θ

*** Transform our integral for g into the sum shown in 7.183**

This is the step that took me a very long time to get done right. We have this integral for g

$$g(r,\varphi|r_0,\pi) = (1/\pi^2) \int_0^\infty d\gamma\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0) \{ \ ch[\gamma(\pi - |\varphi-\pi|)] - ch[\gamma(\varphi-\pi)] / ch(\gamma\pi) \ \}$$

which we write as (using the fact that the entire integrand is even in γ)

$$g(r,\varphi|r_0,\pi) = (1/2\pi^2) \int_{-\infty}^\infty d\gamma \ K_{i\gamma}(kr_<)\, K_{i\gamma}(kr_>)\, f(\gamma)$$

$f(\gamma) = ch[\gamma(|\varphi-\pi|-\pi)] - ch[\gamma(\varphi-\pi)] / ch(\pi\gamma)$ = a function even in γ. // Note f(0) = 0
$= f_E(\gamma) + f_v(\gamma)$

$f_v(\gamma) = - ch[\gamma(\varphi-\pi)] / ch(\pi\gamma)$ $\qquad f_E(\gamma) = ch[\gamma(|\varphi-\pi|-\pi)]$

Note added 8.30.11. Another way to write f(γ):

$f(\gamma) = ch[\gamma(|\varphi-\pi|-\pi)] - ch[\gamma(\varphi-\pi)] / ch(\pi\gamma)$

$= ch(\gamma(|\varphi-\pi|)\, ch(\gamma\pi) - sh(\gamma(|\varphi-\pi|)\, sh(\gamma\pi) - ch[\gamma(\varphi-\pi)] / ch(\pi\gamma)$

$= ch(\gamma(|\varphi-\pi|)\, \{ ch(\gamma\pi) - 1/ ch(\pi\gamma)\} - sh(\gamma(|\varphi-\pi|)\, sh(\gamma\pi)$

$= ch(\gamma(|\varphi-\pi|)\, sh^2(\gamma\pi)/ ch(\pi\gamma)\} - sh(\gamma(|\varphi-\pi|)\, sh(\gamma\pi)$

$= sh(\gamma\pi)\, \{ th(\gamma\pi)\, ch(\gamma(|\varphi-\pi|) - sh(\gamma(|\varphi-\pi|) \}$

so we could write then

$$g(r,\varphi|r_0,\pi) = (1/\pi^2) \int_0^\infty d\gamma\, K_{i\gamma}(kr)\, K_{i\gamma}(kr_0)\, sh(\gamma\pi)\, \{ th(\gamma\pi)\, ch(\gamma(|\varphi-\pi|) - sh(\gamma(|\varphi-\pi|) \}$$

We now use the fact that (here r is a generic argument)

$K_{i\gamma}(kr) = (\pi/2)\,[I_{-i\gamma}(kr) - I_{i\gamma}(kr)] / \sin(i\gamma\pi) = (\pi/2i)\,[I_{-i\gamma}(kr) - I_{i\gamma}(kr)] / \mathrm{sh}(\gamma\pi)$
$= (i\pi/2)\,[I_{i\gamma}(kr) - I_{-i\gamma}(kr)] / \mathrm{sh}(\gamma\pi)$

to obtain (we select the first K for decomposition by the above)

$$g(r,\varphi|r_0,\pi) = (1/2\pi^2) \int_{-\infty}^{\infty} d\gamma\, \{(i\pi/2)\,[I_{i\gamma}(kr_<) - I_{-i\gamma}(kr_<)] / \mathrm{sh}(\gamma\pi)\}\, K_{i\gamma}(kr_>)\, f(\gamma)$$

$$= (i/4\pi) \int_{-\infty}^{\infty} d\gamma\, \{\,[I_{i\gamma}(kr_<) - I_{-i\gamma}(kr_<)] / \mathrm{sh}(\gamma\pi)\}\, K_{i\gamma}(kr_>)\, f(\gamma)$$

$$= (i/4\pi) \int_{-\infty}^{\infty} d\gamma\, [I_{i\gamma}(kr_<) - I_{-i\gamma}(kr_<)]\; K_{i\gamma}(kr_>)\, f(\gamma)/\, \mathrm{sh}(\gamma\pi)$$

$$= g_1 + g_2$$

$$g_1 = (i/4\pi) \int_{-\infty}^{\infty} d\gamma\, I_{i\gamma}(kr_<)\, K_{i\gamma}(kr_>)\, f(\gamma)/\, \mathrm{sh}(\gamma\pi)$$

$$g_2 = -(i/4\pi) \int_{-\infty}^{\infty} d\gamma\, I_{-i\gamma}(kr_<)\, K_{-i\gamma}(kr_>)\, f(\gamma)/\, \mathrm{sh}(\gamma\pi)$$

By taking $\gamma \to -\gamma$, we find that g_2 becomes g_1, so in fact $g_1 = g_2$. Using the g_2 form we then have

$$g(r,\varphi|r_0,\pi) = -(i/2\pi) \int_{-\infty}^{\infty} d\gamma\, I_{-i\gamma}(kr_<)\, K_{-i\gamma}(kr_>)\, f(\gamma)/\, \mathrm{sh}(\gamma\pi)$$

At this point we make the claim that we can close this contour either up or down to form a closed contour which captures the poles of $f(\gamma)/\,\mathrm{sh}(\gamma\pi)$ [I and K are both analytic in order everywhere!] Here is a weak supporting argument for doing this:

10.41 Asymptotic Expansions for Large Order

10.41(i) Asymptotic Forms

If $\nu \to \infty$ through positive real values with $z(\neq 0)$ fixed, then

10.41.1 $$I_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ez}{2\nu}\right)^\nu,$$

10.41.2 $$K_\nu(z) \sim \sqrt{\frac{\pi}{2\nu}} \left(\frac{ez}{2\nu}\right)^{-\nu}.$$

Maybe this result can be trusted for the IK product and we would then have

$I_\nu K_\nu \to (1/2)\,\nu^{-1}$

$I_{i\gamma}(kr)\, K_{i\gamma}(kr_0) \to (1/2)\,(i\gamma)^{-1}$

Then as we to up or down, we are looking at an integrand which is roughly 1/γ * oscillatory, and *maybe* this causes the GC to vanish. It at least has a chance. It seems that the chance is the same up or down, so let's close "up" and then we get the right sense on our pole residues.

Closing up, we then have

$$g(r,\varphi|r_o,\pi) = -(i/2\pi)\int_{up} d\gamma\, I_{-i\gamma}(kr_<)\, K_{-i\gamma}(kr_>)\, f(\gamma)/\,\mathrm{sh}(\gamma\pi)$$

Now we could continue with the full f(γ), and then we would have to deal with 1/sh(γπ) poles for the v piece, and 1/[sh(πγ)ch(πγ)] poles for the E piece. But I already know the E piece from previous work, so I will only do the v piece here. But first lets review the E piece. But before that, notice that f(γ)/ sh(γπ) does not have a pole at γ = 0 because f(0) = 0. Therefore, we will get no reside at γ = 0 which will correspond to n=0 below.

*** Series for the E piece of g.**

First, we have to adjust the E series 7.166 for the fact that we have $\varphi_0 = \pi$ in our knife edge problem, so

$$E = (1/2\pi)\,\Sigma_n\, e^{in(\varphi-\pi)} I_n(kr_<)K_n(kr_>) = (1/2\pi)\,\Sigma_n\,(-1)^n e^{in\varphi}\, I_n(kr_<)K_n(kr_>)$$

We can fold over the negative part of the series,

$$E(r,\varphi|r_o,\pi)_{neg} = (1/2\pi)\,\Sigma_{n=-1}^{-\infty}\,(-1)^n\, e^{in\varphi}\, I_n(kr_<)K_n(kr_>)$$

$$= (1/2\pi)\,\Sigma_{n=1}^{\infty}\,(-1)^n\, e^{-in\varphi}\, I_n(kr_<)K_n(kr_>)$$

So we then have

$$E = (1/2\pi)\,\Sigma_{n=1}^{\infty}\,(-1)^n\,(e^{in\varphi} + e^{-in\varphi})\; I_n(kr_<)K_n(kr_>) + (1/2\pi)\, I_0(kr_<)K_0(kr_>$$

$$= (1/2\pi)\,\Sigma_{n=1}^{\infty}\,(-1)^n\, 2\cos(n\varphi)\; I_n(kr_<)K_n(kr_>) + (1/2\pi)\, I_0(kr_<)K_0(kr_>)$$

$$= (1/\pi)\,\Sigma_{n=1}^{\infty}\,(-1)^n \cos(n\varphi)\, I_n(kr_<)K_n(kr_>) + (1/2\pi)\, I_0(kr_<)K_0(kr_>)$$

But we know that in the g = v+E sum there will be no n=0 term as discussed above, and we can think of that as a term in v cancelling the I_0K_0 term above. So let's just simplify this E sum to read

$$E \approx (1/\pi)\,\Sigma_{n=1}^{\infty}\,(-1)^n \cos(n\varphi)\, I_n(kr_<)K_n(kr_>)$$

where we intentionally throw out the n=0 term. Now anticipating what will happen below, we rewrite this sum by change to m = 2n or n = m/2 in this last sum, and then taking m → n again. This gives

$$E(r,\varphi|r_o,\pi) = (1/\pi)\,\Sigma_{n=2,4,6..}^{\infty}\,(-1)^{n/2}\, I_{n/2}(kr_<)\, K_{n/2}(kr_>)\cos(n\varphi/2)$$

and if we then write $\cos(n\pi/2) = (-1)^{n/2}$ which is valid for n = even integers, we get

$$E(r,\varphi|r_o,\pi) = (1/\pi)\,\Sigma_{n=2,4,6..}^{\infty}\, I_{n/2}(kr_<)\, K_{n/2}(kr_>)\cos(n\varphi/2)\cos(n\pi/2)$$

So this sum is our "E piece", obtained via 7.166 instead of doing the residues. This leaves us only with the "v piece".

*** Series for the v piece of g and then add to get g = v + E**

Our "v piece" is this, since $f_v(\gamma) = -\ ch[\gamma(\varphi-\pi)] / ch(\pi\gamma)$,

$$v(r,\varphi|r_0,\pi) = -(i/2\pi)\int_{up} d\gamma\, I_{-i\gamma}(kr_<)\, K_{-i\gamma}(kr_>)\, f_v(\gamma)/\, sh(\gamma\pi)$$

$$= (i/2\pi)\int_{up} d\gamma\, I_{-i\gamma}(kr_<)\, K_{-i\gamma}(kr_>)\, ch[\gamma(\varphi-\pi)] / [ch(\pi\gamma)\, sh(\gamma\pi)]$$

$$= -(1/2\pi i)\int_{up} d\gamma\, I_{-i\gamma}(kr_<)\, K_{-i\gamma}(kr_>)\, ch[\gamma(\varphi-\pi)] / [ch(\pi\gamma)\, sh(\gamma\pi)]$$

$$= -2\,(1/2\pi i)\int_{up} d\gamma\, I_{-i\gamma}(kr_<)\, K_{-i\gamma}(kr_>)\, ch[\gamma(\varphi-\pi)] / sh(2\gamma\pi)$$

*** Before continuing, we have to clarify the poles of 1/ sh(2γπ) .**

The poles are located here

$$ish(2\pi\gamma_n) = \sin(i2\pi\gamma_n) = 0 \quad \Rightarrow \quad i2\pi\gamma_n = 0, \pm\pi, \pm 2\pi \quad = -n\pi \qquad n = \text{integers}$$

But we need to be very careful here. If you track the contour sense, we wrap the poles in either half plane in the correct sense. However, the pole at the origin is wrapped only half way and in the wrong sense.

So the poles are located at half integers *and* integers on the imaginary axis:

$$\gamma_n = -n\pi/(2\pi i) = ni/2 \qquad -i\gamma_n = n/2$$

Near one of these poles we have $\gamma = \gamma_n + \delta\gamma = ni/2 + \delta\gamma$ so

$$f(\gamma) = f(\gamma_n) + \delta\gamma\, f'(\gamma_n) \qquad f(\gamma) = ish(2\pi\gamma) \Rightarrow \quad f'(\gamma) = 2\pi i\, ch(2\pi\gamma)$$

$$ish(2\pi\gamma) = ish(2\pi\gamma_n) + \delta\gamma\, 2\pi i\, ch(2\pi\gamma_n) = \delta\gamma\, 2\pi i\, ch(2\pi ni/2) = \delta\gamma\, 2\pi i \cos(2\pi n/2)$$

$$= \delta\gamma\, 2\pi i \cos(\pi n) = 2\pi i\,(-1)^n\, \delta\gamma$$

Therefore we have laboriously shown that

$sh(2\pi\gamma) \approx 2\pi (-1)^n (\gamma - \gamma_n)$ for γ near the pole position γ_n

*** The final steps**

We now use the residue theorem to say: (again, we are ignoring the potential pole at $\gamma = 0$)

$$v(r,\varphi|r_0,\pi) = -2 (1/2\pi i) \int_{up} d\gamma\, I_{-i\gamma}(kr_<) K_{-i\gamma}(kr_>) ch[\gamma(\varphi-\pi)] / sh(2\gamma\pi)$$

$$= -2 \Sigma_{n=1,2.3..}^{\infty} I_{-i\gamma n}(kr_<) K_{-i\gamma n}(kr_>) ch[\gamma_n(\varphi-\pi)] / [2\pi (-1)^n]$$

$$= -(1/\pi) \Sigma_{n=1,2.3..}^{\infty} (-1)^n I_{n/2}(kr_<) K_{n/2}(kr_>) ch[\gamma_n(\varphi-\pi)]$$

But now we write

$$ch[\gamma_n(\varphi-\pi)] = ch[ni(\varphi-\pi)/2] = \cos[n(\varphi-\pi)/2]$$

$$= \cos(n\varphi/2)\cos(n\pi/2) + \sin(n\varphi/2)\sin(n\pi/2)$$

$$= \cos(n\varphi/2)\cos(n\pi/2) \quad \text{when n = even integers}$$
$$= \sin(n\varphi/2)\sin(n\pi/2) \quad \text{when n = odd integers}$$

So we now break the v series into its even and odd terms such that $v = v_e + v_o$

$$v_e(r,\varphi|r_0,\pi) = -(1/\pi) \Sigma_{n=2,4.6..}^{\infty} I_{n/2}(kr_<) K_{n/2}(kr_>) \cos(n\varphi/2)\cos(n\pi/2)$$
$$v_o(r,\varphi|r_0,\pi) = +(1/\pi) \Sigma_{n=1,3.5..}^{\infty} I_{n/2}(kr_<) K_{n/2}(kr_>) \sin(n\varphi/2)\sin(n\pi/2)$$

Now recall from above that we showed

$$E(r,\varphi|r_o,\pi) = (1/\pi) \Sigma_{n=2,4,6..}^{\infty} I_{n/2}(kr_<) K_{n/2}(kr_>) \cos(n\varphi/2) \cos(n\pi/2)$$

Thus, when we compute $g = v + E$, we find that v_e exactly cancels E, and we are left with $g = v_o$ so we get our final result

$$g(r,\varphi|r_0,\pi) = +(1/\pi) \Sigma_{n=1,3.5..}^{\infty} I_{n/2}(kr_<) K_{n/2}(kr_>) \sin(n\varphi/2)\sin(n\pi/2)$$

Now since $\sin(n\pi/2)$ vanishes at even values of n, we are allowed to put them into the sum to get

$$g(r,\varphi|r_0,\pi) = (1/\pi) \Sigma_{n=1}^{\infty} I_{n/2}(kr_<) K_{n/2}(kr_>) \sin(n\varphi/2)\sin(n\pi/2)$$

and this agrees with 7.183 !

(i) Hankel Transform (275)

If you try out R(r)Z(z) as a separated form, you get exactly what Stak shows here in 7.184 and 7.185. The two terms are linear in r, so you could factor out rR if you wanted. Now recall our Hankel case from "comparison" in transforms folder.

$$-(xg')' + \nu^2 g/x - \lambda x g = \delta(x-\xi)$$

which gives the Hankel transform completeness $\delta(x-\xi)/x = \int_0^\infty dk\ k\ J_\nu(kx)J_\nu(k\xi)$. Now in 7.185 we don't have any 1/r term, which means $\nu = 0$ and $\lambda' = \lambda+\nu$ where λ' is the Hankel lambda. I think we would then think of $k = \sqrt{\lambda'} = \sqrt{\lambda+\nu}$ and our transform of interest is $\delta(x-\xi)/x = \int_0^\infty dk\ k\ J_0(kx)J_0(k\xi)$.

So as with the previous stuff, there are two ways to go with this problem where we have assumed it is azisym.

• (1) We could "start with z" and, if doing a free-space Green's, we could come up with a continuous spectrum for $\mu = -\nu$, and that would then be induced over into the r equation, and we would associate that z stuff with a **Fourier Integral transform**. We would have z in $(-\infty,\infty)$ say. We would have

$$Z(z) = e^{\pm ikz} \qquad k^2 = \mu \qquad Z'' = -k^2 Z = -\mu Z \text{ and } k = \text{continuous } (0,\infty)$$

Then this induces over, and the r equation becomes

$$-(rR')' - (\lambda-k^2)rR = 0 \qquad \text{solutions of form } R(r) = Z_0(\sqrt{\lambda-k^2}\, r)$$

• (2) Alternatively, we could "start with r" and assuming r in $(0,\infty)$ we have the Hankel situation there with $\lambda \to \lambda-k^2$ and $\nu \to 0$. For example, the Green's function in r world would be

$$(i\pi/2)\, J_0(\sqrt{\lambda-k^2}\, x_<)\, H_0^{(1)}(\sqrt{\lambda-k^2}\, x_>)$$

Atomic form would be like this

$$\text{atom}_k(r,z) = e^{\pm ikz}\, Z_0(\sqrt{\lambda-k^2}\, r) \qquad k \text{ in } (0,\infty), \text{ continuous spectrum}$$

The eigenfunctions here would be $\varphi_k(x) = \sqrt{k}\ J_0(kx)$ and we associate these with the **Hankel Transform of order 0.**

So depending on which variable you start with, you would apply the appropriate transform to your PDE and that would yield a 1D ODE in your starting variable.

Using the ring source as an example, Stak is now going to study a real problem with sources using these two different methods (1) and (2)! This is not a Green's function problem because we don't have a point source, we have a ring source. Notice that all work up to this point has involved only point sources, and

we thus ended up with various expansions for Helmholtz Green's Functions. Now we are going to do something with a distributed source. Also this is a 3D problem that is azisym.

(j) Source Ring in Free Space (276)

The ratio shown in A is simply $\delta^{(3)}(\mathbf{r}-\mathbf{r}_0)$ where $\mathbf{r}_0 = a\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}$ is at angle φ_0. A tiny piece of the ring near φ_0 does indeed have charge $d\varphi_0/2\pi$ if we insist that the total ring have unit charge. So A is then the 3D ρ source density for this chunk of the ring. If we add up the entire ring contribution we find

$$\rho_{ring}(\mathbf{r}) = \delta(r-a)\delta(z)/(2\pi r) \quad = \text{3D source density}$$

We then write a Poisson type equation driven by this source density, but it has the Helmholtz λ term in there and this is 7.187. Our problem is to solve this thing for u(r,z). Presume a BC is that $u \to 0$ when r or z goes to ∞, and u = finite everywhere away from the ring itself.

(k) Hankel Transform of Order Zero on the Radial Coordinate.

So here we are going to "start with" the r equation part of 7.187. We get 7.188 just looking up ∇^2 in our TK in cylindrical coordinates. We now process this PDE equation as shown such that we replace variable r by variable γ, the Hankel conjugate variable. We replace u(r,z) by $U_H(\gamma,z)$, that is to say. Our goal is to get an ODE in variable z for the function $U_H(\gamma,z)$. Doing parts twice on the first term in 7.189 makes use of the fact that $f\,\partial_r(r\partial_r g) = g\,\partial_r(r\partial_r f)$ in the parts sense (throwing out the "parts" presumably because at least one of the functions vanishes at ∞, and hopefully due to the r factor at 0). This double parts thing works in this nice way because L is self-adjoint (I think). So we swap positions of u and $J_0(\gamma r)$ to get B. Then the first term has $u\,\partial_r(r\partial_r J_0(\gamma r))$ and we use 7.185 to replace that with $-(\lambda+\nu)r\,J_0(\gamma r)\,u$ and with the integral this just becomes $-(\lambda+\nu)\,U_H(\gamma,z)$. Now he says $\lambda+\nu = \gamma^2$, how do we interpret that? The number λ is fixed from the H equation, it is thus already defined. γ is the Hankel conjugate variable, so it is already defined. ν is the separation constant, so we are thus setting this to $\nu = \gamma^2-\lambda$. This is like setting the separation constant $\nu = -n^2$ in an earlier problem where n was forced to be an integer. Here the quantum number variable is γ and can have any continuous value. Thus $-(\lambda+\nu)r\,J_0(\gamma r)\,u = -\gamma^2 r\,J_0(\gamma r)\,u \to -\gamma^2\,U_H$ and we have arrived at 7.190 which is a Green's 1D ODE for $U_H(\gamma,z)$ in variable z.

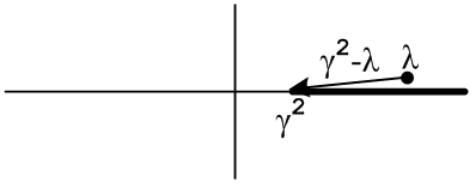
So (take a breath), how do we solve this Green's ODE? If I set $\kappa^2 = \gamma^2-\lambda$ just for a shorthand, then $e^{-\kappa|z|}$ is a viable (at $z = \pm\infty$) solution on both sides of z = 0 (ie, solves the homo ODE). We then have to handle the jump condition at z = 0. Let $g = A\,e^{-\kappa|z|}$. Then $\partial_z g = -|\kappa|A\,e^{-\kappa|z|}$. Integrate both sides of ODE

$$-[\,g'^{+} - g'^{-}] = J_0(\gamma r)/2\pi \qquad \text{or} \qquad -[\,-\kappa A\, - (-\kappa)A] = J_0(\gamma r)/2\pi \;= 2\kappa A \text{ so } A = J_0(\gamma r)/(4\pi\kappa)$$

and this is what p 277 C says. So boom, we have solved pretty easily for $U_H(\gamma,z)$. Then we just use the inverse Hankel which gives u(r,z) as in 7.191 and we are done! We could plot this in r and z in Maple! I do wonder about the sign of $\kappa^2 = \gamma^2-\lambda$. If we knew that $\lambda < 0$ (perhaps $\lambda = -k^2$), then this is not an issue, and probably that is the normal case. I expect this thing to diverge when r = a and z = 0 so u is then sitting right on the source ring. As a first step, we could just set r = a and the RHS of 7.191 becomes the RHS of 7.192. But our unit charge ring is now a unit point source, so the LHS of 7.192 just be the 3D fundy

solution which is 7.52 (Stak has a typo I have noted). Setting $\lambda = -k^2$ gives 7.193 where $\sqrt{\lambda} = \sqrt{-k^2} = +ik$ by some argument we need but I will skip. This gives decay for large k on both sides which must be right, and so we have 7.193.

Now how do we evaluate 7.192 not when $\lambda = -k^2$ but when $\lambda > 0$? We are supposed to use $\lambda + i\varepsilon$ as shown in D in order to evaluate our $\kappa = \sqrt{\gamma^2-\lambda}$ object in this case. I would draw a little picture in the λ plane like this



Then for λ as shown, we have the phase of $\gamma^2-\lambda$ being $-\pi$ so that $\sqrt{\gamma^2-\lambda} = |\sqrt{\gamma^2-\lambda}|\, e^{-i\pi/2} = -i\sqrt{\lambda-\gamma^2}$. For λ on the real axis between 0 and γ^2 we just get the obvious $\sqrt{\gamma^2-\lambda}$. So I have just verified A and B on page 278. I am always pretty good at this kind of thing. So now we just break the 7.192 integral into two pieces and put in the appropriate κ in each piece, and we then get 7.194, all fine. The last 7.195 is the limit of 7.193 when k = 0. You wonder if these various integrals appear in GR7 somewhere? P 708 is close, but hard to look up since combination of Bessel of more complicated argument, algebraic and exponential of algebraic. So none of these are obvious to look up! Maybe that is why Stak is recording them so carefully. Perhaps change variable to $\kappa = \sqrt{\gamma^2-\lambda}$ so $\gamma = \sqrt{\kappa^2+\lambda}$, still not very nice! Ah, you would look this up in Bateman's Hankel transform section!!! Here is something from ET II which looks a lot like 7.193, but note the symmetric $\sqrt{xy}$ stuff etc etc.

24] §2. ПРЕОБРАЗОВАНИЯ ГАНКЕЛЯ НУЛЕВОГО ПОРЯДКА 15

	$f(x)$	$\int_0^\infty f(x)\, J_0(xy)\,(xy)^{1/2}\, dx,\ y > 0$
(24)	$x^{1/2}(x^2+\beta^2)^{-1/2} \times \exp[-\alpha(x^2+\beta^2)^{1/2}],$ $\operatorname{Re}\alpha > 0,\ \operatorname{Re}\beta > 0$	$y^{1/2}(y^2+\alpha^2)^{-1/2} \exp[-\beta(y^2+\alpha^2)^{1/2}]$

Summary of this section: We took our Helmholtz Poisson equation driven by a ring source and we fully solved it for solution u(r,z). We did this by applying the Hankel ν=0 transform to the PDE to remove r in place of γ, then we had a Green's ODE in variable z which we trivially solved, we transformed back, and our final answer was 7.191. We then looked at various special cases of this formula for positions of λ in the λ plane, noting a cut to the right starting at $\lambda = \gamma^2$. The case $\lambda = 0$ roughly quoted in 7.195 should give us the true Laplace theory Poisson solution which appears somewhere in green Jackson. We know to use the Hankel $\nu = 0$ because the ODE in r under separation (7.185) was of this Hankel ODE form (see comparison).

(l) Fourier Transform on z

We now "do it the other way". Start with the PDE and this time replace z by α using a Fourier Transform. We know to do this because our z ODE 7.184 has the Fourier transform form (including range). Then 7.188 becomes p 278 C where we replace u(r,z) by $\hat{u}(r,\alpha)$ as the symbol for the FT. So there is our Green's ODE in variable r with midpoint at r = a, so we need a left and right side function to construct our Green's function shown in E. We now look up in our Bessel doc to find

1. $w(x) = K_\nu(kx)$ solves $-(xw')' + k^2xw + \nu^2 w/x = 0.$

so if we set $k^2 = \alpha^2-\lambda$ and $\nu = 0$, we get p 278 C, so we know solutions are like $K_0(kr)$. As usual we need I_0 on the left for r=0 and K_0 on the right for $r = \infty$. The constant would come as usual from the jump condition which involves the Wronskian of I and K, so I accept the constant in E without checking it. Then the solution for u(r,z) is given by the inverse FT as shown in 7.196, I agree. So boom, the problem is solved. Whereas solution 7.191 was (0,∞) with a pair of J's and other stuff, here we get (-∞,∞) with an I K product of fancy arguments. And as before, we are now going to take some limits.

If $a\to 0$ we know $I_0\to 1$ (Jackson p 75) and we get 7.197 in analogy with 7.192 (both having the same E_3 LHS). Then 7.198 follows on with $\lambda = -k^2$.

Now as before, we have to handle the cut situation in the arg of K_0 in 7.197 and we do this by again breaking the integral into two parts. We can first reflect the α integral range to be (0,∞) and pick up a factor of 2 and replace $e^{-i\alpha z}$ by cos(αz). Then we apply exactly the rule above ($\gamma^2 \to \alpha^2$), and this proves 7.199.

Now a slight topic change. Suppose we take 7.191 which has expo $e^{-\kappa|z|}/\kappa$, and suppose we Fourier Expand that function using Schaum p 177 33.26 which says the FT is then $\alpha./(\alpha^2+\kappa^2)$ [wrong! That is a sine xform, and we need a cos xform for an even function, see below] . This then leads to 7.200 (but I think the factor α/κ is missing, and the $1/4\pi$ should become $1/4\pi * 1/2\pi = 1/8\pi^2$ -- and this is not corrected in the online 2000 edition!) Hard to imagine why 7.200 would be useful.

The rest of page 279 are equations obtained by inverting 7.197 and taking limits. One item is then a common integral rep for K_0.

Check on this since sending in errata! Start with 7.191. It contains this factor

$$f(z) = \exp(-\sqrt{\gamma^2-\lambda}\,|z|) = e^{-b|z|} \qquad b \equiv \sqrt{\gamma^2-\lambda}$$

We want to expand this as a full Fourier transform. Since it is even in z, it would seem that a Fourier Cosine Transform would be appropriate. Schaum page 176 and 178 tells us that (33.39)

$$F_c(\alpha) = \int_0^\infty dz\, \cos(\alpha z)\, f(z) = b/(\alpha^2+b^2)$$

Now we can write

$$\int_0^\infty dz\, \cos(\alpha z)\, f_{even}(z) = (1/2) \int_{-\infty}^\infty dz\, \cos(\alpha z)\, f_{even}(z)$$

$$= (1/2) \int_{-\infty}^{\infty} dz\, [\cos(\alpha z) \pm i\sin(\alpha z)]\; f_{\mathbf{even}}(z)$$

$$F_C(\alpha) = (1/2) \int_{-\infty}^{\infty} dz\, e^{\pm i\alpha z}\; f(z) \qquad \text{since f(z) is indeed even}$$

What is the inverse of this Fourier Transform? From p 175 Schaum this says

$$F(\alpha) = 2\, F_C(\alpha) = \int_{-\infty}^{\infty} dz\, e^{-i\alpha z}\; f(z)$$

and therefore

$$f(z) = (1/2\pi) \int_{-\infty}^{\infty} d\alpha\, e^{+i\alpha z}\, F(\alpha) = (1/2\pi) \int_{-\infty}^{\infty} d\alpha\, e^{+i\alpha z}\, 2\, F_C(\alpha)$$

$$= (1/\pi) \int_{-\infty}^{\infty} d\alpha\, e^{+i\alpha z}\, F_C(\alpha)$$

$$= (1/\pi) \int_{-\infty}^{\infty} d\alpha\, e^{+i\alpha z}\; b/(\alpha^2+b^2) \qquad b \equiv \sqrt{\gamma^2-\lambda}$$

$$= (1/\pi)\sqrt{\gamma^2-\lambda} \int_{-\infty}^{\infty} d\alpha\; /(\alpha^2 + \gamma^2-\lambda)$$

So start with 7.191 which says

$$u(r,z) = (1/4\pi) \int_{\mathbf{0}}^{\infty} d\gamma\; \gamma\; J_0(\gamma r)J_0(\gamma a)\, e^{-b|z|} / b$$

$$= (1/4\pi)(1/b) \int_{\mathbf{0}}^{\infty} d\gamma\; \gamma\; J_0(\gamma r)J_0(\gamma a)\, (1/\pi) \int_{-\infty}^{\infty} d\alpha\, e^{+i\alpha z}\; b/(\alpha^2+b^2)$$

$$= (1/4\pi) \int_{\mathbf{0}}^{\infty} d\gamma\; \gamma\; J_0(\gamma r)J_0(\gamma a)\, (1/\pi) \int_{-\infty}^{\infty} d\alpha\, e^{+i\alpha z}\; 1/(\alpha^2 + \gamma^2-\lambda)$$

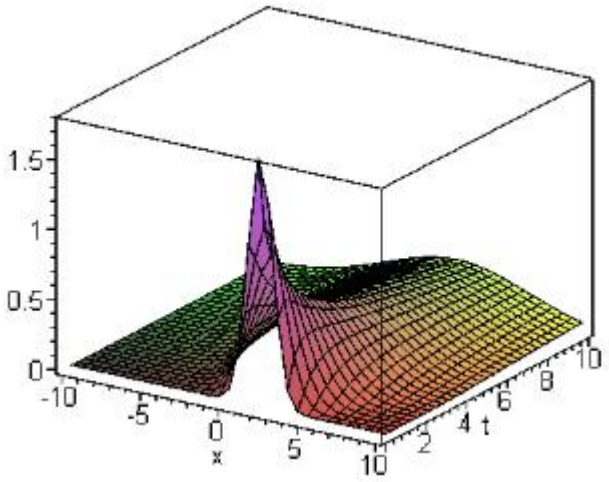
$$= (1/4\pi^2) \int_{\mathbf{0}}^{\infty} d\gamma\; \gamma\; J_0(\gamma r)J_0(\gamma a) \int_{-\infty}^{\infty} d\alpha\, e^{+i\alpha z}\; 1/(\alpha^2 + \gamma^2-\lambda)$$

and his 7.200 is exactly right (either expo sign OK). So delete this erratum!

[280] Another topic change. Recall a result from heat flow in an infinite 1D rod, for example

$u(\mathbf{x},t) = 1/\sqrt{4\pi t} \int_{-\infty}^{\infty} dx_0\, f(\mathbf{x}_0) \exp(-(x-x_0)^2/4t)$ // agrees with (7.16)

Here is a plot of the solution when $f(x) = \delta(x-2)$



The equation has some initial temperature distribution f(x) at t = 0. That is a distribution of thermal energy in the rod at t = 0. In our example, we assume a distribution δ(x-2). Then as time progresses, this thermal energy spreads out and we get some temperature distribution in our rod as a function of time, and that is what this plot is showing. So here f(x) is NOT a heat source that has some time schedule to it, f(x) is an initial condition for the heat problem. Notice by the way that the rest of the integrand is the free-space 1D heat propagator $1/\sqrt{4\pi t} * \exp(-(x-x_0)^2/4t)$

Now we are going to talk here about a similar problem. Instead of a 1D rod, we assume a 3D infinite medium, and near the origin we will have f(**x**) as a distributional initial condition which is a 2D ring. We hold this initial condition for t < 0 and don't let the medium conduct heat. Then suddenly at t = 0 we turn on condition, say, and we ask what the temp looks like in 3D for t > 0. So again, don't think of this ring as a heat source, it is instead an initial condition, it is not a boundary schedule.

So this is the problem being stated in 7.203! Now if we think of a tiny piece of the ring as a point initial condition, its effect on the temperature at some other point in space is given by the 3D free space propagator which he has written down here. The question is: how big is this little initial temp point? If we contract the ring to a point, we want to have a unit thing, meaning just 1 $\delta(\mathbf{r})$ just as we had 1 δ(x-2) in the rod problem noted above. So the spatial integral of our ring of temperature has to be 1 and that is why we have the $d\varphi_0/2\pi$ situation. Integrate and you get 1. OK, I could clarify better, but leave as is, and I now accept p 280 A as the effect on some point in space due to the entire ring (radius a). We write the distance R in cylindricals in the usual way in B, where our observation point is at φ = 0. We then install this R^2 into A and get the first line in 7.204. To do the $d\varphi_0$ integral we look at AS 2010 p 252

10.32.1

$$I_0(z) = \frac{1}{\pi}\int_0^{\pi} e^{\pm z\cos\theta}\, d\theta$$

$= (1/2\pi) \int_0^{2\pi} e^{z\cos\varphi}\, d\varphi$

and this is just the integral we see (including the $1/2\pi$) and thus the second line of 7.204.

So again, boom! We have solved the problem, this gives the temperature at any time. Of course this is an azisym problem.

Now, suppose instead we do time Laplace on our original PDE. Recall the rule for a derivative Schaum p 162 getting sf(s)-F(0). Thus our equation $(\partial_t - \nabla^2)v(t,x)$ becomes $sV(s,x)-v(t=0) - \nabla^2 V(s,x) = 0$ which is what D says. But this little ODE for V is exactly our Helmholtz thing 7.188 with $\lambda = -s$. Thus, we steal our solution 7.191 and replace λ by -s and this is then E ! This is not our heat problem solution yet, it is the Laplace transform of that solution (twiddle). Looking at f(s) shown in F, we see this is a "translation" of the Schaum result p 169 32.108 which is a famous entry. We have $\alpha = |z|$ here and we get all the factors and $e^{-\gamma\gamma t}$ arises from the translation from s to $s+\gamma^2$ as per Schaum 32.5 with $a = -\gamma^2$. So this lets us compute the inverse Laplace of E and we get p 281 A! This is an alternative to result 7.204 which is simpler since it has no integral. It would be interesting to take the $t \to 0+$ limit of 7.204 and show that you get the ring distribution, and the same for 7.205. Those would be in my class assignments, maybe Stak will have them later.

Concluding Comment: This concludes our lengthy 17 page section 7.12 where we took a break from evolution equations to study the Helmholtz equation and "Helmholtz potential theory". And our last effort above was a use of the Helmholtz work to solve a simple heat flow problem (maybe not so simple!). We did a time Laplace to convert the heat equation into a Helmholtz one.

We are now at 79 pages in this raw2 log, so I will start raw3 now! We still have 50 text pages to go in Chapter 7.