The Abel Transform PhL 2.28.11

The work has been done in my Sneddon Chapter 2 raw notes, here I just want to "report out" the results in their various forms.

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1. The trig forms.

Sneddon gives a very general monotonic Srivastav transform form (page 40-41). The theorem is limited to the range $0 < \alpha < 1$ where α is the power appearing in the denominator fo 2.3.1 or 2.3.3. He then applies this theorem to some special cases, one of which I shall call "the trig case". Notice we have a separate transform for for the two possible Volterra endpoint positions. We are thinking about a real interval (a,b):

S1: $\int_a^x dt\, f(t) / [\cos(t) - \cos(x)]^{\alpha} = g(x)$

$\Rightarrow$ $f(t) = \pi^{-1}\sin(\pi\alpha)\, \partial_t \{ \int_a^t du\, \sin(u)g(u) / [\cos(u) - \cos(t)]^{1-\alpha} \}$

S2: $\int_x^b dt\, f(t) / [\cos(x) - \cos(t)]^{\alpha} = g(x)$

$\Rightarrow$ $f(t) = -\pi^{-1}\sin(\pi\alpha)\, \partial_t \{ \int_t^b du\, \sin(u)g(u) / [\cos(t) - \cos(u)]^{1-\alpha} \}$

We can specialize these for $\alpha = 1/2$ which is the "non-generalized" Abel Transform

S1: $\int_a^x dt\, f(t) / [\cos(t) - \cos(x)]^{1/2} = g(x)$

$\Rightarrow$ $f(t) = \pi^{-1}\, \partial_t \{ \int_a^t du\, \sin(u)g(u) / [\cos(u) - \cos(t)]^{1/2} \}$ // which are 2.3.5

S2: $\int_x^b dt\, f(t) / [\cos(x) - \cos(t)]^{1/2} = g(x)$

$\Rightarrow$ $f(t) = -\pi^{-1}\, \partial_t \{ \int_t^b du\, \sin(u)g(u) / [\cos(t) - \cos(u)]^{1/2} \}$ //which are 2.3.6

which we might write one more time as

S1: $\int_a^x dt\, f(t) / \sqrt{\cos t - \cos x} = g(x)$

=> $f(t) = \pi^{-1}\, \partial_t \{ \int_a^t du \sin u\, g(u) / \sqrt{\cos u - \cos t} \}$ // agrees Sned 2.3.5

S2: $\int_x^b dt\, f(t) / \sqrt{\cos x - \cos t} = g(x)$

=> $f(t) = -\pi^{-1}\, \partial_t \{ \int_t^b du \sin u\, g(u) / \sqrt{\cos t - \cos u} \}$ // agrees Sned 2.3.6

2. The quadratic forms.

Similar to the above, we just quote this different special case of Srivastav :

S1: $\int_a^x dt\, f(t) / [x^2 - t^2]^{\alpha} = g(x)$ interval for x,t is [a,b]

=> $f(t) = (2/\pi)\sin(\pi\alpha)\, \partial_t \{ \int_a^t du\, u\, g(u) / [t^2 - u^2]^{1-\alpha} \}$ // which are 2.3.7

S2: $\int_x^b dt\, f(t) / [t^2 - x^2]^{\alpha} = g(x)$ interval for x,t is [a,b]

=> $f(t) = -(2/\pi) \sin(\pi\alpha)\, \partial_t \{ \int_t^b du\, u\, g(u) / [u^2 - t^2]^{1-\alpha} \}$ // which are 2.3.8

and again specialized to $\alpha = 1/2$

S1: $\int_a^x dt\, f(t) / [x^2 - t^2]^{1/2} = g(x)$ interval for x,t is [a,b]

=> $f(t) = (2/\pi)\partial_t \{ \int_a^t du\, u\, g(u) / [t^2 - u^2]^{1/2} \}$

S2: $\int_x^b dt\, f(t) / [t^2 - x^2]^{1/2} = g(x)$ interval for x,t is [a,b]

=> $f(t) = -(2/\pi)\, \partial_t \{ \int_t^b du\, u\, g(u) / [u^2 - t^2]^{1/2} \}$

which we might rewrite as

S1: $\int_a^x dt\, f(t) / \sqrt{x^2 - t^2} = g(x)$

=> $f(t) = (2/\pi)\partial_t \{ \int_a^t du\, u\, g(u) / \sqrt{t^2 - u^2} \}$

S2: $\int_x^b dt\, f(t) / \sqrt{t^2 - x^2} = g(x)$

$$\Rightarrow \quad f(t) = -(2/\pi)\,\partial_t \left\{ \int_t^b du\, u\, g(u) / \sqrt{u^2 - t^2} \right\}$$

3. The linear forms.

I did a little bit in my Sneddon Chap 2 raw notes, but I complete the job in Appendix A below and quote the final results here, where now I do the ∂_t derivatives to get alternate answers:

S1: $$\int_a^x dt\, f(t) \,/\, [x - t]^{\alpha} = g(x)$$

$$\Rightarrow \quad f(t) = (1/\pi)\sin(\pi\alpha)\,\partial_t \left\{ \int_a^t du\, g(u) / [t - u]^{1-\alpha} \right\}$$

$$= (1/\pi)\sin(\pi\alpha) \left\{ g(a) / [t - a]^{1-\alpha} + \int_a^t du\, g'(u) / [t - u]^{1-\alpha} \right\}$$

S2: $$\int_x^b dt\, f(t) / [t - x]^{\alpha} = g(x)$$

$$\Rightarrow \quad f(t) = -(1/\pi)\sin(\pi\alpha)\,\partial_t \left\{ \int_t^b du\, g(u) / [u - t]^{1-\alpha} \right\}$$

$$= (1/\pi)\sin(\pi\alpha) \left\{ g(b) / [b - t]^{1-\alpha} - \int_t^b du\, g'(u) / [u - t]^{1-\alpha} \right\}$$

We can specialize to the case $\alpha = 1/2$ to get

S1: $$\int_a^x dt\, f(t) / \sqrt{x - t} = g(x)$$

$$\Rightarrow \quad f(t) = (1/\pi)\partial_t \left\{ \int_a^t du\, g(u) / \sqrt{t - u} \right\}$$

$$= (1/\pi) \left\{ g(a) / \sqrt{t - a} + \int_a^t du\, g'(u) / \sqrt{t - u} \right\}$$

S2: $$\int_x^b dt\, f(t) / \sqrt{t - x} = g(x)$$

$$\Rightarrow \quad f(t) = -(1/\pi)\,\partial_t \left\{ \int_t^b du\, g(u) / \sqrt{u - t} \right\}$$

$$= (1/\pi) \left\{ g(b) / \sqrt{b - t} - \int_t^b du\, g'(u) / \sqrt{u - t} \right\}$$

Appendix A: Derivation of the linear forms from the quadratic forms.

Part (a) : obtaining the linear forms

We start with A statement of the quadratic forms

S1: $\int_a^x dt\, f(t) / [x^2 - t^2]^\alpha = g(x)$ interval for x,t is [a,b]

$\Rightarrow$ $f(t) = (2/\pi)\sin(\pi\alpha)\, \partial_t \{ \int_a^t du\, u\, g(u) / [t^2 - u^2]^{1-\alpha} \}$ // which are 2.3.7

S2: $\int_x^b dt\, f(t) / [t^2 - x^2]^\alpha = g(x)$ interval for x,t is [a,b]

$\Rightarrow$ $f(t) = -(2/\pi)\sin(\pi\alpha)\, \partial_t \{ \int_t^b du\, u\, g(u) / [u^2 - t^2]^{1-\alpha} \}$ // which are 2.3.8

Let $\chi = x^2$ and $\tau = t^2$ and $\mu = u^2$ and $A = a^2$ and $B = b^2$ and use capital letters in this sense: $G(\mu) = g(u)$. We can then edit all of the above:

$d\mu = 2u\, du \quad \Rightarrow \quad du\, u = d\mu\, (1/2)$

$d\tau = 2tdt = 2\sqrt{\tau}\, dt \quad dt = d\tau/(2\sqrt{\tau}) \quad \partial_t = (2\sqrt{\tau})\partial_\tau$

S1: $\int_A^\chi d\tau/(2\sqrt{\tau})\, F(\tau) / [\chi - \tau]^\alpha = G(\chi)$

$\Rightarrow$ $F(\tau) = (2/\pi)\sin(\pi\alpha)\, (2\sqrt{\tau})\partial_\tau \{ \int_A^\tau d\mu(1/2)\, G(\mu) / [\tau - \mu]^{1-\alpha} \}$

S2: $\int_\chi^B d\tau/(2\sqrt{\tau})\, F(\tau) / [\tau - \chi]^\alpha = G(\chi)$

$\Rightarrow$ $F(\tau) = -(2/\pi)\sin(\pi\alpha)\, (2\sqrt{\tau})\partial_\tau \{ \int_\tau^B d\mu(1/2)\, G(\mu) / [\mu - \tau]^{1-\alpha} \}$

To get these into standard forms, we now define

$\mathcal{F}(\tau) = F(\tau)/(2\sqrt{\tau})$

Then we have

S1: $\int_A^\chi d\tau\, \mathcal{F}(\tau) / [\chi - \tau]^\alpha = G(\chi)$

$\Rightarrow$ $\mathcal{F}(\tau) = (2/\pi)\sin(\pi\alpha)\, \partial_\tau \{ \int_A^\tau d\mu(1/2)\, G(\mu) / [\tau - \mu]^{1-\alpha} \}$

S2: $\int_\chi^B d\tau\, \mathcal{F}(\tau) / [\tau - \chi]^\alpha = G(\chi)$

$\Rightarrow$ $\mathcal{F}(\tau) = -(2/\pi)\sin(\pi\alpha)\, \partial_\tau \{ \int_\tau^B d\mu(1/2)\, G(\mu) / [\mu - \tau]^{1-\alpha} \}$

Now that we have all our results, we first change

$\tau \to t$
$\mu \to u$
$\chi \to x$
$A \to a$
$B \to b$
$\mathcal{F} \to f$
$G \to g$

τ to t everywhere, μ to u everywhere, and write 2(1/2) = 1,

S1: $\int_a^x dt\, f(t) / [x - t]^{\alpha} = g(x)$

$\Rightarrow$ $f(t) = (1/\pi)\sin(\pi\alpha)\, \partial_t \{ \int_a^t du\, g(u) / [t - u]^{1-\alpha} \}$

S2: $\int_x^b dt\, f(t) / [t - x]^{\alpha} = g(x)$

$\Rightarrow$ $f(t) = -(1/\pi)\sin(\pi\alpha)\, \partial_t \{ \int_t^b du\, g(u) / [u - t]^{1-\alpha} \}$

Part (b): Doing the derivatives

We now refer to Appendix A of our Sneddon Chapter 2 raw notes where we show that

$g(x) = \int_a^x dt\, k(x-t)f(t) \Rightarrow \partial_x g(x) = f(a)\, k(x-a) + \int_a^x dt\, k(x-t)f'(t)$

$g(x) = \int_x^b dt\, k(t-x)f(t) \Rightarrow \partial_x g(x) = -f(b)\, k(b-x) + \int_x^b dt\, k(t-x)\, f'(t)$

We of course have $k(z) = 1/z^{\alpha}$ so we rewrite these lines as

$g(x) = \int_a^x dt\, f(t)/ [x - t]^{\alpha} \Rightarrow \partial_x g(x) = f(a) / [x - a]^{\alpha} + \int_a^x dt\, f'(t) / [x - t]^{\alpha}$

$g(x) = \int_x^b dt\, f(t)/ [t - x]^{\alpha} \Rightarrow \partial_x g(x) = -f(b) / [b - x]^{\alpha} + \int_x^b dt\, f'(t) / [t - x]^{\alpha}$

Now on the RHS's replace t→ u everywhere THEN replace x → t everywhere

$g(t) = \int_a^t du\, f(u)/ [t - u]^{\alpha} \Rightarrow \partial_t g(t) = f(a) / [t - a]^{\alpha} + \int_a^t du\, f'(u) / [t - u]^{\alpha}$

$g(t) = \int_t^b du\, f(u)/ [u - t]^{\alpha} \Rightarrow \partial_t g(t) = -f(b) / [b - t]^{\alpha} + \int_t^b du\, f'(u) / [u - t]^{\alpha}$

Now let g→h and THEN let f→g

$$h(t) = \int_a^t du\, g(u)/[t-u]^{\alpha} \quad \Rightarrow \quad \partial_t h(t) = g(a)/[t-a]^{\alpha} + \int_a^t du\, g'(u)/[t-u]^{\alpha}$$

$$h(t) = \int_t^b du\, g(u)/[u-t]^{\alpha} \quad \Rightarrow \quad \partial_t h(t) = -g(b)/[b-t]^{\alpha} + \int_t^b du\, g'(u)/[u-t]^{\alpha}$$

We can then make these replacements (we take $\alpha \rightarrow 1-\alpha$ to match forms above)

$$\partial_t \left\{ \int_a^t du\, g(u)/[t-u]^{1-\alpha} \right\} = g(a)/[t-a]^{1-\alpha} + \int_a^t du\, g'(u)/[t-u]^{1-\alpha}$$

$$\partial_t \left\{ \int_t^b du\, g(u)/[u-t]^{1-\alpha} \right\} = -g(b)/[b-t]^{1-\alpha} + \int_t^b du\, g'(u)/[u-t]^{1-\alpha}$$

Then we can restate our results from the end of part (a)

S1: $\int_a^x dt\, f(t)/[x-t]^{\alpha} = g(x)$

$$\Rightarrow f(t) = (1/\pi)\sin(\pi\alpha)\, \partial_t \left\{ \int_a^t du\, g(u)/[t-u]^{1-\alpha} \right\}$$

$$= (1/\pi)\sin(\pi\alpha) \left[g(a)/[t-a]^{1-\alpha} + \int_a^t du\, g'(u)/[t-u]^{1-\alpha} \right]$$

S2: $\int_x^b dt\, f(t)/[t-x]^{\alpha} = g(x)$

$$\Rightarrow f(t) = -(1/\pi)\sin(\pi\alpha)\, \partial_t \left\{ \int_t^b du\, g(u)/[u-t]^{1-\alpha} \right\}$$

$$= -(1/\pi)\sin(\pi\alpha) \left[-g(b)/[b-t]^{1-\alpha} + \int_t^b du\, g'(u)/[u-t]^{1-\alpha} \right]$$

And let's restate this one more time with some sign changes

S1: $\int_a^x dt\, f(t)/[x-t]^{\alpha} = g(x)$

$$\Rightarrow f(t) = (1/\pi)\sin(\pi\alpha)\, \partial_t \left\{ \int_a^t du\, g(u)/[t-u]^{1-\alpha} \right\}$$

$$= (1/\pi)\sin(\pi\alpha) \left\{ g(a)/[t-a]^{1-\alpha} + \int_a^t du\, g'(u)/[t-u]^{1-\alpha} \right\}$$

S2: $\int_x^b dt\, f(t)/[t-x]^{\alpha} = g(x)$

$$\Rightarrow f(t) = -(1/\pi)\sin(\pi\alpha)\, \partial_t \left\{ \int_t^b du\, g(u)/[u-t]^{1-\alpha} \right\}$$

$$= (1/\pi)\sin(\pi\alpha)\ \{\ g(b)/[b-t]^{1-\alpha} - \int_t^b du\, g'(u)/[u-t]^{1-\alpha}\ \}$$

We can then specilialized these last results to $\alpha = 1/2$ and $\sin(\pi/2) = 1$

S1: $\int_a^x dt\, f(t)\ /[x-t]^{1/2} = g(x)$

$$\Rightarrow\ f(t) = (1/\pi)\partial_t\ \{\ \int_a^t du\, g(u)/\ [t-u]^{1/2}\ \}$$

$$= (1/\pi)\ \{\ g(a)/[t-a]^{1/2} + \int_a^t du\, g'(u)/[t-u]^{1/2}\ \}$$

S2: $\int_x^b dt\, f(t)/[t-x]^{1/2} = g(x)$

$$\Rightarrow\ f(t) = -(1/\pi)\ \partial_t\ \{\ \int_t^b du\, g(u)/\ [u-t]^{1/2}\ \}$$

$$= (1/\pi)\ \{\ g(b)/[b-t]^{1/2} - \int_t^b du\, g'(u)/[u-t]^{1/2}\ \}$$

And one more time with radicals

S1: $\int_a^x dt\, f(t)/\sqrt{x-t} = g(x)$

$$\Rightarrow\ f(t) = (1/\pi)\partial_t\ \{\ \int_a^t du\, g(u)/\sqrt{t-u}\ \}$$

$$= (1/\pi)\ \{\ g(a)/\sqrt{t-a} + \int_a^t du\, g'(u)/\sqrt{t-u}\ \}$$

S2: $\int_x^b dt\, f(t)/\sqrt{t-x} = g(x)$

$$\Rightarrow\ f(t) = -(1/\pi)\ \partial_t\ \{\ \int_t^b du\, g(u)/\ \sqrt{u-t}\ \}$$

$$= (1/\pi)\ \{\ g(b)/\sqrt{b-t} - \int_t^b du\, g'(u)/\sqrt{u-t}\ \}$$

Appendix B: Show that Polyanin's S1 linear form results agree:

Our linear results from above were:

S1: $\int_a^x dt\, f(t)\ /[x-t]^{\alpha} = g(x)$

$$\Rightarrow\ f(t) = (1/\pi)\sin(\pi\alpha)\ \partial_t\ \{\ \int_a^t du\, g(u)/\ [t-u]^{1-\alpha}\ \}$$

$$= (1/\pi)\sin(\pi\alpha)\ \{\ g(a)/[t-a]^{1-\alpha} + \int_a^t du\, g'(u)/[t-u]^{1-\alpha}\ \}$$

S2: $\int_x^b dt\, f(t)/ [t - x]^{\alpha} = g(x)$

$\Rightarrow$ $f(t) = -(1/\pi)\sin(\pi\alpha)\, \partial_t \{ \int_t^b du\, g(u) / [u- t]^{1-\alpha} \}$

$= (1/\pi)\sin(\pi\alpha) \{ g(b) / [b - t]^{1-\alpha} - \int_t^b du\, g'(u) / [u - t]^{1-\alpha} \}$

We shall now make a series of changes to get things into Polyanin form. First, f→y

S1: $\int_a^x dt\, y(t) / [x - t]^{\alpha} = g(x)$

$\Rightarrow$ $y(t) = (1/\pi)\sin(\pi\alpha)\, \partial_t \{ \int_a^t du\, g(u) / [t - u]^{1-\alpha} \}$

$= (1/\pi)\sin(\pi\alpha) \{ g(a) / [t -a]^{1-\alpha} + \int_a^t du\, g'(u) / [t - u]^{1-\alpha} \}$

S2: $\int_x^b dt\, y(t)/ [t - x]^{\alpha} = g(x)$

$\Rightarrow$ $y(t) = -(1/\pi)\sin(\pi\alpha)\, \partial_t \{ \int_t^b du\, g(u) / [u- t]^{1-\alpha} \}$

$= (1/\pi)\sin(\pi\alpha) \{ g(b) / [b - t]^{1-\alpha} - \int_t^b du\, g'(u) / [u - t]^{1-\alpha} \}$

Next, g → f and at the same time lets do α → λ

S1: $\int_a^x dt\, y(t) / [x - t]^{\lambda} = f(x)$

$\Rightarrow$ $y(t) = (1/\pi)\sin(\pi\lambda)\, \partial_t \{ \int_a^t du\, f(u) / [t - u]^{1-\lambda} \}$

$= (1/\pi)\sin(\pi\lambda) \{ f(a) / [t -a]^{1-\lambda} + \int_a^t du\, f'(u) / [t - u]^{1-\lambda} \}$

S2: $\int_x^b dt\, y(t)/ [t - x]^{\lambda} = f(x)$

$\Rightarrow$ $y(t) = -(1/\pi)\sin(\pi\lambda)\, \partial_t \{ \int_t^b du\, f(u) / [u- t]^{1-\lambda} \}$

$= (1/\pi)\sin(\pi\lambda) \{ f(b) / [b - t]^{1-\lambda} - \int_t^b du\, f'(u) / [u - t]^{1-\lambda} \}$

Now in the y(t) = equations of each set, replace t by x

S1: $\int_a^x dt\, y(t) / [x - t]^{\lambda} = f(x)$

$\Rightarrow$ $y(x) = (1/\pi)\sin(\pi\lambda)\, \partial_x \{ \int_a^x du\, f(u) / [x - u]^{1-\lambda} \}$

$$= (1/\pi)\sin(\pi\lambda)\ \{\ f(a) / [x - a]^{1-\lambda} + \int_a^x du\ f'(u) / [x - u]^{1-\lambda}\ \}$$

S2: $\int_x^b dt\ y(t)/ [t - x]^{\lambda} = f(x)$

$$\Rightarrow\ y(x) = -(1/\pi)\sin(\pi\lambda)\ \partial_x\ \{\ \int_x^b du\ f(u) / [u - x]^{1-\lambda}\ \}$$

$$= (1/\pi)\sin(\pi\lambda)\ \{\ f(b) / [b - x]^{1-\lambda} - \int_x^b du\ f'(u) / [u - x]^{1-\lambda}\ \}$$

Now in the y(x) equations, change integration varaible from u to t

S1: $\int_a^x dt\ y(t)\ / [x - t]^{\lambda} = f(x)$

$$\Rightarrow\ y(x) = (1/\pi)\sin(\pi\lambda)\ \partial_x\ \{\ \int_a^x dt\ f(t) / [x - t]^{1-\lambda}\ \}$$

$$= (1/\pi)\sin(\pi\lambda)\ \{\ f(a) / [x - a]^{1-\lambda} + \int_a^x dt\ f'(t) / [x - t]^{1-\lambda}\ \}$$

S2: $\int_x^b dt\ y(t)/ [t - x]^{\lambda} = f(x)$

$$\Rightarrow\ y(x) = -(1/\pi)\sin(\pi\lambda)\ \partial_x\ \{\ \int_x^b dt\ f(t) / [t - x]^{1-\lambda}\ \}$$

$$= (1/\pi)\sin(\pi\lambda)\ \{\ f(b) / [b - x]^{1-\lambda} - \int_x^b dt\ f'(t) / [t - x]^{1-\lambda}\ \}$$

Polyanin gives the form S1, but I could not find the form S2 in his book. The S1 agrees.

S1: $\int_a^x dt\ y(t)\ / [x - t]^{\lambda} = f(x)$

$$\Rightarrow\ y(x) = (1/\pi)\sin(\pi\lambda)\ \partial_x\ \{\ \int_a^x dt\ f(t) / [x - t]^{1-\lambda}\ \}$$

$$= (1/\pi)\sin(\pi\lambda)\ \{\ f(a) / [x - a]^{1-\lambda} + \int_a^x dt\ f'(t) / [x - t]^{1-\lambda}\ \}$$

7. $\displaystyle\int_a^x \frac{y(t)\,dt}{(x-t)^{\lambda}} = f(x), \qquad 0 < \lambda < 1.$

Generalized Abel equation (generalized Abel integral equation).

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{(x-t)^{1-\lambda}} = \frac{\sin(\pi\lambda)}{\pi}\left[\frac{f(a)}{(x-a)^{1-\lambda}} + \int_a^x \frac{f_t'(t)\,dt}{(x-t)^{1-\lambda}}\right].$$

5. $\int_a^x \frac{y(t)\,dt}{\sqrt{x-t}} = f(x).$

Abel equation (Abel integral equation).

Solution:

$$y(x) = \frac{1}{\pi}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{\sqrt{x-t}} = \frac{f(a)}{\pi\sqrt{x-a}} + \frac{1}{\pi}\int_a^x \frac{f_t'(t)\,dt}{\sqrt{x-t}}.$$

Appendix C: is there a SL problem associated with this transform?

I don't remember Sneddon ever asking this question. We are looking for eigenfunctions of this form, where x is a continuous parameter:

$\varphi_x(t) = 1/\sqrt{x-t} \;\; = (x-t)^{-1/2}$

$\varphi_x'(t) = 1/\sqrt{x-t} \;\; = (1/2)(x-t)^{-3/2}$

$\varphi_x''(t) = 1/\sqrt{x-t} \;\; = (3/4)(x-t)^{-5/2}$

So maybe this ODE, where x is just a parameter

$(4/3)\,(x-t)^2\,\varphi_x''(t) + \varphi_x(t) = 0$

$L = (4/3)\,(x-t)^2\,D^2 + 1$

Maybe our interval is t in (a,x) . L looks self adjoint to me. But $\varphi_x(t)$ blows up at the x endpoint. Consider then

$\int_a^x dt\,|\,\varphi_x(t)\,|^2 = \int_a^x dt\; 1/(x-t)$ = log divergent at the upper endpoint

so this is not an L_2 situation, but it is OK as L_1. Can you have SL with L_1 norm?

When you talk about the Laplace Transform, you don't normally think of eigenfunctions but they are there.

Our candidate eigenfunctions are $\varphi_x(t) = 1/\sqrt{x-t}$ but these are pretty clearly not oscillatory, so how can you ever have orthogonality? You would want something like this

$\int_a^x dt\,(1/\sqrt{x-t}\,)(1/\sqrt{y-t}\,) \sim \delta(x-y)$

which seems pretty unlikely. I think the answer is that SL theory does not apply. I will put this question aside for rainy day 4,234,567.