Fourier Transform Notes PhL 7.28.11

While reading Weiner-Hopf, I realized that earlier Stak comments on Fourier Transforms were hazy, so now is the time and place to get this sharpened up.

We given Vol I page 24. At this point we have established the standard issue Fourier Transform theorem where the projection is 5.32 (Stak assumes a certain sign in the exponential, beware!) and the recovery is 5.34. He claims that the theorem is valid for any f(x) that has a finite L^1 norm, meaning the absolute value is integrable on (-∞,∞).

Notation: In vol I, he used the notation $\mathcal{L}_2(a,b)$ to mean the L^2 norm, but here he is just using $\mathcal{L}(a,b)$ to mean the L^1 norm, page 23. Is this a typo? I cannot find "standard notations" in my Riesz book or the web, so we just have to remember that this is what Stak means. I would write $\mathcal{L}_1(a,b)$. I think f'(x) has to be continuous as well to get the standard form, not sure. But I am not worrying right now about the exact conditions for FT to be valid.

Extension to complex frequency. Look at 5.39. We want to define f^(ω) now for complex ω, not just for real ω as we have up to this point. We formally define then f^(ω) as shown on the first line, and we see on the second line that our "thing here" is a standard issue projection of the function e^{-vx} f(x). If we assume that e^{-vx} f(x) is L_1, then we think we can invert this second line. The inversion would be this:

$$e^{-vx}f(x) = (1/2\pi) \int_{-\infty}^{\infty} du\, e^{-iux}\, f\hat{}(u+iv)$$

where we are thinking of u as the variable conjugate to x, and v is just a fixed parameter. We can now rewrite this inversion formula as an ω integration instead of a u integration. We have

$$\omega = u+iv \qquad d\omega = du \qquad u = x \quad \Rightarrow \quad \omega = x + iv$$

so we get

$$e^{-vx}f(x) = (1/2\pi) \int_{-\infty+iv}^{\infty+iv} d\omega\, e^{-i(\omega-iv)x}\, f\hat{}(\omega)$$

$$= (1/2\pi) \int_{-\infty+iv}^{\infty+iv} d\omega\, e^{-i\omega x}e^{-vx}\, f\hat{}(\omega)$$

or

$$f(x) = (1/2\pi) \int_{-\infty+iv}^{\infty+iv} d\omega\, e^{-i\omega x}\, f\hat{}(\omega)$$

So we can restate our Fourier Transform Theorem this way

$$f\hat{}(\omega) = (1/2\pi) \int_{-\infty}^{\infty} dx\, e^{+i\omega x}\, f(x)$$

$e^{-\nu x} f(x)$ has a finite L_1 norm, so is in $\mathcal{L}(-\infty,\infty)$

$$f(x) = (1/2\pi) \int_{-\infty+i\nu}^{\infty+i\nu} d\omega\, e^{-i\omega x} \hat{f}(\omega)$$

Comment: The above transform is valid for any ν for which $e^{-\nu x} f(x)$ has a finite L_1 norm. The idea is that you might have to increase ν to a larger positive value to force this L_1 norm to be OK. Once you obtain such a ν, all is well, except you have to remember that the horizontal recovery contour must lie above Im(ω) = ν. As Stak points out, increasing ν in general does not help you since $e^{-\nu x}$ blows up for negative x. But it DOES help you if you are dealing with one-side functions $f_+(x)$!!! See below.

Analyticity in ω hidden in the works. *Suppose* the function $e^{-\nu x} f(x)$ had a finite L_1 norm for a continuous range of values of $\nu = (\nu_1, \nu_2)$. In this case, we know that the recovery contour can be any infinite horizontal line such that $\nu_1 < \text{Im}(\omega) < \nu_2$. This in turn implies that you are free to move that contour up or down in this range without changing the integral. That in turn implies that $\hat{f}(\omega)$ must be analytic in this strip in the ω plane!

One-sided functions and Analyticity in a half-plane.

Suppose f(x<0) = 0. We refer to such a function as $f_+(x)$ which I think the literature refers to as "a plus function". Stak calls it a "right-sided function" or a "causal function".

Suppose also that $f_+(x) \sim e^{\alpha x}$ for large positive x.

Then consider $e^{-\nu x} f_+(x)$. For large x>0 this then goes as $e^{-(\nu-\alpha)x}$. We can see that if $\nu > \alpha$, this thing has expo decay and is therefore certainly an L_1 function. Therefore, when $\nu > \alpha$, we can apply our adjusted Fourier Theorem above which now says

$$\hat{f}(\omega) = (1/2\pi) \int_{-\infty}^{\infty} dx\, e^{+i\omega x} f_+(x)$$

$$f_+(x) = (1/2\pi) \int_{-\infty+i\nu}^{\infty+i\nu} d\omega\, e^{-i\omega x} \hat{f}(\omega) \qquad \text{for any value of } \nu > \alpha$$

Now we have some comments to add to the soup. The first is that we can write the projection as

$$\hat{f}(\omega) = (1/2\pi) \int_{0}^{\infty} dx\, e^{+i\omega x} f_+(x)$$

For what domain of ω does this projection exist (converge)? The integrand has the limiting large x form

$$e^{+i\omega x} e^{\alpha x} = e^{+i(u+iv)x} e^{\alpha x} = e^{iux} e^{(\alpha-v)x}$$

We can see that if $\nu > \alpha$, the projection exists. So it exists for the range of ν being (α,∞). This means that the projection $\hat{f}(\omega)$ exists in the half plane Im(ω) > α. This is just the Laplace Transform idea that the Laplace projection is analytic in the right-half s plane. Moreover, as noted in the previous section's last paragraph, not only does $\hat{f}(\omega)$ exist in the half plane Im(ω) > α, it is analytic there!

Now let's repeat the above for a "minus function":

Suppose also that $f_-(x) \sim e^{\alpha x}$ for large negative x.

Then consider $e^{-vx} f_-(x)$. For large $x<0$ this then goes as $e^{-(v-\alpha)x}$. We can see that if $v < \alpha$, this thing has expo decay and is therefore certainly an L_1 function. Therefore, when $v < \alpha$, we can apply our adjusted Fourier Theorem above which now says

$$\hat{f}(\omega) = (1/2\pi) \int_{-\infty}^{\infty} dx\, e^{+i\omega x} f_-(x)$$

$$f_-(x) = (1/2\pi) \int_{-\infty+iv}^{\infty+iv} d\omega\, e^{-i\omega x} \hat{f}(\omega) \qquad \text{for any value of } v < \alpha$$

Now we have some comments to add to the soup. The first is that we can write the projection as

$$\hat{f}(\omega) = (1/2\pi) \int_{-\infty}^{0} dx\, e^{+i\omega x} f_-(x)$$

For what domain of ω does this projection exist (converge)? The integrand has the limiting large x form

$$e^{+i\omega x} e^{\alpha x} = e^{+i(u+iv)x} e^{\alpha x} = e^{iux} e^{(\alpha - v)x}$$

We can see that if $v < \alpha$, the projection exists. So it exists for the range of v being $(-\infty,\alpha)$. This means that the projection $\hat{f}(\omega)$ exists in the half plane $\mathrm{Im}(\omega) < \alpha$. This is just the Laplace Transform idea that the Laplace projection is analytic in the right-half s plane. And as noted above, $\hat{f}(\omega)$ is analytic in the entire half plane $\mathrm{Im}(\omega) < \alpha$.

Decomposition of f(x)

Suppose you take an arbitrary f(x) and trivially write it as $f_+(x) + f_-(x)$ as shown on page 28.
And suppose $f_+(x) \sim e^{\alpha x}$ as $x \to +\infty$ and $f_-(x) \sim e^{\beta x}$ as $x \to -\infty$. We know that $\hat{f}_+(\omega)$ will be analytic for $\mathrm{Im}(\omega) > \alpha$ and $\hat{f}_-(\omega)$ will be analytic for $\mathrm{Im}(\omega) < \beta$. In general we have then that

$f_+(x) \sim e^{\alpha x}$ $\quad$ $\hat{f}_+(\omega)$ analytic for $\mathrm{Im}(\omega) > \alpha$
$f_-(x) \sim e^{\beta x}$ $\quad$ $\hat{f}_+(\omega)$ analytic for $\mathrm{Im}(\omega) < \beta$

If $\beta > \alpha$, then there will be a strip on which we have $\alpha < \mathrm{Im}(\omega) < \beta$ in which both projections will be analytic. Otherwise the known regions of analyticity do not overlap.

If we assume the regions do overlap, we may conclude that

$\hat{f}(\omega) = \hat{f}_+(\omega) + \hat{f}_-(\omega)$ $\quad$ exists for ω in the strip $\alpha < \mathrm{Im}(\omega) < \beta$ and is analytic there.

This follows since the FT is linear.

Can we reverse the logic of this decomposition?

Suppose we know that f^(ω) is analytic in the strip $\alpha < \text{Im}(\omega) < \beta$. Can we come up with the above decomposition into two terms? Are they unique?

To answer these questions, we might start off by examining

$$f(x) = (1/2\pi) \int_{-\infty+iv}^{\infty+iv} d\omega\, e^{-i\omega x} \hat{f}(\omega) \qquad \text{for any value of v with } \alpha < v < \beta$$

Does this integral exist? We need to know that $|\hat{f}(\omega)| \to 0$ at least at the two ends of this integral, then it will at least converge. And since the contour runs through a known analytic region, it won't encounter any poles or cuts. So I would say off hand that f(x) exists and is the FT of f^(ω).

Once we have f(x), we could decompose it as in the previous section. But do we meet our conditions for doing the decomposition? Using the above formula for f(x), what happens as $x \to +\infty$? It is not clear.

In general, I don't see how this ties in with Stak p 316-317 where I have ground to a halt.

Comments on pages 316-317.

(1) This section does not really involve Fourier Transforms. There are no exponentials anywhere. So I don't think my notes above apply in any way whatsoever!

(2) When Stak says $a(\omega) = a_+(\omega) + a_-(\omega)$, is he implying that $a_+(\omega)$ is the Fourier Transform of some plus function $f_+(x)$? Or is he not implying that? [no, he is not implying that!]

A Problem. Suppose someone hands us a function f^(ω) which is analytic in the $\text{Im}(\omega) > \alpha$ half plane. Suppose we *know* this is the FT of a right-sided function $f_+(x)$. Can we use any horizontal contour above α to recover $f_+(x)$? Well, if we assume $f_+(x)$ exists, then we know its f^(ω) will be analytic above some line β. If w are told that f^(ω) which is analytic in the $\text{Im}(\omega) > \alpha$, then we conclude that $\beta = \alpha$. And then the answer is yes.