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Fourier transform

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This article specifically discusses Fourier transformation of functions on the real line; for other kinds of Fourier transformation, see Fourier analysis and list of Fourier-related transforms.

Fourier transforms
Continuous Fourier transform
Fourier series
Discrete Fourier transform
Discrete-time Fourier transform
Related transforms

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In mathematics, the **Fourier transform**, named in honor of French mathematician Joseph Fourier, is a certain linear operator that maps functions to other functions. Loosely speaking, the Fourier transform decomposes a function into a continuous spectrum of its *frequency components*, and the inverse transform synthesizes a function from its spectrum of frequency components. A useful analogy is the relationship between a series of pure notes (the frequency components) and a musical chord (the function itself). In mathematical physics, the Fourier transform of a signal $x(t)$ can be thought of as that signal in the "frequency domain." This is similar to the basic idea of the various other Fourier transforms including the Fourier series of a periodic function. (*See also fractional Fourier transform and linear canonical transform for generalizations.*)

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Definitions

There are several common conventions for defining the Fourier transform of a complex-valued Lebesgue integrable function, x. In communications and signal processing, for instance, it is often the function:

$$X(f) = \int_{-\infty}^{\infty} x(t)\ e^{-i2\pi ft}\,dt, \quad \text{for every real number } f.$$

When the independent variable t represents *time* (with SI unit of seconds), the transform variable f represents ordinary frequency (in hertz). The complex-valued function, X, is said to represent x in the **frequency domain**. I.e., if x is a continuous function, then it can be reconstructed from X by the **inverse transform:**

$$x(t) = \int_{-\infty}^{\infty} X(f)\ e^{i2\pi ft}\,df, \quad \text{for every real number } t.$$

Other notations for $X(f)$ are: $\hat{x}(f)$ and $\mathcal{F}\{x\}(f)$.

The interpretation of X is aided by expressing it in polar coordinate form: $X(f) = A(f)\ e^{i\phi(f)}$, where:

$A(f) = |X(f)|$, the amplitude
$\phi(f) = \angle X(f)$, the phase.

Then the inverse transform can be written:

$$x(t) = \int_{-\infty}^{\infty} A(f)\ e^{i(2\pi ft + \phi(f))}\,df,$$

which is a recombination of all the **frequency components** of $x(t)$. Each component is a complex sinusoid of the form $e^{i2\pi ft}$ whose amplitude is $A(f)$ and whose initial phase angle (at $t = 0$) is $\varphi(f)$.

In mathematics, the Fourier transform is commonly written in terms of angular frequency: $\omega = 2\pi f$, whose units are radians per second.

The substitution $f = \dfrac{\omega}{2\pi}$ into the formulas above produces this convention:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\ e^{-i\omega t}\,dt \text{ [1]}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\ e^{i\omega t}\,d\omega,$$

which is also a bilateral Laplace transform evaluated at $s = i\omega$.

The 2π factor can be split evenly between the Fourier transform and the inverse, which leads to another popular convention:

$$X(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x(t)\ e^{-i\omega t}\,dt$$

$$x(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} X(\omega)\ e^{i\omega t}\,d\omega.$$

This convention and the $X(f)$ convention are **unitary** transforms.

Variations of all three conventions can be created by conjugating the complex-exponential kernel of both the forward and the reverse transform. The signs must be opposites. Other than that, the choice is (again) a matter of convention.

Summary of popular forms of the Fourier transform

angular frequency ω (rad/s)	**unitary**	$X_1(\omega) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)\ e^{-i\omega t}\,dt = \frac{1}{\sqrt{2\pi}} X_2(\omega) = \frac{1}{\sqrt{2\pi}} X_3\left(\frac{\omega}{2\pi}\right)$ $x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X_1(\omega)\ e^{i\omega t}\,d\omega$
	non-unitary	$X_2(\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x(t)\ e^{-i\omega t}\,dt = \sqrt{2\pi}\ X_1(\omega) = X_3\left(\frac{\omega}{2\pi}\right)$ $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(\omega)\ e^{i\omega t}\,d\omega$
ordinary frequency f (hertz)	**unitary**	$X_3(f) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x(t)\ e^{-i2\pi f t}\,dt = \sqrt{2\pi}\ X_1(2\pi f) = X_2(2\pi f)$ $x(t) = \int_{-\infty}^{\infty} X_3(f)\ e^{i2\pi f t}\,df$

Generalization

There are several ways to define the Fourier transform pair. The "forward" and "inverse" transforms are always defined so that the operation of both transforms in either order on a function will return the original function. In other words, the composition of the transform pair is defined to be the identity transformation. Using two arbitrary real constants a and b, the most general definition of the forward 1-dimensional Fourier transform is given by

$$X(\omega) = \sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{+\infty} x(t) e^{-ib\omega t}\,dt$$

and the inverse is given by

$$x(t) = \sqrt{\frac{|b|}{(2\pi)^{1+a}}} \int_{-\infty}^{+\infty} X(\omega) e^{ib\omega t}\,d\omega$$

Note that the transform definitions are symmetric; they can be reversed by simply changing the signs of a and b.

The convention adopted in this article is $(a,b) = (0,1)$. The choice of a and b is usually chosen so that it is geared towards the context in which the transform pairs are being used. The non-unitary convention above is $(a,b) = (1,1)$. Another very common definition is $(a,b) = (0,2\pi)$ which is often used in signal processing applications. In this case, the angular

frequency ω becomes ordinary frequency *f*. If *f* (or ω) and *t* carry units, then their product must be dimensionless. For example, *t* may be in units of time, specifically seconds, and *f* (or ω) would be in hertz (or radian/s).

Properties

In this section, all the results are derived for the following definition (normalization) of the Fourier transform:

$$F(\omega) = \mathcal{F}\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

See also the "Table of important Fourier transforms" section below for other properties of the continuous Fourier transform.

Completeness

We define the Fourier transform on the set of compactly-supported complex-valued functions of **R** and then extend it by continuity to the Hilbert space of square-integrable functions with the usual inner-product. Then $\mathcal{F}$: $L^2(\mathbf{R}) \to L^2(\mathbf{R})$ is a unitary operator. That is. $\mathcal{F}^* = \mathcal{F}^{-1}$ and the transform preserves inner-products (see Parseval's theorem, also described below). Note that, $\mathcal{F}^*$ refers to adjoint of the Fourier Transform operator.

Moreover we can check that,

$$\mathcal{F}^2 = \mathcal{J}, \quad \mathcal{F}^3 = \mathcal{F}^* = \mathcal{F}^{-1}, \quad \text{and} \quad \mathcal{F}^4 = \mathcal{I}$$

where $\mathcal{J}$ is the Time-Reversal operator defined as,

$$||\mathcal{J}\{f\}(t) - f(-t)||_2 = 0$$

and $\mathcal{I}$ is the Identity operator defined as,

$$||\mathcal{I}\{f\}(t) - f(t)||_2 = 0$$

Extensions

The Fourier transform can also be extended to the space of integrable functions defined on $\mathbf{R}^n$

$$\mathcal{F} : L^1(\mathbb{R}^n) \to C(\mathbb{R}^n).$$

where,

$$L^1(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} \mid \int_{\mathbb{R}^n} |f(x)|\, dx < \infty\}.$$

and $C(\mathbf{R}^n)$ is the space of continuous functions on $\mathbf{R}^n$.

In this case the definition usually appears as

$$\mathcal{F}\{f\}(w) \stackrel{\text{def}}{=} \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\mathbb{R}^n} f(x) e^{-i\omega \cdot x}\, dx.$$

where $\omega \in \mathbf{R}^n$ and $\omega \cdot x$ is the inner product of the two vectors ω and *x*.

One may now use this to define the continuous Fourier transform for compactly supported smooth functions, which are dense in $L^2(\mathbf{R}^n)$. The Plancherel theorem then allows us to extend the definition of the Fourier transform to functions on $L^2(\mathbf{R}^n)$ (even those not compactly supported) by continuity arguments. All the properties and formulas listed on this page apply to the Fourier transform so defined.

Unfortunately, further extensions become more technical. One may use the Hausdorff-Young inequality to define the Fourier transform for $f \in L^p(\mathbf{R}^n)$ for $1 \le p \le 2$. The Fourier transform of functions in L^p for the range $2 < p < \infty$ requires the study of distributions, since the Fourier transform of some functions in these spaces is no longer a function, but rather a distribution.

The Plancherel theorem and Parseval's theorem

It should be noted that depending on the author either of these theorems might be referred to as the Plancherel theorem or as Parseval's theorem.

If $f(t)$ and $g(t)$ are square-integrable and $F(\omega)$ and $G(\omega)$ are their Fourier transforms, then we have Parseval's theorem:

$$\int_{\mathbb{R}^n} f(t)\bar{g}(t)\,dt = \int_{\mathbb{R}^n} F(\omega)\bar{G}(\omega)\,d\omega,$$

where the bar denotes complex conjugation. Therefore, the Fourier transformation yields an isometric automorphism of the Hilbert space $L^2(\mathbf{R}^n)$.

The Plancherel theorem, which is equivalent to Parseval's theorem, states that

$$\int_{\mathbb{R}^n} |f(t)|^2\ dt = \int_{\mathbb{R}^n} |F(\omega)|^2\ d\omega.$$

This theorem is usually interpreted as asserting the unitary property of the Fourier transform. See Pontryagin duality for a general formulation of this concept in the context of locally compact abelian groups.

Localization property

As a rule of thumb: the more concentrated $f(t)$ is, the more spread out $F(\omega)$ is. In particular, if we "squeeze" a function in t, it spreads out in ω and vice-versa; and we cannot arbitrarily concentrate both the function and its Fourier transform.

Therefore a function which equals its Fourier transform strikes a precise balance between being concentrated and being spread out. It is easy in theory to construct examples of such functions (called self-dual functions) because the Fourier transform has order 4 (that is, iterating it four times on a function returns the original function). The sum of the four iterated Fourier transforms of any function will be self-dual. There are also some explicit examples of self-dual functions, the most important being constant multiples of the Gaussian function

$$f(t) = \exp\left(\frac{-t^2}{2}\right).$$

This function is related to Gaussian distributions, and in fact, is an eigenfunction of the Fourier transform operators. Again, it is worth stressing that the mere fact that the Gaussian is self-dual does not make it in any way special: many self-dual functions exist.

The trade-off between the compaction of a function and its Fourier transform can be formalized in the form of a **Fourier Uncertainty Principle**. Suppose $f(t)$ and $F(\omega)$ are a Fourier transform pair for a finite-energy (i.e. square-integrable) function. Without loss of generality, we assume that $f(t)$ is normalized:

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = 1.$$

It follows from Parseval's theorem that $F(\omega)$ is also normalized.

Define the expected *location*[2] of a particle (with probability density $|f(t)|^2$) as

$$u_f \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} t|f(t)|^2\,dt.$$

and the expectation value of the *momentum*[2] of the particle (with probability density $|f(\omega)|^2$) as

$$\xi_F \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \omega|F(\omega)|^2\,d\omega.$$

Also define the variances around the above-defined average values as

$$\sigma_f^2 \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} (t-u_f)^2|f(t)|^2\,dt$$

and

$$\sigma_F^2 \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} (\omega-\xi_F)^2|F(\omega)|^2\,d\omega.$$

Then it can be shown that

$$\sigma_f^2\,\sigma_F^2 \geq \frac{1}{4}.$$

The equality is achieved for the Gaussian function listed above, which shows that the Gaussian function is maximally concentrated in "time-frequency".

The most famous practical application of this property is found in quantum mechanics. Following from the axioms of quantum mechanics, the momentum and position wave functions are Fourier transform pairs to within a factor of $h/2\pi$ and are normalized to unity. The above expression then becomes a statement of the Heisenberg uncertainty principle.

The Fourier transform also translates between smoothness and decay. If $f(t)$ is several times differentiable, then $F(\omega)$ decays rapidly towards zero for $\omega \to \pm\infty$.

Analysis of differential equations

Fourier transforms, and the closely related Laplace transforms are widely used in solving differential equations. The Fourier transform is compatible with differentiation in the following sense: if $f(t)$ is a differentiable function with Fourier transform $F(\omega)$, then the Fourier transform of its derivative is given by $i\omega\,F(\omega)$. This can be used to transform differential equations into algebraic equations. Note that this technique only applies to problems whose domain is the whole set of real numbers. By extending the Fourier transform to functions of several variables (as outlined below), partial differential equations with domain $\mathbf{R}^n$ can also be translated into algebraic equations.

Convolution theorem

Main article: Convolution theorem

The Fourier transform translates between convolution and multiplication of functions. If $f(t)$ and $h(t)$ are integrable functions with Fourier transforms $F(\omega)$ and $H(\omega)$ respectively, and if the convolution of f and h exists and is absolutely integrable, then the Fourier transform of the convolution is given by the product of the Fourier transforms $F(\omega)$ $H(\omega)$ (possibly multiplied by a constant factor depending on the Fourier normalization convention).

In the current normalization convention, this means that if

$$g(t) = \{f * h\}(t) = \int_{-\infty}^{\infty} f(s)h(t-s)\,ds$$

where * denotes the convolution operation; then

$$G(\omega) = \sqrt{2\pi} \cdot F(\omega)H(\omega).$$

The above formulas hold true for functions defined on both one- and multi-dimension real space. In linear time invariant (LTI) system theory, it is common to interpret $h(t)$ as the impulse response of an LTI system with input $f(t)$ and output $g(t)$, since substituting the unit impulse for $f(t)$ yields $g(t)=h(t)$. In this case, $H(\omega)$ represents the frequency response of the system.

Conversely, if $f(t)$ can be decomposed as the product of two other functions $p(t)$ and $q(t)$ such that their product $p(t)q(t)$ is integrable, then the Fourier transform of this product is given by the convolution of the respective Fourier transforms $P(\omega)$ and $Q(\omega)$, again with a constant scaling factor.

In the current normalization convention, this means that if $f(t) = p(t)\ q(t)$ then:

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\Big(P(\omega) * Q(\omega)\Big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} P(\alpha)Q(\omega-\alpha)\,d\alpha.$$

Cross-correlation theorem

In an analogous manner, it can be shown that if $g(t)$ is the cross-correlation of $f(t)$ and $h(t)$:

$$g(t) = (f \star h)(t) = \int_{-\infty}^{\infty} \bar{f}(s)\,h(t+s)\,ds$$

then the Fourier transform of $g(t)$ is:

$$G(\omega) = \sqrt{2\pi}\,\overline{F}(\omega)\,H(\omega)$$

where capital letters are again used to denote the Fourier transform.

Tempered distributions

The most general and useful context for studying the continuous Fourier transform is given by the tempered distributions; these include all the integrable functions mentioned above and have the added advantage that the Fourier transform of any tempered distribution is again a tempered distribution and the rule for the inverse of the Fourier transform is universally valid. Furthermore, the useful Dirac delta is a tempered distribution but not a function; its Fourier transform is a constant function (whose specific value depends upon the form of the Fourier transform used). Distributions can be differentiated and the above mentioned compatibility of the Fourier transform with differentiation and convolution remains true for tempered distributions.

Table of important Fourier transforms

The following table records some important Fourier transforms. G and H denote Fourier transforms of $g(t)$ and $h(t)$, respectively. g and h may be integrable functions or tempered distributions. Note that the two most common unitary conventions are included.

Functional relationships

	Signal	Fourier transform unitary, angular frequency $G(\omega) \stackrel{\text{def}}{=}$	Fourier transform unitary, ordinary frequency $G(f) \stackrel{\text{def}}{=}$	Remarks
	$g(t)$	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)e^{-i\omega t}\,dt$	$\int_{-\infty}^{\infty} g(t)e^{-i2\pi f t}\,dt$	
101	$a\cdot g(t)+b\cdot h(t)$	$a\cdot G(\omega)+b\cdot H(\omega)$	$a\cdot G(f)+b\cdot H(f)$	Linearity
102	$g(t-a)$	$e^{-ia\omega}G(\omega)$	$e^{-i2\pi af}G(f)$	Shift in time domain
103	$e^{iat}g(t)$	$G(\omega-a)$	$G\left(f-\frac{a}{2\pi}\right)$	Shift in frequency domain, dual of 2
104	$g(at)$	$\frac{1}{\lvert a\rvert}G\left(\frac{\omega}{a}\right)$	$\frac{1}{\lvert a\rvert}G\left(\frac{f}{a}\right)$	If $\lvert a\rvert$ is large, then $g(at)$ is concentrated around 0 and $\frac{1}{\lvert a\rvert}G\left(\frac{\omega}{a}\right)$ spreads out and flattens. It is interesting to consider the limit of this as $\lvert a\rvert$ tends to infinity - the delta function
105	$G(t)$	$g(-\omega)$	$g(-f)$	Duality property of the Fourier transform. Results from swapping "dummy" variables of t and ω.
106	$\frac{d^n g(t)}{dt^n}$	$(i\omega)^n G(\omega)$	$(i2\pi f)^n G(f)$	Generalized derivative property of the Fourier transform
107	$t^n g(t)$	$i^n\frac{d^n G(\omega)}{d\omega^n}$	$\left(\frac{i}{2\pi}\right)^n\frac{d^n G(f)}{df^n}$	This is the dual of 106
108	$(g*h)(t)$	$\sqrt{2\pi}G(\omega)H(\omega)$	$G(f)H(f)$	$g*h$ denotes the convolution of g and h — this rule is the convolution theorem
109	$g(t)h(t)$	$\frac{(G*H)(\omega)}{\sqrt{2\pi}}$	$(G*H)(f)$	This is the dual of 108
110	$g(t)$ is purely real, and an even function	$G(\omega)$ and $G(f)$ are purely real, and even functions		
111	$g(t)$ is purely real, and an odd function	$G(\omega)$ and $G(f)$ are purely imaginary, and odd functions		

Square-integrable functions

Signal	Fourier transform unitary, angular frequency	Fourier transform unitary, ordinary frequency	Remarks
$g(t)$	$G(\omega) \stackrel{\text{def}}{=}$	$G(f) \stackrel{\text{def}}{=}$	

	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)e^{-i\omega t}\mathrm{d}t$	$\int_{-\infty}^{\infty} g(t)e^{-i2\pi ft}\mathrm{d}t$	
201 $\mathrm{rect}(at)$	$\frac{1}{\sqrt{2\pi a^2}}\cdot\mathrm{sinc}\left(\frac{\omega}{2\pi a}\right)$	$\frac{1}{\lvert a\rvert}\cdot\mathrm{sinc}\left(\frac{f}{a}\right)$	The rectangular pulse and the normalized sinc function
202 $\mathrm{sinc}(at)$	$\frac{1}{\sqrt{2\pi a^2}}\cdot\mathrm{rect}\left(\frac{\omega}{2\pi a}\right)$	$\frac{1}{\lvert a\rvert}\cdot\mathrm{rect}\left(\frac{f}{a}\right)$	Dual of rule 201. The rectangular function is an idealized low-pass filter, and the sinc function is the non-causal impulse response of such a filter.
203 $\mathrm{sinc}^2(at)$	$\frac{1}{\sqrt{2\pi a^2}}\cdot\mathrm{tri}\left(\frac{\omega}{2\pi a}\right)$	$\frac{1}{\lvert a\rvert}\cdot\mathrm{tri}\left(\frac{f}{a}\right)$	*tri* is the triangular function
204 $\mathrm{tri}(at)$	$\frac{1}{\sqrt{2\pi a^2}}\cdot\mathrm{sinc}^2\left(\frac{\omega}{2\pi a}\right)$	$\frac{1}{\lvert a\rvert}\cdot\mathrm{sinc}^2\left(\frac{f}{a}\right)$	Dual of rule 203.
205 $e^{-\alpha t^2}$	$\frac{1}{\sqrt{2\alpha}}\cdot e^{-\frac{\omega^2}{4\alpha}}$	$\sqrt{\frac{\pi}{\alpha}}\cdot e^{-\frac{(\pi f)^2}{\alpha}}$	Shows that the Gaussian function $\exp(-\alpha t^2)$ is its own Fourier transform. For this to be integrable we must have $\mathrm{Re}(\alpha)>0$.
206 $e^{iat^2}=e^{-\alpha t^2}\big\vert_{\alpha=-ia}$	$\frac{1}{\sqrt{2a}}\cdot e^{-i\left(\frac{\omega^2}{4a}-\frac{\pi}{4}\right)}$	$\sqrt{\frac{\pi}{a}}\cdot e^{-i\left(\frac{\pi^2 f^2}{a}-\frac{\pi}{4}\right)}$	common in optics
207 $\cos(at^2)$	$\frac{1}{\sqrt{2a}}\cos\left(\frac{\omega^2}{4a}-\frac{\pi}{4}\right)$	$\sqrt{\frac{\pi}{a}}\cos\left(\frac{\pi^2 f^2}{a}-\frac{\pi}{4}\right)$	
208 $\sin(at^2)$	$\frac{-1}{\sqrt{2a}}\sin\left(\frac{\omega^2}{4a}-\frac{\pi}{4}\right)$	$-\sqrt{\frac{\pi}{a}}\sin\left(\frac{\pi^2 f^2}{a}-\frac{\pi}{4}\right)$	
209 $\mathrm{e}^{-a\lvert t\rvert}$	$\sqrt{\frac{2}{\pi}}\cdot\frac{a}{a^2+\omega^2}$	$\frac{2a}{a^2+4\pi^2 f^2}$	*a>0*
210 $\frac{1}{\sqrt{\lvert t\rvert}}$	$\frac{1}{\sqrt{\lvert\omega\rvert}}$	$\frac{1}{\sqrt{\lvert f\rvert}}$	the transform is the function itself
211 $J_0(t)$	$\sqrt{\frac{2}{\pi}}\cdot\frac{\mathrm{rect}\left(\frac{\omega}{2}\right)}{\sqrt{1-\omega^2}}$	$\frac{2\cdot\mathrm{rect}(\pi f)}{\sqrt{1-4\pi^2 f^2}}$	$J_0(t)$ is the Bessel function of first kind of order 0
212 $J_n(t)$	$\sqrt{\frac{2}{\pi}}\frac{(-i)^n T_n(\omega)\mathrm{rect}\left(\frac{\omega}{2}\right)}{\sqrt{1-\omega^2}}$	$\frac{2(-i)^n T_n(2\pi f)\mathrm{rect}(\pi f)}{\sqrt{1-4\pi^2 f^2}}$	it's the generalization of the previous transform; $T_n(t)$ is the Chebyshev polynomial of the first kind.
213 $\frac{J_n(t)}{t}$	$\sqrt{\frac{2}{\pi}}\frac{i}{n}(-i)^n\cdot U_{n-1}(\omega)\cdot\sqrt{1-\omega^2}\mathrm{rect}\left(\frac{\omega}{2}\right)$	$\frac{2\mathrm{i}}{n}(-i)^n\cdot U_{n-1}(2\pi f)\cdot\sqrt{1-4\pi^2 f^2}\mathrm{rect}(\pi f)$	$U_n(t)$ is the Chebyshev polynomial of the second kind
214 $\mathrm{sech}(at)$	$\frac{1}{a}\sqrt{\frac{\pi}{2}}\mathrm{sech}\left(\frac{\pi}{2a}\omega\right)$	$\frac{\pi}{a}\mathrm{sech}\left(\frac{\pi^2}{a}f\right)$	Hyperbolic secant is its own Fourier transform

Distributions

	Signal	Fourier transform unitary, angular frequency	Fourier transform unitary, ordinary frequency	Remarks
		$G(\omega) \stackrel{\text{def}}{=}$	$G(f) \stackrel{\text{def}}{=}$	
	$g(t)$	$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) e^{-i\omega t}\, dt$	$\int_{-\infty}^{\infty} g(t) e^{-i2\pi f t}\, dt$	
301	1	$\sqrt{2\pi} \cdot \delta(\omega)$	$\delta(f)$	$\delta(\omega)$ denotes the Dirac delta distribution.
302	$\delta(t)$	$\frac{1}{\sqrt{2\pi}}$	1	Dual of rule 301.
303	e^{iat}	$\sqrt{2\pi} \cdot \delta(\omega - a)$	$\delta(f - \frac{a}{2\pi})$	This follows from and 103 and 302.
304	$\cos(at)$	$\sqrt{2\pi}\frac{\delta(\omega-a)+\delta(\omega+a)}{2}$	$\frac{\delta(f-\frac{a}{2\pi})+\delta(f+\frac{a}{2\pi})}{2}$	Follows from rules 101 and 303 using Euler's formula: $\cos(at) = (e^{iat} + e^{-iat})/2.$
305	$\sin(at)$	$i\sqrt{2\pi}\frac{\delta(\omega+a)-\delta(\omega-a)}{2}$	$i\frac{\delta(f+\frac{a}{2\pi})-\delta(f-\frac{a}{2\pi})}{2}$	Also from 101 and 303 using $\sin(at) = (e^{iat} - e^{-iat})/(2i).$
306	t^n	$i^n\sqrt{2\pi}\delta^{(n)}(\omega)$	$\left(\frac{i}{2\pi}\right)^n \delta^{(n)}(f)$	Here, n is a natural number. $\delta^n(\omega)$ is the n-th distribution derivative of the Dirac delta. This rule follows from rules 107 and 302. Combining this rule with 1, we can transform all polynomials.
307	$\frac{1}{t}$	$-i\sqrt{\frac{\pi}{2}}\,\mathrm{sgn}(\omega)$	$-i\pi \cdot \mathrm{sgn}(f)$	Here $\mathrm{sgn}(\omega)$ is the sign function; note that this is consistent with rules 107 and 302.
308	$\frac{1}{t^n}$	$-i\sqrt{\frac{\pi}{2}} \cdot \frac{(-i\omega)^{n-1}}{(n-1)!}\,\mathrm{sgn}(\omega)$	$-i\pi\frac{(-i2\pi f)^{n-1}}{(n-1)!}\,\mathrm{sgn}(f)$	Generalization of rule 307.
309	$\mathrm{sgn}(t)$	$\sqrt{\frac{2}{\pi}} \cdot \frac{1}{i\ \omega}$	$\frac{1}{i\pi f}$	The dual of rule 307.
310	$u(t)$	$\sqrt{\frac{\pi}{2}}\left(\frac{1}{i\pi\omega} + \delta(\omega)\right)$	$\frac{1}{2}\left(\frac{1}{i\pi f} + \delta(f)\right)$	Here *u*(*t*) is the Heaviside unit step function; this follows from rules 101 and 309.
311	$e^{-at}u(t)$	$\frac{1}{\sqrt{2\pi}(a + i\omega)}$	$\frac{1}{a + i2\pi f}$	*u*(*t*) is the Heaviside unit step function and $a > 0$.
312	$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{\sqrt{2\pi}}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$	$\frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right)$	The Dirac comb — helpful for explaining or understanding the transition from continuous to discrete time.

Fourier transform properties

Notation: $f(t) \quad \stackrel{\mathcal{F}}{\Longleftrightarrow} \quad F(\omega)$ denotes that *f*(*t*) and *F*(ω) are a Fourier transform pair.

Linearity

$$af_1(t) + bf_2(t) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad aF_1(\omega) + bF_2(\omega)$$

Convolution

$$f_1(t) * f_2(t) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad F_1(\omega)F_2(\omega)$$

Conjugation

$$\overline{f(t)} \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad \overline{F(-\omega)}$$

Scaling

$$f(at) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad \frac{1}{|a|}F\left(\frac{\omega}{a}\right), \qquad a \in \mathbb{R}, a \neq 0$$

Time reversal

$$f(-t) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad F(-\omega)$$

Time shift

$$f(t - t_0) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad e^{-i\omega t_0}F(\omega)$$

Modulation (multiplication by complex exponential)

$$f(t) \cdot e^{i\omega_0 t} \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad F(\omega - \omega_0) \qquad \omega_0 \in \mathbb{R},$$

Multiplication by sin $\omega_0 t$

$$f(t)\sin\omega_0 t \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad \frac{i}{2}[F(\omega + \omega_0) - F(\omega - \omega_0)]$$

Multiplication by cos $\omega_0 t$

$$f(t)\cos\omega_0 t \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad \frac{1}{2}[F(\omega + \omega_0) + F(\omega - \omega_0)]$$

Integration

$$\int_{-\infty}^{t} f(u)\,du \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad \frac{1}{i\omega}F(\omega) + \pi F(0)\delta(\omega)$$

Parseval's theorem

$$\int_{\mathbb{R}} f(t) \cdot \overline{g(t)}\, dt = \int_{\mathbb{R}} F(\omega) \cdot \overline{G(\omega)}\, d\omega$$

See also

- Fourier series
- Fast Fourier transform
- Laplace transform
- Discrete Fourier transform
- Fractional Fourier transform
- Linear canonical transform
- Fourier sine transform
- Short-time Fourier transform

Notes

1. ^ *X*(*f*) and *X*(ω) represent different, but related, functions, as shown in the table labeled *Summary of popular forms of the Fourier transform*.
2. ^ [a] [b] Location, momentum and particle do not have any physical meaning here; they are simply convenient monikers chosen with analogy to the interpretation used in the Heisenberg Uncertainty Principle.

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External links

- Tables of Integral Transforms (http://eqworld.ipmnet.ru/en/auxiliary/aux-inttrans.htm) at EqWorld: The World of Mathematical Equations.
- *Eric W. Weisstein, Fourier Transform (http://mathworld.wolfram.com/FourierTransform.html) at MathWorld.*
- Fourier Transform Module by John H. Mathews (http://math.fullerton.edu/mathews/c2003/FourierTransformMod.html)
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