Schwartz Functions PhL 4.1.11

These are notes taken while reading a PDF entitled "Math 395 Fall 2009" by Mitya Boyarchenko at Ann Arbor. I like the writing very much.

[1] In his presentation, the FT was first presented (elsewhere) for L^1 functions. Notation is

$$\hat{f}(\xi) = FT(f(x)) = (\mathcal{F}f)(\xi)$$

He shows some measure dm(x) in both directions, perhaps for Cartesians this is just the usual dx, perhaps he has some deeper meaning, but that is all in some previous notes. He points out that $\mathcal{F}^{-1}$ in his inversion formula is not really the inverse of $\mathcal{F}$. If you have some domain D(F) for $\mathcal{F}$ within L^2, you presumably have some range ("image") R(F) also in L^2. Going the other way we would have some $D(F^{-1})$ and $R(F^{-1})$. He says in general $D(F^{-1}) \neq R(F)$ so there might be points in R(F) which have no inverses. So the point so far is to keep in mind that you have to worry about one to one when talking inverses and that means looking at domains and ranges.

He then goes back to the L^1 world and asks: **can we find some S within L**^{1} which is dense in L^1 and which is invariant under F, meaning F:S→S. We note that $e^{-x^2/2}$ as a single function would define such a space S, but is not dense in L^1. He notes that L^1 is further problematical because F can take an L^1 function to something outside L_1, he gives an example.

So instead he says let's try $C_0^\infty(R)$ functions where the subscript 0 means "**compact support"** and I know what that means. The ∞ means infinitely differentiable to this author. And C means our function is allow to map f:R→C. He notes that for R, compact support and bounded support are the same by the Heine-Borel theorem, good.

[2] He claims that $C_0^\infty(R)$ is dense in L^1 and in L^p. **But** $C_0^\infty(R)$ is not a candidate for S because $\hat{f}(\xi)$ never has compact support even if f(x) does, and this is obvious to me. That is to say, $\hat{f}(\xi)$ exists for all ξ in C and thus cannot be said to have compact support. He also claims that $\hat{f}(\xi)$ is analytic in all ξ . Some theorem he quotes says $\hat{f}(\xi)$ can only have discrete 0's in ξ and these zeros are countable. I would say that if $\hat{f}(\xi)$ is analytic, it can be done as a power series which can be factored into these discrete countable zeros. I don't see at this point why he is mentioning these zeros, the main idea is that $\hat{f}(\xi)$ does not have compact support.

Lemma 36.1 and its induction generalization: if $x^n f(x)$ is "integrable" (perhaps in L^p sense?) for all integers n, then $\hat{f}(\xi)$ is infinitely differentiable. This was a homework assignment, I think it would be easy to show, a matter of parts integration. I suspect that a "measurable function" is perhaps an LI function.

Lemma 36.2 . Suppose f(x) and f'(x) are both integrable and f→0 at ±∞ and f(x) is "continuously differentiable". The claim is that $\xi\,\hat{f}(\xi) \to 0$. (in the previous theorem our comment might be that $\xi\,\hat{f}(\xi)$ was integrable, but certainly if it is integrable, we must have $\xi\,\hat{f}(\xi) \to 0$). Generalizing by induction again, if $f^{(m)}(x)$ is integrable for all m, then $\xi^m\,\hat{f}(\xi) \to 0$ for all m.

He then claims that if f(x) is bounded and $x^n f(x) \to 0$ for all n, then $x^n f(x)$ is integrable for all n. How about the function $f(x) = x^{-1/2}$ (n includes 0)? Well, your f(x) has to be L^p to start with. OK, I accept this claim for now.

[3] **A Schwartz function** is (1) infinitely differentiable; (2) $x^m f^{(n)}(x) \to 0$ for all $m,n \geq 0$. He calls this space of functions $\mathcal{S}(R)$. He notes that $C_0^\infty(R)$ lies within $\mathcal{S}(R)$ and tells you one kind of Schwartz function class (those with compact support). The other class are gaussians and variations of gaussians.

Lemma 36.5: F:S→S ! A quick proof is given based on earlier results. He then claims this little theorem

$$\int dx\, f(x)\, \hat{g}(x) = \int dx\, \hat{f}(x)\, g(x) \qquad \text{f and g in } L^1$$

He outlines a proof using Fubini's theorem and mention of Lebesgue measure, and of course the same is true of you replace ^ with a symbol meaning inverse FT as in 36.2

Fourier Inversion Theorem: The claim here is that if you compute $\hat{f}(x)$, you successfully recover f(x) with the inverse FT, assuming f(x) is a Schwartz function. In other words, with this restriction the inverse of F always exists. We are still talking L^1 here.

[4] He expends half of this page proving the above theorem in sort of dense Stak fashion.

Corollary 36.7. He claims first that F:S→S is a linear isomorphism, meaning F is a linear operator, and meaning everything is 1 to 1 in the mapping. Secondly, $F^2 f(x) \to f(-x)$, which seems to dispute my claim of an inverse? Well I have only claimed that if f(x) is Schwartz, then F exists and F^{-1} exists and F^{-1} brings you back to f(x). I have not claimed that $F^{-1} = F$. So, F:S→S is "bijective" which means it is 1 to 1, and it is "onto" meaning it hits all of the set S as range. For example, let $\hat{f}(x) = F f(x)$ be in the range of F. As we let f(x) span the domain S, we get all possible range elements. If we pick a range element and come back with it, we have $F^2 f(x)$ back in the domain of F. Somehow this shows that we hit all elements in S with F, onto, fudge for now. He notes that $F^4 = (2\pi)^2 1$.

Theorem 36.8. If $f \to \hat{f}$ and both are L^1, then $f = F^{-1}(\hat{f})$. He gives a proof. Notice that in general if f is L^1, you won't get $\hat{f}$ being L^1.

[5] Lemma 36.9 says that if f is L^1 and its integral with all test functions φ is 0, then f = 0 a.e.. Note that the letters a.e. mean Almost Everywhere, something Stak talks about somewhere.

I think up to this point, he has only been talking L^1, so the whole Schwartz function set S is within that world. Here for the first time he talks L^2 and gives

Plancherel's Formula: $\int dx\, f(x)g(x)^* = \int dx\, \hat{f}(x)\, \hat{g}\,(x)^*$ provided f and g are in S.

So the PDF ends at this point. So what are the main points of this PDF?

(1) If you limit your interest to "functions of slow growth" meaning perhaps Schwartz functions, then the FT always gives a result and is 1 to 1.

(2) If you have a "function of slow growth" on one side of the FT (this makes the integral converge), then on the other side your function will be infinitely differentiable. Conversely, if on one side you have a function f(x) which is infinitely differentiable, then $\hat{f}(\xi)$ will be a function of slow growth so the inverse FT will converge.

The writer is pretty good, though I would say I followed about 40% of what he had to say.