The Poisson Summation Formula PhL 3.30.11

Overview: This formula is in the context of the regular Fourier Transform. If we define this transform using frequency instead of angular frequency as the variable

$$\varphi^{\#}(\nu) \equiv \int_{-\infty}^{\infty} dy\, \varphi(y) e^{i2\pi\nu y} \qquad \varphi(y) = (1/2\pi) \int_{-\infty}^{\infty} d\nu\, \varphi^{\#}(\nu) e^{-i2\pi\nu y}$$

Then the Poisson Summation formula says this:

$$\Sigma_{n=-\infty}^{\infty} \varphi(n) = \Sigma_{n=-\infty}^{\infty} \varphi^{\#}(n)$$

I have about 5 inches of comment on this in the raw Chap 1 notes, but let's do it again here from scratch.

(a) It all begins back on page 39 where he considers, in distribution theory, the derivative of a piecewise differentiable function and he shows that

$$f' = [f'] + \Sigma_k\ \Delta f_k\, \delta(x-a_k)$$

where $[f']$ is the derivative away from the "jumps" which are Δf_k located at $x = a_k$.

(b) On page 44-45, he applies this idea to the sawtooth function on page 45 and he concludes that, for this function,

$$f'(x) = 1 - \Sigma_{k=-\infty}^{\infty} \delta(x-k)$$

where $[f'] = 1$ in this case, and $\Delta f_k = 1$ as well. So this is one expression for $f'(x)$. But we can get another expression by doing a Fourier Series expansion of the sawtooth which tells us that

$$f(x) = -(2/\pi) \Sigma_{n=1}^{\infty} \sin(2\pi nx)/(2n)$$

and this is a routine fact that I won't derive in detail, Schaum p 131 or whatever. So we can differentiate this to get our "other" expression for $f'(x)$:

$$f'(x) = -2 \Sigma_{n=1}^{\infty} \cos(2\pi nx) \qquad // \; = 1 - \Sigma_{k=-\infty}^{\infty} \delta(x-k)$$

We now set our two expressions equal to learn that (old Neumann's factor ε_n appears here!)

$$\Sigma_{k=-\infty}^{\infty} \delta(x-k) = 1 + 2 \Sigma_{n=1}^{\infty} \cos(2\pi nx) = \Sigma_{n=0}^{\infty} \varepsilon_n \cos(2\pi nx) \qquad 1.22$$

(c) We now move to page 46 and Example 7. The first action is to rewrite the above using

$\cos(2\pi nx) = (1/2)\,(e^{i2\pi nx} + e^{-i2\pi nx})$ so that

$$\Sigma_{k=-\infty}^{\infty} \delta(x-k) = 1 + \Sigma_{n=1}^{\infty} (e^{i2\pi nx} + e^{-i2\pi nx}) = \Sigma_{n=-\infty}^{\infty} e^{i2\pi nx} \qquad \text{p 46 A}$$

We next set $x = y/\lambda$ to get

$$\Sigma_{k=-\infty}^{\infty} \delta(y/\lambda-k) = \Sigma_{n=-\infty}^{\infty} e^{i2\pi ny/\lambda} \qquad \text{p 46 B}$$

$$= \Sigma_{k=-\infty}^{\infty} |\lambda| \, \delta(y-k\lambda)$$

Multiply both sides by $\varphi(y)$ and integrate on y to get

$$\int_{-\infty}^{\infty} dy \, \varphi(y) \Sigma_{k=-\infty}^{\infty} |\lambda| \, \delta(y-k\lambda) = \int_{-\infty}^{\infty} dy \, \varphi(y) \Sigma_{n=-\infty}^{\infty} e^{i2\pi ny/\lambda}$$

$$\Sigma_{k=-\infty}^{\infty} |\lambda| \int_{-\infty}^{\infty} dy \, \varphi(y) \, \delta(y-k\lambda) = \Sigma_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dy \, \varphi(y) \, e^{i2\pi ny/\lambda}$$

$$\Sigma_{k=-\infty}^{\infty} \varphi(k\lambda) = (1/|\lambda|) \Sigma_{n=-\infty}^{\infty} \Phi(2\pi n/\lambda) \qquad \text{page 47 1.23b}$$

This thing is the Poisson Summation Formula. We have used the Fourier Integral definition

$$\int_{-\infty}^{\infty} dy \, \varphi(y) e^{iky} \equiv \Phi(k) \qquad \varphi(y) = (1/2\pi) \int_{-\infty}^{\infty} dk \, \Phi(k) e^{-iky}$$

Using the Volume 2 notation $\varphi\hat{}$ for a Fourier Integral Transform we can write 1.23b as

$$\Sigma_{k=-\infty}^{\infty} \varphi(k\lambda) = (1/|\lambda|) \Sigma_{n=-\infty}^{\infty} \varphi\hat{}(2\pi n/\lambda)$$

and λ can be an arbitrary number. If we set $\lambda = 1$ we get

$$\Sigma_{k=-\infty}^{\infty} \varphi(k) = \Sigma_{n=-\infty}^{\infty} \varphi\hat{}(2\pi n)$$

and this is Vol II page 212 7.47 ! We could alternatively selected $\lambda = 2\pi$ to get

$$\Sigma_{k=-\infty}^{\infty} \varphi(k/2\pi) = (1/2\pi) \Sigma_{n=-\infty}^{\infty} \varphi\hat{}(n)$$

Now in my Spectral Notes page 11 we know there is something called Parseval's Theorem

$$\int_{-\infty}^{\infty} dy \, |\varphi(y)|^2 = (1/2\pi) \int_{-\infty}^{\infty} dk \, |\varphi\hat{}(k)|^2$$

which has a similar look to our series result, but it involves the squares of the functions! There is a Fourier Series version of this theorem that is this: (wiki on Parseval)

$$A(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx} \qquad \sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |A(x)|^2 dx,$$

and there are discrete Fourier transform versions of this as well.

For me, the Poisson Summation Formula is completely new.

(d) Fourier Integral Transform in Frequency Versus Angular Frequency.

Above we had the angular thing like so:

$$\int_{-\infty}^{\infty} dy\ \varphi(y) e^{iky} \equiv \Phi(k) \qquad \varphi(y) = (1/2\pi) \int_{-\infty}^{\infty} dk\ \Phi(k) e^{-iky}$$

Now we can write $\omega = 2\pi f$ which here would be $k = 2\pi\nu$ where ν is waves per meter say. Then we get

$$\int_{-\infty}^{\infty} dy\ \varphi(y) e^{i2\pi\nu y} \equiv \Phi(2\pi\nu) \qquad \varphi(y) = (1/2\pi) \int_{-\infty}^{\infty} d\nu\ \Phi(2\pi\nu) e^{-i2\pi\nu y}$$

We would in this case define a new symbol, let's say $\varphi^{\#}(\nu) \equiv \Phi(2\pi\nu)$ and then we have

$$\int_{-\infty}^{\infty} dy\ \varphi(y) e^{i2\pi\nu y} \equiv \varphi^{\#}(\nu) \qquad \varphi(y) = (1/2\pi) \int_{-\infty}^{\infty} d\nu\ \varphi^{\#}(\nu) e^{-i2\pi\nu y}$$

and this is our Fourier Integral Transform using the x and ν as the conjugate variables.

(e) We had from above

$$\Sigma_{k=-\infty}^{\infty} \varphi(k\lambda) = (1/|\lambda|) \Sigma_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dy\ \varphi(y)\ e^{i2\pi n y/\lambda}$$

So now we can write

$$\int_{-\infty}^{\infty} dy\ \varphi(y)\ e^{i2\pi n y/\lambda} = \varphi^{\#}(n/\lambda)$$

and then our Poisson Sum gizmo says

$$\Sigma_{k=-\infty}^{\infty} \varphi(k\lambda) = (1/|\lambda|) \Sigma_{n=-\infty}^{\infty} \varphi^{\#}(n/\lambda)$$

and now we can set $\lambda = 1$ to get

$$\Sigma_{k=-\infty}^{\infty} \varphi(k) = \Sigma_{n=-\infty}^{\infty} \varphi^{\#}(n)$$

which is certainly a compact looking result! In wiki it appears this way

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \hat{f}\,(k)$$

What is this Poisson thing saying? If you add up the values of f(x) at all the integers, you get the same thing you get by adding up $f^{\#}(\nu)$ at all its integers! This really seems amazing, but I just derived it above following Stak. Now we started with a test function, so you have to make sure that both series converge when you apply something like this.