

36. SCHWARTZ FUNCTIONS, FOURIER INVERSION, PLANCHEREL FORMULA

36.1. Goal. We would like to discuss Fourier transform on some classes of functions other than $L^1(\mathbb{R})$; in particular, we would like to extend it to $L^2(\mathbb{R})$. As a key ingredient we will study the space of Schwartz functions on $\mathbb{R}$ (which is contained in $L^p(\mathbb{R})$ for every $1 \leq p \leq \infty$, and is dense in $L^p(\mathbb{R})$ when $1 \leq p < \infty$). As a by-product of our discussion we will also obtain a proof of the Fourier inversion theorem, which was stated earlier.

36.2. Recollections. Let us recall the definitions of the Fourier and the inverse Fourier transforms. If $f = f(x) \in L^1(\mathbb{R})$, the *Fourier transform* of f is defined to be the function

$$\widehat{f}(\xi) = (\mathcal{F}f)(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) dm(x), \quad \xi \in \mathbb{R}.$$

If $g = g(\xi) \in L^1(\mathbb{R})$, the *inverse Fourier transform* of g is defined to be the function

$$(\mathcal{F}^{-1}g)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} g(\xi) dm(\xi), \quad x \in \mathbb{R}.$$

As I mentioned earlier, $\mathcal{F}^{-1}$ is *not* the inverse map to $\mathcal{F}$ in the strict sense: for example, the domain of $\mathcal{F}^{-1}$ (which is the space $L^1(\mathbb{R})$, by definition) is not equal to the image of $\mathcal{F}$ (which is hard to describe explicitly, but certainly contains many functions that are not integrable on $\mathbb{R}$). So for the time being it is better to think of $\mathcal{F}$ and $\mathcal{F}^{-1}$ as two separate maps, even though they are “inverse to each other in some sense.”

36.3. Motivation. As a motivation behind the notion of a Schwartz function, let us consider the following question. Is there a convenient subspace $\mathcal{S} \subset L^1(\mathbb{R})$ that is invariant under the Fourier transform, in the sense that if $f \in \mathcal{S}$, then $\widehat{f} \in \mathcal{S}$? The adjective “convenient” (which is not a mathematical term) means that $\mathcal{S}$ should have a fairly explicit description, and that it should be *dense* in $L^1(\mathbb{R})$ with respect to the L^1 norm.

For example, the 1-dimensional space spanned by the function $f(x) = e^{-x^2/2}$ is invariant under $\mathcal{F}$, because, as we saw in an earlier lecture, $\widehat{f}(\xi) = \sqrt{2\pi} \cdot e^{-\xi^2/2}$. However, this space is not dense in $L^1(\mathbb{R})$, and hence is not a good candidate.

Note also that the space $L^1(\mathbb{R})$ itself is not going to work because there are many functions $f \in L^1(\mathbb{R})$ such that $\widehat{f} \notin L^1(\mathbb{R})$, an easy example being $f(x) = \mathbb{1}_{[0,1]}(x)$.

36.4. Smooth functions with compact support. As a next candidate, let us try the space of infinitely differentiable functions with compact support on $\mathbb{R}$. By definition, this is the space $C_0^\infty(\mathbb{R})$ consisting of infinitely differentiable functions $f : \mathbb{R} \rightarrow \mathbb{C}$ for which there exists a compact subset $K \subset \mathbb{R}$ (depending on f) such that¹ $f(x) = 0$ for all $x \notin K$.

¹In general, the *support* of a function f is defined as the closure of the set of points x where $f(x) \neq 0$. Thus, for a function on $\mathbb{R}^n$, having compact support is equivalent to having bounded support, by the Heine-Borel theorem.

By this week's homework, $C_0^\infty(\mathbb{R})$ is dense in $L^1(\mathbb{R})$ (in fact, it is dense in $L^p(\mathbb{R})$ for every $1 \leq p < \infty$). However, this space is *not* invariant under the Fourier transform. Even worse: if $f \in C_0^\infty(\mathbb{R})$ and $f \not\equiv 0$, then $\widehat{f}$ can *never* have compact support on $\mathbb{R}$. The reason is as follows. One can easily check that if $f \in C_0^\infty(\mathbb{R})$, then $\widehat{f}(\xi)$ is defined for all $\xi \in \mathbb{C}$, and $\widehat{f}$ is an *entire* function, that is, $\widehat{f}$ is holomorphic ($\iff$ analytic) on the whole complex plane. This implies (by a general result of complex analysis) that if $\widehat{f} \not\equiv 0$, then the set of zeroes of $\widehat{f}$ in the complex plane is *discrete*, and therefore at most countable. In particular, the restriction of $\widehat{f}$ to $\mathbb{R}$ certainly cannot have compact support.

36.5. Schwartz functions. It turns out that there is a space $\mathcal{S}(\mathbb{R})$ that “sits between” $C_0^\infty(\mathbb{R})$ and $L^1(\mathbb{R})$ that is invariant under $\mathcal{F}$. This is precisely the space of Schwartz functions, which we will define soon. As motivation for the definition, let us consider the following two statements, which are part of this week's homework.

Lemma 36.1. *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a measurable function such that both $f(x)$ and $x \cdot f(x)$ are integrable. Then $\widehat{f}(\xi)$ is differentiable w.r.t. ξ , and, in fact,*

$$\frac{d\widehat{f}}{d\xi} = \mathcal{F}(-ixf(x)).$$

Using this lemma, one immediately checks by induction that if $f : \mathbb{R} \rightarrow \mathbb{C}$ is a measurable function such that $x^n \cdot f(x)$ is integrable for every integer $n \geq 0$, then the Fourier transform $\widehat{f} : \mathbb{R} \rightarrow \mathbb{C}$ is an infinitely differentiable function.

Lemma 36.2. *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a continuously differentiable function such that both $f(x)$ and $f'(x)$ are integrable, and such that $\lim_{x \rightarrow \pm\infty} f(x) = 0$. Then $\lim_{\xi \rightarrow \pm\infty} \xi \cdot \widehat{f}(\xi) = 0$.*

In the situation of the last lemma, one can check (using integration by parts) that

$$\xi \cdot \widehat{f}(\xi) = -i \cdot \mathcal{F}(f'(x)).$$

By induction, we see that if $f : \mathbb{R} \rightarrow \mathbb{C}$ is infinitely differentiable and $f^{(m)}(x)$ is integrable for every $m \geq 0$ (where $f^{(m)}$ denotes the m -th derivative of f), then $\xi^m \cdot \widehat{f}(\xi) \rightarrow 0$ as $\xi \rightarrow \pm\infty$ for each $m \geq 0$.

Now, one should view the property that $x^n \cdot f(x)$ is integrable for each $n \geq 0$ as a kind of “fast decay at infinity” condition. In particular, if a measurable function $f(x)$ is bounded and satisfies $\lim_{x \rightarrow \pm\infty} x^n \cdot f(x) = 0$ for all $n \geq 0$, it follows that $x^n \cdot f(x)$ is integrable for all $n \geq 0$, by observing that $x^n \cdot f(x) = (x^{n+2} \cdot f(x)) \cdot x^{-2}$ for all $x \neq 0$.

The upshot of the discussion above is that if a function decays very quickly at infinity, then its Fourier transform is infinitely differentiable. Conversely, if a function is infinitely differentiable and all of its derivatives are integrable, then the Fourier transform of the function decays very quickly at infinity. It is therefore not surprising that by imposing a combination of these two properties, we obtain a space of functions that is invariant under the Fourier transform:

Definition 36.3. A function $f : \mathbb{R} \rightarrow \mathbb{C}$ is said to be a *Schwartz function* if f is infinitely differentiable and for all integers $m, n \geq 0$, we have $\lim_{x \rightarrow \pm\infty} x^m \cdot f^{(n)}(x) = 0$. The space of all Schwartz functions on $\mathbb{R}$ is denoted by $\mathcal{S}(\mathbb{R})$ (it is a vector space over $\mathbb{C}$).

Examples 36.4. There are essentially two main types of examples of Schwartz functions. On the one hand, $C_0^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$. On the other hand, the function $f(x) = e^{-x^2}$ is also a Schwartz function (but it does not have compact support). As far as I know, most interesting examples of Schwartz functions that do not have compact support are obtained as simple variations of the example e^{-x^2} .

36.6. Fourier transform of Schwartz functions. We can now prove

Lemma 36.5. *If $f \in \mathcal{S}(\mathbb{R})$, then $\widehat{f} \in \mathcal{S}(\mathbb{R})$.*

Proof. The comments in §36.5 above imply that if $f \in \mathcal{S}(\mathbb{R})$, then $\widehat{f}(\xi)$ is infinitely differentiable. They also imply that $\xi^m \cdot \widehat{f}(\xi) \rightarrow 0$ as $\xi \rightarrow \pm\infty$ for every $m \geq 0$. It remains to check that the same remains true if $\widehat{f}(\xi)$ is replaced with $\widehat{f}^{(n)}(\xi)$ for any $n \geq 1$. However, this also follows by induction. Indeed, it is easy to see that if $f \in \mathcal{S}(\mathbb{R})$, then $x \cdot f(x) \in \mathcal{S}(\mathbb{R})$ (and, in fact, $x^n \cdot f(x) \in \mathcal{S}(\mathbb{R})$ for each $n \geq 1$). We already saw that $\widehat{f}^{(n)}(\xi)$ is the Fourier transform of $(-ix)^n \cdot f(x)$ for each $n \geq 0$, which completes the proof. $\square$

36.7. A key observation. The following observation will play an important role in the proofs of the results of this section. We claim that:

$$(36.1) \quad \int_{\mathbb{R}} f(x) \widehat{g}(x) \, dm(x) = \int_{\mathbb{R}} g(y) \widehat{f}(y) \, dm(y) \quad \forall f, g \in L^1(\mathbb{R}).$$

We note that both sides are well defined. Indeed, we know that the function $\widehat{g}(x)$ is continuous (in particular, measurable) and bounded. Hence the product $f \cdot \widehat{g}$ is integrable (because f is integrable). Similarly, $g \cdot \widehat{f}$ is integrable.

Identity (36.1) is very easy to prove. First one checks, using Fubini's theorem, that the function $F(x, y) = e^{-ixy} f(x) g(y)$ is integrable with respect to the Lebesgue measure on $\mathbb{R}^2$. Next one uses Fubini's theorem to evaluate $\int_{\mathbb{R}^2} F \, dm$ in two different ways, which implies that $\int_{\mathbb{R}^2} F \, dm$ is equal to both the left hand side and the right hand side of (36.1).

A very similar argument also shows that

$$(36.2) \quad \int_{\mathbb{R}} f(x) \cdot (\mathcal{F}^{-1}g)(x) \, dm(x) = \int_{\mathbb{R}} g(y) \cdot (\mathcal{F}^{-1}f)(y) \, dm(y) \quad \forall f, g \in L^1(\mathbb{R}).$$

36.8. Fourier inversion theorem. Our next goal is to sketch a proof of the Fourier inversion theorem for L^1 functions. We begin with a version for Schwartz functions:

Theorem 36.6. *Let $f \in \mathcal{S}(\mathbb{R})$. Then for every $x \in \mathbb{R}$, we have*

$$(36.3) \quad f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \widehat{f}(\xi) \, dm(\xi) \stackrel{\text{def}}{=} (\mathcal{F}^{-1}\widehat{f})(x).$$

Note that by Lemma 36.5, we have $\widehat{f} \in \mathcal{S}(\mathbb{R})$, and, in particular, $\widehat{f}$ is integrable, so the inverse Fourier transform of $\widehat{f}$ is defined. In addition, there is no need to say “for almost every $x \in \mathbb{R}$ ” in Theorem 36.6 because Schwartz functions are continuous.

Proof of Theorem 36.6. First, using a linear change of variables in (36.1), one shows that for all $f, g \in L^1(\mathbb{R})$ and every number $\lambda > 0$, we have

$$(36.4) \quad \int_{\mathbb{R}} f\left(\frac{x}{\lambda}\right) \cdot \widehat{g}(x) \, dm(x) = \int_{\mathbb{R}} g\left(\frac{y}{\lambda}\right) \cdot \widehat{f}(y) \, dm(y).$$

Next suppose that $f, g \in \mathcal{S}(\mathbb{R})$. Then for a fixed $x \in \mathbb{R}$, we have $f(x/\lambda) \rightarrow f(0)$ as $\lambda \rightarrow +\infty$ because f is continuous. Moreover, f is also bounded, so there is a constant $C > 0$ such that $|f(x/\lambda)| \leq C$ for all $x \in \mathbb{R}$ and all $\lambda > 0$. Hence the product $f\left(\frac{x}{\lambda}\right) \cdot \widehat{g}(x)$ is bounded by $C \cdot |\widehat{g}(x)|$, and we already know that $\widehat{g}$ is integrable because $\widehat{g}$ is a Schwartz function. Hence the Dominated Convergence Theorem allows us to interchange the limit as $\lambda \rightarrow \infty$ with the integral on the left hand side of (36.4). Similarly, we can also interchange the limit with the integral on the right hand side of (36.4). Therefore

$$f(0) \cdot \int_{\mathbb{R}} \widehat{g}(x) \, dm(x) = g(0) \cdot \int_{\mathbb{R}} \widehat{f}(y) \, dm(y) \quad \forall f, g \in \mathcal{S}(\mathbb{R}).$$

Let us now substitute the function $g(y) = e^{-y^2/2}$ into the identity above. We saw earlier that $\widehat{g}(x) = \sqrt{2\pi} \cdot e^{-x^2/2}$, and hence $\int_{\mathbb{R}} \widehat{g}(x) \, dm(x) = 2\pi$. We conclude that

$$f(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(y) \, dm(y) \quad \forall f \in \mathcal{S}(\mathbb{R})$$

(which is in fact a special case of the Fourier inversion formula). Finally, to complete the proof, let $t \in \mathbb{R}$ be arbitrary and consider the function $f_t(x) = f(x+t)$. It is clear that $f_t \in \mathcal{S}(\mathbb{R})$, and one checks using the change of variables $x+t \leftrightarrow x$ that $\widehat{f}_t(y) = e^{ity} \cdot \widehat{f}(y)$. Moreover, $f_t(0) = f(t)$, so replacing f with f_t in the identity above yields (36.3). $\square$

Corollary 36.7. *The Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$ is a linear isomorphism. Moreover, if $f \in \mathcal{S}(\mathbb{R})$, then $\mathcal{F}^2(f) \stackrel{\text{def}}{=} \mathcal{F}(\mathcal{F}(f)) = 2\pi \cdot \check{f}$, where $\check{f}(x) = f(-x)$.*

Proof. By Theorem 36.6, we have $f = \mathcal{F}^{-1}(\mathcal{F}(f))$ for all $f \in \mathcal{S}(\mathbb{R})$. Moreover, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are related to each other via $(\mathcal{F}f)(y) = 2\pi(\mathcal{F}^{-1}f)(-y)$. It follows that $\mathcal{F}(\mathcal{F}(f)) = 2\pi \cdot \check{f}$, which in turn implies that $\mathcal{F} : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$ is bijective, because the map $f(x) \mapsto f(-x)$ is obviously bijective. (In fact, $\mathcal{F} \circ \mathcal{F} \circ \mathcal{F} \circ \mathcal{F}$ equals $4\pi^2$ times the identity map.) $\square$

Theorem 36.8. *Let $f \in L^1(\mathbb{R})$ be such that $\widehat{f} \in L^1(\mathbb{R})$ as well. Then $f = \mathcal{F}^{-1}(\widehat{f})$ a.e.*

Proof. Let us write $f_0 = \mathcal{F}^{-1}(\widehat{f})$ for the time being. We will first check that $\int_{\mathbb{R}} f \cdot h \, dm = \int_{\mathbb{R}} f_0 \cdot h \, dm$ for every $h \in \mathcal{S}(\mathbb{R})$. Given $h \in \mathcal{S}(\mathbb{R})$, we know by Theorem 36.6 and Corollary 36.7 that $h = \widehat{g}$, where $g := \mathcal{F}^{-1}(h) \in \mathcal{S}(\mathbb{R})$. Therefore

$$\int f h \, dm = \int f \widehat{g} \, dm = \int g \widehat{f} \, dm = \int (\mathcal{F}^{-1}h) \widehat{f} \, dm = \int \mathcal{F}^{-1}(\widehat{f}) h \, dm,$$

where in the second equality we used (36.1) and in the fourth one we used (36.2). So we see that $\int_{\mathbb{R}} (f - f_0)h \, dm = 0$ for every $h \in \mathcal{S}(\mathbb{R})$. In particular, this holds for every $h \in C_0^\infty(\mathbb{R})$. Now the lemma below shows that $f = f_0$ a.e. and completes the proof. $\square$

Lemma 36.9. *If $f \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} f \cdot \varphi \, dm = 0$ for all $\varphi \in C_0^\infty(\mathbb{R})$, then $f = 0$ a.e.*

A slightly more general statement is proved in the next section of these lecture notes.

36.9. Plancherel's formula. Our last topic is Plancherel's formula and the extension of the Fourier transform to the space $L^2(\mathbb{R})$. Let us recall (from Problem Set 7) that $L^2(\mathbb{R})$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) \cdot \overline{g(x)} \, dm(x),$$

where the bar denotes complex conjugation.

Theorem 36.10 (Plancherel's formula). *If $f, g \in \mathcal{S}(\mathbb{R})$, then $\langle \widehat{f}, \widehat{g} \rangle = 2\pi \cdot \langle f, g \rangle$.*

Proof. Using (36.1), we obtain $\langle \widehat{f}, \widehat{g} \rangle = \int_{\mathbb{R}} \widehat{f} \cdot \overline{\widehat{g}} \, dm = \int_{\mathbb{R}} f \cdot \mathcal{F}(\widehat{g})$. Next, one easily checks that for any $h \in L^1(\mathbb{R})$, we have $\mathcal{F}(\widehat{h}) = 2\pi \cdot \overline{\mathcal{F}^{-1}(h)}$. Since $\mathcal{F}^{-1}(\widehat{g}) = g$ by Theorem 36.6, we find that $\langle \widehat{f}, \widehat{g} \rangle = 2\pi \cdot \langle f, g \rangle$, as claimed. $\square$

In particular, we see that $(2\pi)^{-1/2} \cdot \mathcal{F} : \mathcal{S}(\mathbb{R}) \xrightarrow{\sim} \mathcal{S}(\mathbb{R})$ is an *isometry* with respect to the L^2 norm: for any $f \in \mathcal{S}(\mathbb{R})$, we have $\left\| \frac{\widehat{f}}{\sqrt{2\pi}} \right\|_2 = \|f\|_2$. Moreover, by this week's homework, $C_0^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, and hence, *a fortiori*, $\mathcal{S}(\mathbb{R})$ is also dense in $L^2(\mathbb{R})$. By homework problem 10.1, it follows that $(2\pi)^{-1/2} \cdot \mathcal{F}$ extends uniquely to an isometry $L^2(\mathbb{R}) \xrightarrow{\sim} L^2(\mathbb{R})$. Thus we can now define $\mathcal{F}$ as a linear map from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$.

- Remarks 36.11.* (1) Earlier we had a definition of $\mathcal{F} = \mathcal{F}_{old}$ for functions $f \in L^1(\mathbb{R})$. Now we have another Fourier transform, $\mathcal{F} = \mathcal{F}_{new} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$. By construction, $\mathcal{F}_{old}(f) = \mathcal{F}_{new}(f)$ for every $f \in \mathcal{S}(\mathbb{R})$. In fact, one can show that the same is true whenever $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Thus, the notation $\mathcal{F}$ remains unambiguous: if $\mathcal{F}(f)$ can be defined for a given function f in two different ways, the results are the same.
- (2) However, there is an important difference between $\mathcal{F}_{old}$ and $\mathcal{F}_{new}$. For any $f \in L^1(\mathbb{R})$, the values $\widehat{f}(\xi)$ are unambiguously defined for every $\xi \in \mathbb{R}$. On the other hand, if $f \in L^2(\mathbb{R})$, then $\mathcal{F}(f)$ is only well defined as an element of $L^2(\mathbb{R})$. In particular, it does not make sense to speak of the value of $\mathcal{F}(f)$ at a given point of $\mathbb{R}$, because $\mathcal{F}(f)$ is only defined up to modifying it on a set of measure zero.