**Comparison of KL and Hankel transforms** PhL 7.18.11

I have updated this comparison better in " a review of generic singular BC...". as of 8.29.11

I will try a side-by-side comparison here: (note that the interval in both cases is 0,∞ )

| | The KL case | The Hankel case |
|---|---|---|
| ODE: | $-(xg')' +\nu^2 xg -\lambda g/x = \delta(x-\xi)$ | $-(xg')' + \nu^2\, g/x - \lambda xg = \delta(x-\xi)$ s |
| Green's: | $I_{-ik}(\nu x_<)\, K_{-ik}(\nu x_>)$, $k \equiv \sqrt{\lambda}$ | $(i\pi/2)\, J_\nu(kx_<)\, H_\nu^{(1)}(kx_>)$, $k = \sqrt{\lambda}$ |
| | $I_{-i\sqrt{\lambda}}(\nu x_<)\ K_{-i\sqrt{\lambda}}(\nu x_>)$ | $(i\pi/2)\, J_\nu(\sqrt{\lambda}\, x_<)\, H_\nu^{(1)}(\sqrt{\lambda}\, x_>)$ |
| Lg: | $Lg = -(xg')' +\nu^2 xg$<br>$p(x) = x$, $q(x) = \nu^2 x$ | $Lg = -(xg')' + \nu^2 g/x$<br>$p(x) = x$, $q(x) = \nu^2/x$ |
| [L-λs(x)]g: | $-(xg')' +\nu^2 xg - \lambda g/x$ $\quad s(x)=1/x$ | $-(xg')' + \nu^2 g/x - \lambda xg$ $\quad s(x)=x$ |
| completeness: | $x\,\delta(x-\xi)$<br>$= (2/\pi^2) \int_0^\infty dk\, k \sinh(\pi k) K_{ik}(\nu x)\, K_{ik}(\nu\xi)$ | $\delta(x-\xi)/x$<br>$= \int_0^\infty dk\, k\ J_\nu(kx) J_\nu(k\xi)$ |

eigenfunctions:

$\varphi_k(x) = (\sqrt{2}/\pi)\ \sqrt{k \sinh(\pi k)}\ K_{ik}(\nu x)$ //wrong $\qquad$ $\varphi_k(x) = \sqrt{k}\ J_\nu(kx)$ //wrong

Note that $K_{-\nu}(z) = K_\nu(z)$ so $K_{-ik}(\nu x)$ OK.

This last we know as follows. Looking at vol I p 304 we have (for a general spectrum situation)

$$-(1/2\pi i) \oint d\lambda\, g(x|\xi;\lambda) = \delta(x-\xi)/s(x) = \Sigma_n\, \varphi_n(x)\, \bar{\varphi}_n(\xi) + \int d\alpha\, \varphi_\alpha(x)\, \bar{\varphi}_\alpha(\xi)$$

Suppose there is only some continuous spectrum we label by α say in (0,∞). Then we have

$$\delta(x-\xi) = s(x) \int_0^\infty d\alpha\, \varphi_\alpha(\xi)\, \bar{\varphi}_\alpha(x)$$

Apply $\int dx\ \varphi_{\alpha'}(x)$ to both sides to get

$$\int dx\ \varphi_{\alpha'}(x)\, \delta(x-\xi) = \int dx\, s(x)\, \varphi_{\alpha'}(x) \int_0^\infty d\alpha\, \varphi_\alpha(\xi)\, \bar{\varphi}_\alpha(x)$$

$$\Rightarrow \qquad \varphi_{\alpha'}(\xi) = \int_0^\infty d\alpha\, \varphi_\alpha(\xi) \int dx\, s(x)\, \varphi_{\alpha'}(x)\, \bar{\varphi}_\alpha(x)$$

This is consistent with this normalization:

$$\int dx\, s(x)\, \varphi_{\alpha'}(x)\, \bar{\varphi}_\alpha(x) = \delta(\alpha-\alpha')$$

Now, in the KL case if we think of $\alpha = k$ as our spectrum label, then we can write

$$\delta(x-\xi) = s(x) \int_0^\infty d\alpha\, \varphi_\alpha(\xi)\, \bar{\varphi}_\alpha(x)$$

$$= (1/x) \int_0^\infty dk\, \varphi_k(\xi)\, \bar{\varphi}_k(x)$$

$$= (1/x)\,(2/\pi^2) \int_0^\infty dk\, k \sinh(\pi k) K_{ik}(\nu x)\, K_{ik}(\nu\xi)$$

from which we conclude that ( remember K order symmetry!)

$$\varphi_k(x) = (\sqrt{2}/\pi)\, \sqrt{k \sinh(\pi k)}\, K_{ik}(\nu x) \qquad \text{// wrong!}$$

with normalization given by

$$\int dx\, (1/x)\, \varphi_{k'}(x)\, \bar{\varphi}_k(x) = \delta(k-k')$$

In the Hankel case we again think of $\alpha = k$ and we have

$$\delta(x-\xi) = s(x) \int_0^\infty d\alpha\, \varphi_\alpha(\xi)\, \bar{\varphi}_\alpha(x)$$

$$= x \int_0^\infty dk\, \varphi_k(\xi)\, \bar{\varphi}_k(x)$$

$$= x \int_0^\infty dk\, k\, J_\nu(kx) J_\nu(k\xi)$$

from which we conclude that

$$\varphi_k(x) = \sqrt{k}\, J_\nu(kx) \qquad \text{// wrong}$$

with normalization given by

$$\int dx\, (x)\, \varphi_{k'}(x)\, \bar{\varphi}_k(x) = \delta(k-k')$$

or

$$\int dx\, (x) \sqrt{k'}\, J_\nu(k'x) \sqrt{k}\, J_\nu(kx) = \delta(k-k')$$

or

$$\int dx\, x\, J_\nu(k'x)\, J_\nu(kx) = \delta(k-k')/\sqrt{k'}\sqrt{k} = \delta(k-k')/k$$

which agrees with my transforms.doc grouping which I quote here:

$$\varphi_k(x;\nu) = \sqrt{k}\, J_\nu(kx) \qquad \text{// complete orthonormal set}$$

$$f(\rho) = \int_0^\infty dk\, k\, J_\nu(k\rho)\, F_\nu(k) \qquad \text{// expansion}$$

$$F_\nu(k) = \int_0^\infty d\rho\, \rho\, J_\nu(k\rho)\, f(\rho) \qquad \text{// projection}$$

$$\int_0^\infty d\rho\, \rho\, J_\nu(k\rho)\, J_\nu(k'\rho) = \delta(k-k')/k \qquad \text{// orthogonality}$$

$$\int_0^\infty dk\, k\, J_\nu(k\rho)\, J_\nu(k\rho') = \delta(\rho-\rho')/\rho \qquad \text{// completeness}$$

**Statement of the KL transform.**

We have from above that

$$\varphi_k(x;\nu) = (\sqrt{2}/\pi)\ \sqrt{k\, sh(\pi k)}\ K_{ik}(\nu x)$$

and then we can write orthogonality this way

$$\int dx\, (x)\, \varphi_{k'}(x)\, \bar{\varphi}_k(x) = \delta(k-k')$$

$$(2/\pi^2) \int dx\, x \sqrt{k\, sh(\pi k)}\ K_{ik}(\nu x) \sqrt{k\, sh(\pi k)}\ K_{ik}(\nu x) = \delta(k-k')$$

Now in the KL doc we wrote

$$-(xg')' + \mu x g - \lambda x^{-1} g = \delta(x-\xi) \qquad \lambda = \gamma^2\ ,\ \mu = k^2$$

$$f(x) = (2/\pi^2) \int_0^\infty \kappa d\kappa\, \sinh(\pi\kappa)\, K_{i\kappa}(\sqrt{\mu}\, x)\, F(\kappa) \qquad \text{which is 4.139.}$$

$$F(\kappa) \equiv \int_0^\infty dx\, f(x)\, K_{i\kappa}(\sqrt{\mu}\, x)/x$$

which we can translate to say

$$-(xg')' + \mu x g - \lambda x^{-1} g = \delta(x-\xi) \qquad \lambda = \gamma^2\ ,\ \mu = k^2$$

$$f(x) = (2/\pi^2) \int_0^\infty \kappa d\kappa\, \sinh(\pi\kappa)\, K_{i\kappa}(\sqrt{\mu}\, x)\, F(\kappa) \qquad \text{which is 4.139.}$$

$$F(\kappa) \equiv \int_0^\infty dx\, f(x)\, K_{i\kappa}(\sqrt{\mu}\, x)/x$$

So in our current document we will then have $\mu \rightarrow \nu^2$ and

$$-(xg')' + \mu x g - \lambda x^{-1} g = \delta(x-\xi)$$

$$f(x) = (2/\pi^2) \int_0^\infty \kappa d\kappa \sinh(\pi\kappa) K_{i\kappa}(\sqrt{\mu}\, x) F(\kappa) \qquad \text{which is 4.139.}$$

$$F(\kappa) \equiv \int_0^\infty dx\, f(x) K_{i\kappa}(\sqrt{\mu}\, x)/x$$

And here are the corresponding K-L results

$$f(x) = (2/\pi^2) \int_0^\infty d\kappa\, \kappa \sinh(\pi\kappa) K_{i\kappa}(\sqrt{\mu}\, x) F(\kappa) \qquad \text{// expansion}$$

$$F(\kappa) \equiv \int_0^\infty dx\, f(x) K_{i\kappa}(\sqrt{\mu}\, x)/x \qquad \text{// projection}$$

$$(2/\pi^2) \int dx\, (1/x) \sqrt{k \sh(\pi k)}\; K_{ik}(\nu x)\; \sqrt{k' \sh(\pi k')}\; K_{ik'}(\nu x) = \delta(k-k') \qquad \text{// orthogonality}$$

$$(2/\pi^2) \int_0^\infty dk\, k \sinh(\pi k) K_{ik}(\nu x) K_{ik}(\nu x') = x\delta(x-x') \qquad \text{// completeness}$$

**Comments:**

- The ODE's are the same if we just swap $\nu^2 \leftrightarrow -\lambda$ . This rule converts either into the other.

- If we make this swap, either Green's function becomes the other Green's function (as must be the case)

$$I_{-i\sqrt{\lambda}}(\nu x_<)\; K_{-i\sqrt{\lambda}}(\nu x_>) = I_\nu(-i\sqrt{\lambda}\, x_<) K_\nu(-i\sqrt{\lambda}\, x_>)$$

$$= i^{-\nu} J_\nu(i * -i\sqrt{\lambda}\, x_<)\; (\pi/2)\, i^{\nu+1} H_\nu^{(1)}(i*-i\sqrt{\lambda}\, x_>)$$

$$= (i\pi/2) J_\nu(\sqrt{\lambda}\, x_<) H_\nu^{(1)}(\sqrt{\lambda}\, x_>)$$

- This swap does NOT convert one completeness relation into the other. One has an integration of a factor in the Bessel function order, the other an integration of a factor in the Bessel function argument.

• The underlying L operators are different. The s(x) and q(x) functions are different.

• Both transforms are associated with a Bessel equation with two parameters I have called $\nu^2$ and $\lambda$. And both transforms involve the interval $(0,\infty)$. Both SL problems have a $\lambda$ spectrum of the positive real axis.

• The two associated eigenvalue problems are different even though the spectrum is the same:

$L_\nu u_\nu = \lambda s(x) u_\nu$ :

| | | |
|---|---|---|
| KL: | $L_\nu u = -(xu')' + \nu^2 xu$ | $s(x) = 1/x$ |
| Hankel: | $L_\nu u = -(xu')' + \nu^2 (1/x)u$ | $s(x) = x$ |

• Question: would you refer to either of these L's as a Bessel Operator? I guess they both are such, in the sense that if you consider the EV problem associated with that L, and if you pick the right s(x), you arrive at a 2-parameter Bessel ODE in either case. It is not the same ODE, however, since $-\lambda$ and $\nu^2$ are swapped.

• The official Bessel's equation [ Schaum p 136 with $\nu=n$ ] has one numerator parameter $\nu^2$ which is the order. But we can write a "scaled Bessel's equation" which has another numerator parameter $k^2$. Then we go on two different paths if we take one or the other of these two parameters to be the $\lambda$ eigenvalue parameter of a SL problem. Each path has its own q(x) and s(x). Each path is a different SL problem and each path results in a different transform. In either path, the functions involved are always something in the "family of Bessel functions".