The Hankel Transform PhL 7.18.11

According to our Bessel Functions document Theorem 3, we know that

$w(x) = J_\nu(kx)$ solves $-(xw')' - k^2xw + \nu^2\, w/x = 0.$

or, swapping order of the last two terms

$-(xw')' + \nu^2\, w/x - k^2xw = 0$

We now want to think of k^2 as being our official λ, so we then have

$-(xw')' + \nu^2\, w/x - \lambda xw = 0 \qquad \lambda = k^2 \qquad k = \sqrt{\lambda}$

This ODE is in agreement with Stak Vol I p 316 where Stak treats the Hankel transform. I presume that the Green's function defining equation is this

$-(xg')' + \nu^2\, g/x - \lambda xg = \delta(x-\xi)$

or

$Lg - \lambda s(x) g = \delta(x-\xi) \qquad Lg = -(xg')' + \nu^2\, g/x \qquad s(x) = x \qquad x \text{ in } (0,\infty)$

This L is self adjoint and we have that $p(x) = x$ and $q(x) = \nu^2/x$ as per Vol I page 268.

On page 316 Stak claims that this is the solution free-space Green's function

$$g(x|\xi;\lambda) = (i\pi/2)\, J_\nu(\sqrt{\lambda}\, x_<)\, H_\nu^{(1)}(\sqrt{\lambda}\, x_>)$$

We know at once that the large and small x limits are good, and we know we could confirm the jump constant using the Wronksian for these two functions.

Big Question: what is the situation in the λ plane for $g(x|\xi;\lambda)$?When we first look at g, we are inclined to say that the singularities are all due to the function $k(\lambda) \equiv \sqrt{\lambda}$ and our normal instinct when encountering $f(z) = z^{1/2}$ is to draw the cut off to the left. But in Green's Function Theory, we know that we need to take the cut off to the right. We know that the spectrum in λ is always on the positive real axis. In this case, we know there would be poles on the right for a cylinder space of finite radius, and as that radius is increased, we know the poles on the right coalesce into a cut on the right.

We can compute the discontinuity across this cut on the right of the Green's function as follows:

$\lambda^+ = \lambda \qquad \sqrt{\lambda^+} = \sqrt{\lambda} \qquad \lambda > 0$ everywhere on the cut

$\lambda^- = \lambda e^{i2\pi} \qquad \sqrt{\lambda^-} = -\sqrt{\lambda}$

We then have for our discontinuity across the cut which exists for $\lambda>0$,

$$\text{disc}_\lambda[g(x|\xi;\lambda)] = g(x|\xi;\lambda+) - g(x|\xi;\lambda-)$$

$$= (i\pi/2)\, J_\nu(\sqrt{\lambda}\, x_<)\, H_\nu^{(1)}(\sqrt{\lambda}\, x_>) \; - \; (i\pi/2)\, J_\nu(-\sqrt{\lambda}\, x_<)\, H_\nu^{(1)}(-\sqrt{\lambda}\, x_>)$$

$$= -(i\pi/2)\,[J_\nu(-a)\, H_\nu^{(1)}(-b) - J_\nu(a)\, H_\nu^{(1)}(b)] \qquad a = \sqrt{\lambda}\, x_< \qquad a = \sqrt{\lambda}\, x_>$$

Now nicely Stak gives us the required symmetry rules right on page 316 so we continue

$$= -(i\pi/2)\,[e^{-i\pi\nu} J_\nu(a)\,\{e^{-i\pi\nu}(\, H_\nu^{(1)}(b) - 2J_\nu(b)\,)\} \; - J_\nu(a)\, H_\nu^{(1)}(b)]$$
$$= -\,(i\pi/2)\,[J_\nu(a)\,\{\, H_\nu^{(1)}(b) - 2J_\nu(b)\,\} \; - J_\nu(a)\, H_\nu^{(1)}(b)]$$
$$= -\,(i\pi/2)\,[J_\nu(a)\, H_\nu^{(1)}(b) - 2\, J_\nu(a)\, J_\nu(b) \; - J_\nu(a)\, H_\nu^{(1)}(b)]$$
$$= -\,(i\pi/2)\,[J_\nu(a)\,\{-\,2J_\nu(b)\,\}] \;\; = \;\; (i\pi)\, J_\nu(a) J_\nu(b) \qquad a = \sqrt{\lambda}\, x_< \qquad a = \sqrt{\lambda}\, x_>$$

So we can then say, where C is a contour wrapping the positive λ axis in the CCW direction, $s(x) = x$,

Completeness: $\quad -(1/2\pi i) \oint d\lambda\, g(x|\xi;\lambda) \;= \delta(x-\xi)/s(x) \;= \Sigma_n\, \varphi_n(x)\, \bar{\varphi}_n(\xi)$

Orthogonality: $\quad \langle \sqrt{s}\, \varphi_n, \sqrt{s}\, \varphi_m \rangle \;= K_n\, \delta_{n,m} \;\; // \; K_n = 1$!!

$$\delta(x-\xi)/x = -\,(1/2\pi i) \int_C d\lambda\, g(x|\xi;\lambda)$$
$$= +\,(1/2\pi i) \int_0^\infty d\lambda\, \mathrm{disc}_\lambda[g(x|\xi;\lambda)] \qquad // \text{ sign due to sense of contour}$$
$$= +\,(1/2\pi i) \int_0^\infty d\lambda\; (i\pi)\, J_\nu(\sqrt{\lambda}\, x_<) J_\nu(\sqrt{\lambda}\, x_>)$$
$$= +\,(1/2) \int_0^\infty d\lambda\; J_\nu(\sqrt{\lambda}\, x_<) J_\nu(\sqrt{\lambda}\, x_>)$$

Now replace $\lambda = k^2$ with $d\lambda = 2kdk$ and get

$$= +\,(1/2) \int_0^\infty 2kdk\; J_\nu(kx_<) J_\nu(kx_>)$$
$$= + \int_0^\infty dk\, k\; J_\nu(kx_<) J_\nu(kx_>)$$

so we have then obtained the completeness relation

$$\delta(x-\xi)/x \;=\; \int_0^\infty dk\, k\; J_\nu(kx_<) J_\nu(kx_>)$$

Now let's define the following projection of a function f(x) defined on $(0,\infty)$,

$$F_\nu(k) \equiv \int_0^\infty dx\; x\, J_\nu(kx)\, f(x)$$

Then apply $\int_0^\infty dk\ k\ J_\nu(kx')$ to both sides to get

$$\int_0^\infty dk\ k\ J_\nu(kx')\ F_\nu(k) = \int_0^\infty dk\ k\ J_\nu(kx') \int_0^\infty dx\ x\ J_\nu(kx)\ f(x)$$

$$= \int_0^\infty dx\ x\ f(x) \int_0^\infty dk\ k\ J_\nu(kx')\ J_\nu(kx) = \int_0^\infty dx\ x\ f(x)\ \delta(x-x')/x = f(x')$$

so we have then obtained the Hankel transform:

$$F_\nu(k) = \int_0^\infty dx\ x\ J_\nu(kx)\ f(x)$$

$$f(x) = \int_0^\infty dk\ k\ J_\nu(kx)\ F_\nu(k)$$

and the thing is perfectly symmetric. This then leads to my transforms.doc symmetric statement of the Hankel transform

$$f(\rho) = \int_0^\infty dk\ k\ J_\nu(k\rho)\ F_\nu(k) \qquad \text{// expansion}$$

$$F_\nu(k) = \int_0^\infty d\rho\ \rho\ J_\nu(k\rho)\ f(\rho) \qquad \text{// projection}$$

$$\int_0^\infty d\rho\ \rho\ J_\nu(k\rho)\ J_\nu(k'\rho) = \delta(k-k')/k \qquad \text{// orthogonality}$$

$$\int_0^\infty dk\ k\ J_\nu(k\rho)\ J_\nu(k\rho') = \delta(\rho-\rho')/\rho \qquad \text{// completeness}$$